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Some Results of Axiomatic Quantum Field Theory in Noncommutative Spacetime*

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We present this work for the special volume of the Acta Physica issued in honour of our friend and colleague Professor Peter Prešnajder on the occasion of his 75th Birthday. This specific work we have selected as our contribution dedicated to him, since it has been first Peter Prešnajder who has introduced us to the concept of Noncommutative Geometry and in particular to the subject of Quantum Field Theory in Noncommutative Spacetime, as early as in 1998 when we started to work with him as our first common work on this subject [1]. After that, thanks to the initiative attitude and the original ideas Peter had, we started to produced a series of works on the subject, among them our joint works [2]–[4], ..., the last our joint work with Peter on this subject being [5]. Of course Peter had previously been working on diverse aspects of Noncommutative Geometry since the beginning of 1990's.

Abstract: Classical results of the axiomatic quantum field theory - irreducibility of the set of field operators, Reeh and Schlieder's theorems and generalized Haag's theorem are proven in SO(1,1) invariant quantum field theory, of which an important example is noncommutative quantum field theory. In SO(1,3) invariant theory new consequences of generalized Haag's theorem are obtained. It has been proven that the equality of four-point Wightman functions in two theories leads to the equality of elastic scattering amplitudes and thus the total cross-sections in these theories.

1. Introduction

Quantum field theory (QFT) as a mathematically rigorous and consistent theory was formulated in the framework of the axiomatic approach in the works of Wightman, Jost, Bogoliubov, Haag and others ([6]–[10]).

Within the framework of this theory on the basis of most general principles such as Poincaré invariance, local commutativity and spectrality, a number of fundamental physical results, for example, the CPT-theorem and the spin-statistics theorem were proven [6]–[9].

Noncommutative quantum field theory (NC QFT) being one of the generalizations of standard QFT has been intensively developed during the past years (for reviews, see [11, 12]). The idea of such a generalization of QFT ascends to Heisenberg and it was initially developed in Snyder's work [13]. The present development in this direction is connected with the construction of noncommutative geometry [14] and new physical

*Dedicated to Professor Peter Prešnajder on the occasion of his 70th birthday
†Professor Yuri Vernov passed away in Moscow on 27 May 2015

arguments in favour of such a generalization of QFT [15]. Essential interest in NC QFT is also due the fact that in some cases it is a low-energy limit of string theory [16]. The simplest and at the same time most studied version of noncommutative field theory is based on the following Heisenberg-like commutation relations between coordinates:

$$[x \;, x \;] \quad i \quad , \tag{1}$$

where is a constant antisymmetric matrix

It is known that the construction of NC QFT in a general case (^{0i} 0) meets serious difficulties with unitarity and causality [17]–[20]. For this reason the version with ^{0i} 0 (space-space noncommutativity), in which there do not appear such difficulties and which is a low-energy limit of the string theory, draws special attention. Then always there is a system of coordinates, in which only ^{12} ^{21} 0 [21]. Thus, when ^{0i} 0, without loss of generality it is possible to choose coordinates x^0 and x^3 as commutative and coordinates x^1 and x^2 as noncommutative.

The relation (1) breaks the Lorentz invariance of the theory, while the symmetry under the SO(1,1) SO(2) subgroup of the Lorentz group survives [19]. Translational invariance is still valid. Below we shall consider the theory to be SO(1,1) invariant with respect to coordinates x^0 and x^3. Besides these classical groups of symmetry, in the paper [22] it was shown, that the noncommutative field theory with the commutation relation (1) of the coordinates, and built according to the Weyl-Moyal correspondence, has also a quantum symmetry, i.e. twisted Poincaré invariance.

In the works [23], [24] the Wightman approach was formulated for NC QFT. For scalar fields the CPT theorem and the spin-statistics theorem were proven in the case ^{0i} 0.

In [23] it was proposed that Wightman functions in the noncommutative case can be written down in the standard form

$$W(x_1,\ldots,x_n) \quad \langle \;_0, \; (x_1)\ldots \; (x_n) \;_0 \rangle, \tag{2}$$

where $_0$ is the vacuum state. However, unlike the commutative case, these Wightman functions are only SO(1,1) SO(2) invariant. In fact in [23] the CPT theorem has been proven in the commutative theory, where Lorents invariance is broken up to SO(1,1) SO(2) symmetry, as in the noncommutative theory it is necessary to use the -product at least in coinciding points.

In [24] it was proposed that in the noncommutative case the usual product of operators in the Wightman functions has to be replaced by the Moyal-type product both in coinciding and different points:

$$(x_1) \;\ldots\; (x_n) = \prod_{a\; b\; n} \exp \frac{i}{2} \quad \frac{}{x_a} \frac{}{x_b} \quad (x_1)\ldots \; (x_n),$$

$$a \quad 1,2,\ldots n \quad 1. \tag{3}$$

Such a product of operators is compatible with the twisted Poincaré invariance of the theory [25] and also reflects the natural physical assumption, that noncommutativity should change the product of operators not only in coinciding points, but also in different ones. This follows also from another interpretation of NC QFT in terms of a quantum shift operator [26].

In [27] it was shown that in the derivation of axiomatic results, the concrete type of product of operators in various points is insignificant. It is essential only that from the appropriate spectral condition (see formula (10)), the analyticity of Wightman functions with respect to the commutative variables x^0 and x^3 follows, while x^1 and x^2 remain real. In accordance with Eq. (3) the Wightman functions can be written down as follows:

$$W\,(x_1,\ldots,x_n)\quad \langle\ {}_0,\ (x_1)\ \ \ldots\ \ \ (x_n)\ {}_0\rangle . \tag{4}$$

Note that actually there is no field operator defined in a point [28], (see also [8]). Only the smoothed operators written symbolically as

$$_f\qquad (x) f(x)\, dx\,, \tag{5}$$

where $f(x)$ are test functions, can be rigorously defined.

In QFT the standard assumption is that all $f(x)$ are test functions of tempered distributions. On the contrary, in the NC QFT the corresponding generalized functions can not be tempered distributions as the -product contains infinite number of derivatives. It is well-known (see, for example, [6]) that there could be only a finite number of derivatives in any tempered distribution.

The formal expression (4) actually means that the scalar product of the vectors $_k\quad {}_{f_k}\cdots\ {}_{f_1}\ {}_0$ and $_n\quad {}_{f_{k\ 1}}\cdots\ {}_{f_n}\ {}_0$ is the following:

$$\langle\ {}_k,\ {}_n\rangle\qquad W(x_1,\ldots,x_n)\overline{f_1(x_1)}\ \ \ldots\ \ \ \overline{f_k(x_k)}\,dx_1\ldots dx_n;$$

$$W(x_1,\ldots,x_n)\quad \langle\ {}_0,\ (x_1)\ldots\ (x_n)\ {}_0\rangle . \tag{6}$$

In paper [29] it was shown that the series

$$f(x)\quad f(y)\quad \exp\frac{i}{2}\qquad \frac{}{x}\ \frac{}{y}\quad f(x)f(y) \tag{7}$$

converges if $f(x)\quad S\ ,\qquad 1/2,\ S$ is a Gel̓fand-Shilov space [30]. The similar result was obtained also in [31].

The difference of noncommutative case from commutative one is that action of the operator $_f$ is defined by the -product.

In [27] it was shown that, besides the above-mentioned theorems, in NC QFT (with $_{0i}$ 0) a number of other classical results of the axiomatic theory remain valid. In [25] on the basis of the twisted Poincaré invariance of the theory the Haag's theorem was obtained [33, 34] (see also [6] and references therein).

The present work deals with further development of the axiomatic approach in NC QFT. In fact, our results are valid for the wide class of SO(1,1) invariant four-dimensional field theories.

At first we formulate the basic properties of Wightman functions in spacespace NC QFT.

In the present work, analogues of some known results of the axiomatic approach in quantum field theory are obtained for the SO(1,1) invariant field theory, of which an important example is NC QFT. We prove that classical results, such as the irreducibility of the set of field operators, the theorems of Reeh and Schlieder [6]–[9] remain valid in the noncommutative case. It should be emphasized that the results obtained in this paper do not depend on the SO(2) invariance of the theory in the variables x^1 and x^2 and therefore can be extended to more general cases. The irreducibility of the set of field operators remain valid in any theory, which is translation invariant in commutative variables, if only Eq. (27) is fulfilled. The first theorem of Reeh and Schlieder is valid, if the Wightman functions are analytical in the variables x^0 and x^3 in the primitive domains of analyticity ("tubes").

In the SO(1,3) invariant theory new consequences of the generalized Haag's theorem are found, without analogues in NC QFT. At the same time it is proven that the basic physical conclusion of Haag's theorem is valid also in the SO(1,1) invariant theory, and it is sufficient that spectrality, local commutativity condition and translational invariance be fulfilled only for the transformations concerning the commutating coordinates. The analysis of Haag's theorem reveals essential distinctions between commutative and noncommutative cases, more precisely between the SO(1,3) and SO(1,1) invariant theories. In the commutative case, the conditions (59) and (60), whose consequence is generalized Haag's theorem, lead to the equality of Wightman functions in two theories up to four-point ones. In the present paper it is shown that in the SO(1,1) invariant theory, unlike the commutative case, only two-point Wightman functions are equal and it is shown that from the equality of two-point Wightman functions in two theories it follows that if in one of them the current is equal to zero, it is equal to zero in the other as well and under weaker conditions than the standard ones. It is also shown that for the derivation of Eq. (60) it is sufficient to assume that the vacuum vector is a unique normalized vector, invariant under translations along the axis x^3. It is proven that from the equality of four-point Wightman functions in two theories, the equality of their elastic scattering amplitudes follows and, owing to the optical theorem, the equality of total cross sections as well. In derivation of this result LCC is not used.

The study of Wightman functions leads still to new nontrivial consequences also in the commutative case[1].

The paper is arranged as follows. In section 2 the basic properties of Wightman functions in space-space NC QFT are formulated; in section 3 the irreducibility of the set of field operators is proven; in section 4 generalizations of the theorems of Reeh and Schlieder to NC QFT are obtained; section 5 is devoted to generalized Haag's theorem; in section 6 it is shown that in the commutative case, the conditions of weak local commutativity (WLCC) and of local commutativity (LCC), which are valid in the noncommutative case ((24) and (22)), appear to be equivalent to the usual WLCC and LCC, respectively.

2. Basic Properties of Wightman Functions in Space-space NC QFT

As in the commutative case, we assume that every vector of J can be approximated with arbitrary accuracy by the vectors of the type

$$ {}_{f_1}\cdots\ {}_{f_n}\ {}_0 \ . \tag{8}$$

In other words the vacuum vector ${}_0$ is cyclic, i.e. the axiom of cyclicity of vacuum is fulfilled.

Let us note that the vectors of the type (8) can be written formally as follows:

$$ {}_{f_1}\cdots\ {}_{f_n}\ {}_0 \qquad (x_1)\ldots\ (x_n)\ {}_0 f_1(x_1)\ \ \ldots\ \ f_n(x_n)dx_1\ldots dx_n\ . \tag{9}$$

[1]A partial result on the subject had been previously communicated in [35].

It is naturally to assume that Wightman functions are tempered distributions with respect to commutative coordinates as the -product contains derivatives with respect to noncommutative coordinates only. In accordance with this assumption we can use the standard arguments to prove Wightman functions analyticity in "tubes" and extended "tubes".

It is well known that in commutative case Wightman functions analyticity in tubes is a consequence of the spectral condition, which implies that complete system of physical states (in gauge theories also nonphysical ones) does not contain tahyon states in momentum space. It means that momentum P_m for every state satisfies the condition: $P_n^0 \quad |\vec{P}_n|$. This condition is usually written as $P_n \quad \bar{V}$. As Wightman functions in the noncommutative case are analytical function only in commutative variables, it is sufficient to assume the weaker condition of spectrality. Precisely, we assume that any vector in p space, belonging to the complete system of these vectors, is time-like with respect to momentum components P_n^0 and P_n^3, i.e. that

$$P_n^0 \quad \left|P_n^3\right|. \tag{10}$$

The condition (10) is conveniently written as $P_n \quad \bar{V}_2$, where $\bar{V}_2$ is the set of the four-dimensional vectors satisfying the condition $P^0 \quad \left|P^3\right|$.

For the results obtained below, translational invariance only in commuting coordinates is essential, therefore we write down the Wightman functions as:

$$W(x_1,\ldots,x_n) \quad W(\ _1,\ldots,\ _{n\ 1},X), \tag{11}$$

where X designates the set of noncommutative variables $x_i^1, x_i^2,\ i \quad 1,\ldots n$, and $\ _j \qquad \ _j^0,\ _j^3$, where $\ _j^0 \quad x_j^0 \quad x_{j\ 1}^0,\ \ _j^3 \quad x_j^3 \quad x_{j\ 1}^3,\ j \quad 1,\ldots,n \quad 1.$

Thus at arbitrary X we can express scalar product (6) as follows:

$$\langle \ _k,\ _n\rangle \qquad W(\ _1,\ldots,\ _{n\ 1},X)\, f(\ _1,\ldots,\ _{n\ 1},X)\, d\ _1\ldots d\ _{n\ 1} \tag{12}$$

and use the completeness of the system of vectors $\ _{P_m}$, where $P_m \quad \{P_n^0, P_n^t\}$ is the two-dimensional momentum corresponding to the commutative coordinates, multiindex n denotes all other characteristics of the state. So

$$\langle\ ,\ \rangle \quad \ _n \quad dP_m \langle\ ,\ _{P_m}\rangle\langle\ _{P_m},\ \rangle. \tag{13}$$

From the condition (10) and Eq. (13) it follows that

$$da\, e^{\ ipa}\langle\ ,U(a)\ \rangle \quad 0,\ \text{if}\ p \quad \bar{V}_2\ , \tag{14}$$

where $a \quad \{a^0, a^3\}$ is a two-dimensional vector, $U(a)$ is a translation in the plane x^0, x^3, and and are arbitrary vectors. The equality (14) is similar to the corresponding equality in the standard case ([6], Chap. 2.6). A direct consequence of the equality (14) is the spectral property of Wightman functions:

$$W(P_1\ldots,P_{n\ 1},X) \quad \frac{1}{(2\ )^{n\ 1}} \quad e^{iP_j\ j}\, W(\ _1,\ldots,\ _{n\ 1},X)\, d\ _1\ldots d\ _{n\ 1} \quad 0, \tag{15}$$

if $P_j \quad \bar{V}_2$. The proof of the equality (15) is similar to the proof of the spectral condition in the commutative case [6], [9]. Recall that in the latter case the equality (15) is valid, if $P_j \quad \bar{V}$. Having written down $W(\ _1,\ldots,\ _{n\ 1},X)$ as

$$W(\ _1,\ldots, \ _{n\ 1},X) \quad \frac{1}{(2\ \ )^{n\ 1}} \quad e^{\ iP_j\ j} W(P_1,\ldots,P_{n\ 1},X)dP_1\ldots dP_{n\ 1}\,, \tag{16}$$

and taking into account that Wightman functions are tempered distributions with respect to the commutative variables, we obtain that, due to the condition (15), $W(\ _1,\ldots,\ _{n\ 1},X)$ is analytical in the "tube" T_n :

$$_i \quad T_n\ , \text{ if } \quad _i \quad _i \ i \ _i\,, \quad _i \quad V_2\ , \quad _i \quad \{\ _i^0,\ _i^3\}. \tag{17}$$

It should be stressed that the noncommutative coordinates x_i^1, x_i^2 remain always real.

Owing to SO(1,1) invariance and according to the Bargmann-Hall-Wightman theorem [6]-[9], $W(\ _1,\ldots,\ _{n\ 1},X)$ is analytical in the domain T_n

$$T_n \quad \bigcup \ _c T_n\ , \tag{18}$$

where $_c \quad SO_c(1,1)$ is the two-dimensional analogue of the complex Lorentz group. This expansion is similar to the transition from tubes to expanded tubes in the commutative case. Just as in the commutative case, the expanded domain of analyticity contains real points x_i, which are the noncommutative Jost points, satisfying the condition $x^i \sim x_j, \quad i, j,$ which means that

$$(x_i^0 \quad x_j^0)^2 \quad (x_i^3 \quad x_j^3)^2 \quad 0\,. \tag{19}$$

It should be emphasized that the noncommutative Jost points are a subset of the set of Jost points of the commutative case, when

$$(x^i \quad x_j)^2 \quad 0 \qquad i, j\,. \tag{20}$$

Let us proceed to the LCC in noncommutative space-space QFT.

First let us recall this condition in commutative case. In the operator form this condition is

$$[\ _{f_1},\ _{f_2}] \quad 0\,, \text{ if } \ O_1 \sim O_2\,, \tag{21}$$

where O_1= supp f_1, O_2= supp f_2. The condition $O_1 \sim O_2$ means that $(x - y)^2 < 0 \quad x \quad O_1$ and $y \quad O_2$. The condition (21) is equivalent to the following property of Wightman functions.

$$W(x_1,\ldots,x_i,x_{i\ 1},\ldots,x_n) \quad W(x_1,\ldots,x_{i\ 1},x_i,\ldots,x_n)\,, \tag{22}$$

if supp $f_i \quad O_i$, supp $f_{i\ 1} \quad O_{i\ 1}$, $O_i \sim O_{i\ 1}$.

In the noncommutative case we have the similar condition, but now $O_1 \sim O_2$ means that

$(x^0 \quad y^0)^2 \quad (x^3 \quad y^3)^2 \quad 0, \quad x \quad O_1$ and $y \quad O_2$.

In terms of Wightman functions this condition means that

$$W(x_1,\ldots,x_i,x_{i\ 1},\ldots,x_n) f(x_1) \ \ldots \ f(x_i) \ \ f(x_{i\ 1}) \ \ldots \ f(x_n) dx_1 \ldots dx_n$$

$$W(x_1,\ldots,x_{i\ 1},x_i,\ldots,x_n) f(x_1) \ \ldots \ f(x_{i\ 1}) \ \ f(x_i) \ \ldots \ f(x_n) dx_1 \ldots dx_n\,, \tag{23}$$

where $W(x_1,\ldots,x_n) \qquad _0,\ (x_1)\ldots\ (x_n)\ \ _0$.

Let us point out that in the noncommutative case WLCC

$$W(x_1,\ldots,x_n) \quad W(x_n,\ldots,x_1)\,, \text{ if } \ x_i \sim x_j \qquad i, j\,. \tag{24}$$

has the same form as in the local theory with the same difference as for LCC.

3. Irreducibility of the set of field operators $_f$ in NC QFT

The irreducibility of a set of field operators $_f$ implies that, from the condition

$$A \;\; _{f_1} \cdots \;\; _{f_n} \;\; _0 \qquad _{f_1} \cdots \;\; _{f_n} \;\; _0 \tag{25}$$

where $f_i \quad f_i(x_i)$ are arbitrary test functions and A is a bounded operator, follows that

$$A \quad C\mathbb{I} \quad C \quad \mathbb{C} \tag{26}$$

where $\mathbb{I}$ is the identity operator.

In the noncommutative case the condition of irreducibility of the set of operators $_f$ is valid as well as in commutative case. The point is that for this it is sufficient to have the translational invariance in the variable x^0 and the spectral condition, which can be weakened up to the condition

$$P_n^0 \quad 0\,. \tag{27}$$

Using condition (25) and the invariance of the vacuum vector with respect to the translations $U(a)$ on the axis x^0, we obtain the following chain of equalities

$$\left\langle A^* \;\; _0,\, U(a) \;\; _{f_1} \cdots \;\; _{f_n} \;\; _0 \right\rangle \quad \left\langle \;\; _0, A\, U(a) \;\; _{f_1} \cdots \;\; _{f_n} \;\; _0 \right\rangle \quad \left\langle \;\; _0,\;\; _{f_1(x_1 \quad a)} \cdots \;\; _{f_n(x_n \quad a)} A \;\; _0 \right\rangle$$

$$\left\langle \;\; _{\bar{f}_n(x_n \quad a)} \cdots \;\; _{\bar{f}_1(x_1 \quad a)} \;\; _0, A \;\; _0 \right\rangle \quad \left\langle U(\quad a) \;\; _{\bar{f}_n(x_n \quad a)} \cdots \;\; _{\bar{f}_1(x_1 \quad a)} \;\; _0, U(\quad a) A \;\; _0 \right\rangle$$

$$\left\langle \;\; _{\bar{f}_n} \cdots \;\; _{\bar{f}_1} \;\; _0, U(\quad a) A \;\; _0 \right\rangle. \tag{28}$$

So

$$\left\langle A^* \;\; _0, U(a) \;\; _{f_1} \cdots \;\; _{f_n} \;\; _0 \right\rangle \quad \left\langle \;\; _{\bar{f}_n} \cdots \;\; _{\bar{f}_1} \;\; _0, U(\quad a) A \;\; _0 \right\rangle. \tag{29}$$

In accordance with the Eq. (14)

$$da\, e^{\;\; ip^0 a} \left\langle A^* \;\; _0, U(a) \;\; _{f_1} \cdots \;\; _{f_n} \;\; _0 \right\rangle \quad 0\,,$$

only if $p^0 \quad 0$. However,

$$da\, e^{\;\; ip^0 a} \left\langle \;\; _{\bar{f}_n} \cdots \;\; _{\bar{f}_1} \;\; _0, U(\quad a) A \;\; _0 \right\rangle \quad 0\,,$$

only if $p^0 \quad 0$. Hence, the equality (28) can be fulfilled only when $p^0 \quad 0$. As we assume the absence of vectors noncollinear to the vacuum one and satisfying the condition $p^0 \quad 0$, there is no vector distinct from the vacuum one, which contributes to both left and right parts of Eq. (28) simultaneously. Taking into account the completeness of the system of vectors $_{P_n}$ we come to conclusion that

$$A \;\; _0 \quad C \;\; _0\,, \tag{30}$$

as $_{f_1} \cdots \;\; _{f_n} \;\; _0$ is an arbitrary vector. Thus owing to (25) and (30)

$$A \;\; _{f_1} \cdots \;\; _{f_n} \;\; _0 \quad C \;\; _{f_1} \cdots \;\; _{f_n} \;\; _0\,. \tag{31}$$

The required equality (26) follows from Eq. (31) in accordance with the boundedness of the operator A and cyclicity of the vacuum vector.

4. Cluster properties and their consequences

It is known [6, 8] that in commutative theory Wightman functions satisfy the following cluster properties:

$$W(x_1,\ldots x_k,x_{k\ 1} \quad a,\ldots x_n \quad a) \quad W(x_1,\ldots x_k)W(x_{k\ 1},\ldots x_n)\,, \tag{32}$$

if and a^2 1. Let us show how classical proof (see [6]) can be extended to space-space NC QFT.

First let us point out that in commutative case translation vector can be arbitrary, but in noncommutative case this vector has to belong to commutative plane. Surely in commutative case we also can chose translation vector in this plane. If we do this the proof in NC QFT is similar to the corresponding proof in usual QFT. As in [6] we give the proof for theories with mass gap. In commutative case we use the following properties of Wightman functions:

i corresponding Wightman functions are tempered distributions;

ii LCC is valid.

But if in commutative case we make shift in the plane, which in noncommutative case is commutative plane, then LCC coincide in commutative and noncommutative cases. Let us stress that it is sufficient to do translation in only one direction as translation vector is not in the final result. Taking into account that corresponding test functions in noncommutative case are tempered distributions in respect with commutative variables, we see that two crucial points in derivation of cluster properties coincide in commutative and non-commutative cases in above mentioned case of choosing translation vector.

Eq. (32) can be refined (see [6]). Namely, if we consider the theory, where only massive particles exist, in addition to the Eq. (32) we have:

$$\left|W(x_1,\ldots x_k,x_{k\ 1} \quad a,\ldots x_n \quad a) \quad W(x_1,\ldots x_k)W(x_{k\ 1},\ldots x_n)\right| \quad \frac{C}{\ ^n}\,, \tag{33}$$

where n is arbitrary.

If the theory contains the particle with zero mass, then in inequality (33) n 2.

The first case corresponds to the theories with short-range interaction, the second - to long-range ones. For Coulomb law n = 2 in inequality (33) [32].

Let us pass to the proof.

We consider two functions:

$$F_1 \quad W(x_1,\ldots x_k,x_{k\ 1} \quad a,\ldots x_n \quad a) \quad W(x_1,\ldots x_k)W(x_{k\ 1},\ldots x_n) \tag{34}$$

and

$$F_2 \quad W(x_{k\ 1} \quad a,\ldots,x_n \quad a,x_1,\ldots x_k) \quad W(x_1,\ldots x_k)W(x_{k\ 1},\ldots x_n)\,. \tag{35}$$

If and a^2 1 and a x^0,x^3 , then owing LCC, in space-space NC QFT

$$F_1 \quad F_2\,. \tag{36}$$

The simplest choice is: $a = \{0, 1\}$ and it would be our choice. It is easy to see that F_1 $F_2 = 0$ at any if P^2 M^2 as we consider theories with mass gap. Indeed let us put the whole system of vectors $_{P,n}$ between points x_k and $x_{k\ 1}$.

Then we have:

$$\sum_n dP\left\langle\ _0,\ (x_1)\ldots\ (x_k)\ \ _{P,n}\right\rangle\left\langle\ _{P,n},\ (x_{k\ 1} \quad a)\ldots\ (x_n \quad a)\ \ _0\right\rangle, \tag{37}$$

where n denotes all other quantum numbers. Then we have

$$\sum_n dP\left\langle\ _0,\ (x_1)\ldots\ (x_k)\ \ _{P,n}\right\rangle\left\langle\ _{P,n},U(\ a)\ (x_{k\ 1})\ldots\ (x_n)\ \ _0\right\rangle, \tag{38}$$

$U(a)$ is a translation operator. Let us recall that $U(a)\ \ _0 \quad \ _0$. Then

$$\int_{n} dP \langle \ _0, \ (x_1) \ldots \ (x_k) \ _{P,n} \rangle \langle U(\ a) \ _{P,n} \ (x_{k \ 1}) \ldots \ (x_n) \ _0 \rangle$$

$$\int_{n} dP \exp(\ i \ ap) \langle \ _0, \ (x_1) \ldots \ (x_k) \ _{P,n} \rangle \langle \ _{P,n}, \ (x_{k \ 1}) \ldots \ (x_n) \ _0 \rangle . \tag{39}$$

Thus using as before translation along axis x_3 we see that

$$F_1 \ \int_{n} dP \exp(\ i \ P_3) \langle \ _0, \ (x_1) \ldots \ (x_k) \ _{P,n} \rangle \langle \ _{P,n}, \ (x_{k \ 1}) \ldots \ (x_n) \ _0 \rangle$$
$$W(x_1, \ldots, x_k) W(x_{k \ 1}, \ldots, x_n) . \tag{40}$$

As $P_3 \ 0$ for $\ _0$, we see that $F_1 \ 0$ only if $P^2 \ M^2$. The same is true for function F_2.

Now let us take into account that Wightman functions in space-space NC QFT are tempered distributions in respect with commutative coordinates. It means that

$$F(x_1, \ldots, x_n) h(x_1, \ldots, x_n) \, dx_1 \ldots dx_n$$

$$D^m G(\ , x_1, \ldots, x_n) \, h(x_1, \ldots, x_n) \, dx_1 \ldots dx_n , \tag{41}$$

where $F \ F_1 \ F_2$. As $F_1 \ F_2 \ 0$ at , then $D^m \ 0$ if . Let us show that actually

$$D^m G(\ , x_1, \ldots, x_n) \ 0 , \tag{42}$$

if $R^2 \ R_0^2$, where $R^2 \ \sum_{j \ 1}^{n} [(x_j^0)^2 \ (x_j^3)^2]$ and $R_0 \ \frac{1}{4}$.

Indeed,

$$(x_i \ x_k \ a)^2 \ (x_i^0 \ x_k^0)^2 \ (x_i^3 \ x_k^3)^2 \ ^2 \ 2 \ (x_i^3 \ x_k^3)$$

$$2((x_i^0)^2 \ (x_k^0)^2) \ 2 \ (|x_i^3| \ |x_k^3|) \ ^2 \ 2R^2 \ 2 \ R \ ^2 \ 0$$

if, for example, $R_0 \ \frac{1}{4}$ at . (43)

So

$$F \ \int_{R_0} D^m G(\ , x_1, \ldots, x_n) h(x_1, \ldots, x_n) \, dx_1 \ldots dx_n . \tag{44}$$

As R_0 at and integral in question converges, then $F \ 0$ at . In order to see that also $F_1 \ 0$ at let us exchange $h(x_1, \ldots, x_n)$ for $\tilde{h}(x_1, \ldots, x_n)$, where

$$\tilde{h}(x_1, \ldots, x_n) \ h(x_1, \ldots, x_n) . \tag{45}$$

Here is infinitely differentiable function of variable $P \ \sum_{j \ 1}^{k} p_k$ such that 1 if $P^2 \ M^2$, $P_0 \ 0$; 0, if $P_0 \ 0$.

In order to make the last step it is sufficient to notice that in accordance with spectral properties of Wightman functions in space-space NC QFT $F_1 \ 0$ only if $P_0 \ 0$ and $F_2 \ 0$ only if $P_0 \ 0$. Indeed,

$$W(x_1, \ldots, x_k, x_{k \ 1} \ a, \ldots, x_n \ a) \ \langle \ _0, \ (x_1) \ldots \ (x_k) U(\ a) \ (x_{k \ 1}) \ldots \ (x_n) \ _0 \rangle \tag{46}$$

and

$$W(x_{k \ 1} \ a, \ldots, x_n \ a, x_1, \ldots, x_k)$$
$$\langle \ _0, \ (x_n) \ldots \ (x_{k \ 1}) U(\ a) \ (x_1) \ldots \ (x_k) \ _0 \rangle . \tag{47}$$

So

$$F_2(x_1,\ldots,x_n)\tilde{h}(x_1,\ldots,x_n)\,dx_1\ldots dx_n \quad 0\,.$$

Thus equation (44) is valid also for F_1 and cluster properties of Wightman functions in space-space NC QFT are proved.

In order to obtain the stronger result (33) we have to do the calculations similar with ones given in [6].

Let us recall that from cluster properties of Wightman functions important physical consequences follow. One of them is the uniqueness of the vacuum state, that is the uniqueness of a translation invariant state.

Let us show that this statement is valid also in space-space NC QFT. Precisely we show that only one translation invariant state in respect with commutative coordinates can exist. In fact, if there exist two vacuum states $_0$ and $_0$, we can always put $\langle\ _0,\ _0\rangle\ 1$, $\langle\ _0,\ _0\rangle\ 1$, $\langle\ _0,\ _0\rangle\ 0$. Then using cluster properties in respect with commutative coordinates, we have $\langle\ _0,\ _0\rangle\ \lim\langle\ _0, U(\ a)\ _0\rangle\ \langle\ _0,\ _0\rangle\langle\ _0,\ _0\rangle\ 0$, if $a_0^2\ \ a_3^2\ \ 1.$

The proof is completed if $_0$ is a finite linear combination of vectors $_{f_1}\cdots\ _{f_n}\ _0$. If $_0$ is an infinite set of above mentioned vectors then:

$$_0 \quad \sum_0^n c_k\ _{f_1}\cdots\ _{f_k}\ _0 \quad _n,\quad _n \quad 0,\ \text{ if } n \quad . \tag{48}$$

As $U(a\ )\ _0\ \ _0$, then Eq. (48) is valid also for $U(a\ )\ _0$. Owing to Eqs. (48) and (32)

$$\langle\ _0, U(a\ )\ _0\rangle\ \ 1 \quad _n,\quad _n \quad 0,\ \text{ if} n \quad .$$

Thus we come to the same contradiction as in the first case.

So we have proved that cluster properties in respect only with commutative coordinates lead to the uniqueness of vacuum state just as cluster properties in respect with all coordinates in commutative case. Another important consequence from cluster properties of Wightman functions, which is valid in space-space NC QFT, is the statement that if $_f$ satisfies LCC, but

$$\{\ _{f_1},\ ^*_{f_2}\}\quad 0,\ \ \{x,y\}\quad xy\quad yx, \tag{49}$$

then $_f\ \ 0$ [8]. It gives us the possibility to extend the proof of spin-statistic theorem given in [24] on complex scalar fields.

In conclusion let us show how cluster properties can be obtained in the NC QFT if LCC is absent. To demonstrate this let us repeat the proof of cluster properties in the book of Strocchi [32]. The only remained problem is that this proof is valid for usual functions, not for distributions. In order to overcome this difficulty we have first to consider cluster properties in tubes. Then we use the possibility to go to zero in the imaginary parts of corresponding variables, and thus extend cluster properties on real variables. Let us point out that as before we have the above mentioned consequence of cluster properties.

5. Theorems of Reeh and Schlieder in NC QFT

In the following we shall prove the analogues of the theorems of Reeh and Schlieder [6, 7] for the noncommutative case.

Theorem 1 *Let supports of functions $\tilde{f}_i$ belong to $\tilde{O}\quad R^2$, where $\tilde{O}$ is any open domain on variables x_i^0 and x_i^3.*

Then there is no vector distinct from zero, which is orthogonal to all vectors of the type $_{\tilde{f}_1} \cdots _{\tilde{f}_n}\ _0$, supp $\tilde{f}_i \quad \tilde{O} \quad R^2$. *First let us consider two vectors*

$$\tilde{}_n \quad _{\tilde{f}_1} \cdots _{\tilde{f}_n}\ _0, \text{ supp } \tilde{f}_i \quad \tilde{O} \quad R^2 \quad i, \tag{50}$$

$$_m \quad _{f_m} \cdots _{f_1}\ _0\,. \tag{51}$$

On sup f_i no restrictions are imposed. We shall prove that $_m$ 0, if for any vector $_n$

$$\langle\ _m, \tilde{}_n \rangle \quad 0\,. \tag{52}$$

For the proof it is sufficient to notice that the corresponding Wightman function
$\langle\ _0, \ (y_1) \ldots \ (y_m) \ (x_1) \ldots \ (x_n)\ _0 \rangle \quad W(y_1, \ldots y_m, x_1, \ldots, x_n)$
is an analytical function in the variables $x_1^0 \quad i \ _0^0, \quad x_1^3 \quad i \ _0^3, \ _i \quad _i \ i \ _i, i \quad 1, \ldots, n \quad 1$, if $_i \quad V_2$. According to the condition (52), this function is equal to zero on the border, if $x_i \quad \tilde{O} \quad R^2$. As $\tilde{O}$ is an open domain, $W(y_1, \ldots, y_m, x_1, \ldots, x_n) \quad 0$. Thus the vector is orthogonal to all vectors of the type (8) and, according to the cyclicity of the vacuum vector, $_m$ 0. Taking into account that space J is a span of these vectors we obtain that

$$\langle\ _m, \ \rangle \quad 0\,, \tag{53}$$

where is arbitrary. As space J is nondegenerate, this equality implies that $_m$ 0.

To prove the absence of any vector orthogonal to all vectors of the type (50) it is sufficient to notice that function $\langle\ , \ _m \rangle$ is analytical in T_n , and then use the arguments given above.

Remark that for the proof of the Theorem 1 only the analyticity of the Wightman functions in the domain T_n has been used.

Theorem 2 *Let the support of* $f \quad O \quad R^2$, *where O is such a domain of commutative variables, for which domain* $\tilde{O} \sim O$, *satisfying the condition of the Theorem 1, exists. Then the condition*

$$_f\ _0 \quad 0 \tag{54}$$

implies that

$$_f \quad 0\,, \tag{55}$$

if the operator $_f$ *satisfies the LCC.*

In accordance with LCC

$$_f\ \tilde{}_n \quad 0\,, \tag{56}$$

if vector $\tilde{}_n$ is defined as in Eq. (50). Hence, for any vector belonging to the domain of definition of the Hermitian operator $_f$,

$$\langle\ _f \ , \tilde{}_n \rangle \quad \langle\ , \ _{\bar{f}}\ \tilde{}_n \rangle \quad 0\,. \tag{57}$$

According to the Theorem 1, the condition (57) means that $_f$ 0. As the domain of definition of the operator $_f$ is dense in J, this equality means the validity of the equality (55).

Remark *Theorem 2 remains true for any densely defined operator* $_f$, *mutually local with* $_{\tilde{f}}$, *i.e. if*

$$_f\ _{\tilde{f}} \qquad _{\tilde{f}}\ _f\,, \tag{58}$$

if supp f O R^2, supp $\tilde{f}$ $\tilde{O}$ R^2, $O \sim \tilde{O}$, *vector belongs to the domain of definition of operators* $_{\tilde{f}}$ *and* $_f$.

6. Generalized Haag's Theorem

Recall the formulation of the generalized Haag's theorem in the commutative case ([6], Theorem 4.17):

Let $^1_f(t)$ *and* $^2_f(t)$, supp f R^3 *be two irreducible sets of operators, for which the vacuum vectors* $^1_0$ *and* $^2_0$ *are cyclic. Further, let the corresponding Wightman functions be analytical in the domain* T^2_n.

Then the two-, three- and four-point Wightman functions coincide in the two theories if there is a unitary operator V, such that

1) $^2_f(t) \quad V\, ^1_f(t) V^*$, (59)

2) $^2_0 \quad CV\, ^1_0, \; C \quad \mathbb{C}, \; |C| \quad 1$. (60)

It should be emphasized that actually the condition 2) is a consequence of condition 1) with rather general assumptions (see the **Statement** below). In the formulation of Haag's theorem it is assumed that the formal operators $_i(t, \vec{x}_n)$ can be smeared only on the spatial variables. This assumption is natural also in noncommutative case if ^{0i} 0.

Let us consider Haag's theorem in the SO(1,1) invariant field theory and show that the corresponding equality is true only for two-point Wightman functions.

For the proof we first note that in the noncommutative case, just as in the commutative one, from conditions 1) and 2) it follows that the Wightman functions in the two theories coincide at equal times

$$\langle\, ^1_0, \; _1(t,\vec{x}_1) \tilde{} \ldots \; _1(t,\vec{x}_n) \; ^1_0 \rangle \quad \langle\, ^2_0, \; _2(t,\vec{x}_1) \tilde{} \ldots \; _2(t,\vec{x}_n) \; ^2_0 \rangle . \tag{61}$$

Having written down the two-point Wightman functions $W_i(x_1, x_2)$, i 1,2 as $W_i(u_1, \; _1; u_2, \; _2)$, where u_i $\{x_i^0, x_i^3\}$, $_i$ $\{x_i^1, x_i^2\}$ we can write for them equality (61) as:

$$W_1(0, \; ^3; \; _1, \; _2) \quad W_2(0, \; ^3; \; _1, \; _2), \tag{62}$$

where u_1 u_2, $_1$ and $_2$ are arbitrary vectors. Now we notice that, due to the SO(1,1) invariance,

$$W_i(0, \; ^3; \; _1, \; _2) \quad W_i(\tilde{} ; \; _1, \; _2) \tag{63}$$

hence,

$$W_1(\tilde{} ; \; _1, \; _2) \quad W_2(\tilde{} ; \; _1, \; _2), \tag{64}$$

where $\tilde{}$ is any Jost point. Due to the analyticity of the Wightman functions in the commuting variables they are completely determined by their values at the Jost points. Thus at any from the equality (64), it follows that

$$W_1(\; ; \; _1, \; _2) \quad W_2(\; ; \; _1, \; _2). \tag{65}$$

As $_1$ and $_2$ are arbitrary, the formula (65) means the equality of two-point Wightman functions at all values of arguments.

Thus, for the equality of the two-point Wightman functions in two theories related by the conditions (59) and (60), the SO(1,1) invariance of the theory and corresponding spectral condition are sufficient.

It is impossible to extend this proof to three-point Wightman functions. Indeed, let us write down $W_i(x_1, x_2, x_3)$ as $W_i(u_1, u_2, u_3;\ _1,\ _2,\ _3)$, where vectors u_i and $_i$ are determined as before. Equality (62) means that

$$W_1(0,\ ^3_1, 0,\ ^3_2;\ _1,\ _2,\ _3)\quad W_2(0,\ ^3_1, 0,\ ^3_2;\ _1,\ _2,\ _3)\,. \tag{66}$$

$_1,\ _2,\ _3$ are arbitrary. In order to have equality of the three-point Wightman functions in the two theories from the SO(1,1) invariance, the existence of transformations SO(1,1) connecting the points $(0,\ ^3_1)$ and $(0,\ ^3_2)$ with an open vicinity of Jost points is necessary. That would be possible, if there exist two-dimensional vectors $\tilde{}_1$ and $\tilde{}_2$, $(\tilde{}_i\quad (0,\ ^3_i))$, satisfying the inequalities:

$$(\tilde{}_1)^2\quad 0,\quad (\tilde{}_2)^2\quad 0,\quad \left|(\tilde{}_1, \tilde{}_2)\right|\quad \sqrt{(\tilde{}_1)^2(\tilde{}_2)^2}\,.$$

These inequalities are similar to the corresponding inequalities in the commutative case (see equation (4.87) in [6]). However, it is easy to check that the last of these inequalities can not be fulfilled, while the first two are fulfilled.

Let us show now that the condition (60) actually is a consequence of the condition (59).

Statement *Condition (60) is fulfilled, if the vacuum vectors* $^i_0$ *are unique, normalized, translationally invariant vectors with respect to translations* $U_i(a)$ *along the axis* x^3.

It is easy to see that the operator $U_1^{\ 1}(a)V\ ^1U_2(a)V$ commutes with operators $^1_f(t)$ and, owing to the irreducibility of the set of these operators, it is proportional to the identity operator. Having considered the limit $a\quad 0$, we see that

$$U_1^{\ 1}(a)V\ ^1U_2(a)V\quad \mathbb{I}\,. \tag{67}$$

From the equality (67) it follows directly that if

$$U_1(a)\ ^1_0\qquad ^1_0,\quad \text{then} \tag{68}$$

$$U_2(a)V\ ^1_0\quad V\ ^1_0, \tag{69}$$

i.e. the condition (60) is fulfilled. If the theory is translationally invariant in all variables, the equality (69) is true, if the vacuum vector is unique, normalized, translationally invariant in the spatial coordinates.

The most important consequence of the generalized Haag theorem is the following statement: if one of the two fields related by conditions (59) and (60) is a free field, the other is also free. In deriving this result the equality of the two-point Wightman functions in the two theories and LCC are used. In [25] it is proved that this result is valid also in the noncommutative theory, if $^{0i}\quad 0$.

Here we obtain the close result in the SO(1,1) symmetric theory using the spectral conditions and translational invariance only with respect to the commutating coordinates. In this case the equality of the two-point Wightman functions in the two theories leads to the conclusion that if LCC (22) is fulfilled and the current in one of the theories is equal to zero, for example, $j^1_f\quad 0$, then $j^2_f\quad 0$ as well; $j^i_f\quad (\Box + m^2)\ ^i_f$. Indeed as $W_1(x^1, x^2)\quad W_2(x^1, x^2)$,

$$\left\langle\ ^1_0, j^1_{\bar f} j^1_f\ ^1_0\right\rangle\quad \left\langle\ ^2_0, j^2_{\bar f} j^2_f\ ^2_0\right\rangle\quad 0, \tag{70}$$

since $j^1_f\quad 0$. Hence, $j^2_f\ ^2_0\quad 0$.

Here we assume that J is a positive metric space. It is sufficient to take advantage of the Theorem 2 from which follows that $j_f^2 \quad 0$ (see the Remark after **Theorem 2**), since LC implies mutual local commutativity of a field operator and the corresponding current.

Let us proceed now to the SO(1,3) symmetric theory. In this case we show that from the equality of the four-point Wightman functions for the fields $\,^1_f(t)$ and $\,^2_f(t)$, related by the conditions (59) and (60), which takes place in the commutative theory, an essential physical consequence follows. Namely, for such fields the elastic scattering amplitudes of the corresponding theories coincide, hence, due to the optical theorem, the total cross-sections coincide as well. In particular, if one of these fields, for example, $\,^1_f$ is a trivial field, i.e. the corresponding S matrix is equal to unity, also the field $\,^2_f$,is free. In the derivation of this result the local commutativity condition is not used. The statement follows directly from the Lehmann-Symanzik-Zimmermann reduction formulas [36]. Here and below dealing with the commutative case in order not to complicate formulas we consider operators $\,_1(x)$ and $\,_2(x)$ as they are given in a point.

Let $\langle p_3, p_4 \,|\, p_1, p_2 \rangle_i$, $i \quad 1, 2$ be an elastic scattering amplitudes for the fields $\,_1(x)$ and $\,_2(x)$ respectively. Owing to the reduction formulas,

$$\langle p_3, p_4 \,|\, p_1, p_2 \rangle_i \sim \quad dx_1 \ldots dx_4 e^{i(\quad p_1 x_1 \quad p_2 x_2 \quad p_3 x_3 \quad p_4 x_4)} .$$

$$\,^4_{j \quad 1} (\sim_j \quad m^2) \langle 0 | T \quad _i(x_1) \ldots \quad _i(x_4) | 0 \rangle , \tag{71}$$

where $T \quad _i(x_1) \cdots \quad _i(x_4)$ is the chronological product of operators. From the equality

$$W_2(x_1, \ldots, x_4) \quad W_1(x_1, \ldots, x_4)$$

it follows that

$$\langle p_3, p_4 | p_1, p_2 \rangle_2 \quad \langle p_3, p_4 | p_1, p_2 \rangle_1 \tag{72}$$

for any p_i. Having applied this equality for the forward elastic scattering amplitudes, we obtain that, according to the optical theorem, the total cross-sections for the fields $\,_1(x)$ and $\,_2(x)$ coincide. If now the S-matrix for the field $\,_1(x)$ is unity, then it is also unity for field $\,_2(x)$. We stress that the equality of the four-point Wightman functions in the two theories related by the conditions (59) and (60) are valid only in the commutative field theory but not in the noncommutative case.

7. Equivalence of various conditions of local commutativity in QFT

Let us show that in the commutative case, when Wightman functions are analytical ones in the usual domain, the conditions (24) and (22) are equivalent to the standard conditions of WLC and LC, i.e. the latter remain valid if the condition (20) is fulfilled. In effect, (24) is a sufficient condition for the theory to be CPT invariant [23]. However, in the commutative case, from CPT invariance the standard condition of WLC follows [6]–[9].

The equivalence of LCC (22) with the standard one follows from the fact that, for the validity of usual LCC its validity on arbitrary small spatially divided domains is sufficient (see [9], Proposal 9.12). Indeed, validity of "noncommutative" LCC (22) in the commutative case means validity of standard LCC in the domain $(x^0 \quad y^0)^2 \quad (x^3 \quad y^3)^2 \quad 0$,

$x^k, y^k, k \quad 1, 2$ are arbitrary. This domain satisfies the requirements of the above mentioned statement.

Besides we can replace (22) with the formally weaker condition, requiring that it is valid only when

$$(x_i^0 \quad x_j^0)^2 \quad (x_i^3 \quad x_j^3)^2 \qquad l^2, \qquad i, j, \tag{73}$$

where l is any fixed fundamental length. Indeed, in the commutative theory, according to the results of Wightman, Petrina and Vladimirov (see [37], Chapter 5 and references therein) the condition

$$[\ (x), \ (y)] \quad 0, \ (x \quad y)^2 \qquad l^2, \tag{74}$$

for any finite l, is equivalent to standard LCC ($l = 0$). Similarly if (22) is fulfilled at (73), then it is fulfilled also at $l = 0$.

Thus, the analysis of Wightman functions in NC QFT, carried out in this and our previous works [24], [27], [25], shows that the basic axiomatic results are valid (or have analogues) in NC QFT as well, at least in the case when $^{0i} \quad 0$.

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