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Localisation in Quantum Field Theory*

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Abstract: In nonrelatistic quantum mechanics, Born's principle of localistion is as follows: For a single particle, if a wave function $_K$ vanishes outside a spatial region K, it is said to be localised in K. In particular if a spatial region K is disjoint from K, a wave function $_K$ localised in K is orthogonal to $_K$.

Such a principle of localisation does not exist compatibly with relativity and causality in quantum field theory (Newton and Wigner) or interacting point particles (Currie, Jordan and Sudarshan). It is replaced by symplectic localisation of observables as shown by Brunetti, Guido and Longo, Schroer and others. This localisation gives a simple derivation of the spin-statistics theorem and the Unruh effect, and shows how to construct quantum fields for anyons and for massless particles with 'continuous' spin.

This review outlines the basic principles underlying symplectic localisation and shows or mentions its deep implications. In particular, it has the potential to affect relativistic quantum information theory and black hole physics.

This article is a brief review of work done by Brunetti, Guido and Longo [1, 2], Schroer and Fassarella and Schroer [3], Mund, Schroer and Yngvaso [4] and others on localisation problems in relativistic quantum field theories and their deep implications. The review is informal in that mathematical rigour is not attempted. It is at a level accessible to most quantum field theorists. It is dedicated to Peter Presnajder, wonderful friend and close collaborator.

1. Introduction

Locality in quantum field theory is often used in the sense that test functions have support in a localised region of a spatial slice or in spacetime. This interpretation is suggested by Born's interpretation of wave functions psi of a particle: $|\ (x)|^2 d^3x$ is the probabilty of finding the particle in a voume d^3x around x.

While this interpretation is adequate for a first approach, it becomes incomplete when the requirements of relativistic invariance and causality are brought in. A more sophisticated approach becomes necessary.

In theories with mass gap and no gauge invariance, the full set of axioms for local relativistic quantum physics has been developed by Haag and Kastler and discussed in Haag's book [1]. The notes that follow will not discuss the Haag-Kastler approach, but will borrow ideas therefrom to describe this more refined approach.

As we explain below, the idea of localisation of wave functions requires the existence of a position operator. That is problematic in relativistic quantum physics. Instead, for relativistic free fields, a new concept of localisation, which localises observables instead of states, has been formulated by Brunetti, Guido and Longo [2] and by Schroer and colleagues [3, 4]. It gives new insights about particles obeying braid statistics, and those transforming by massless "continuous spin" representations.

*Dedicated to Professor Peter Prešnajder on the occasion of his 70th birthday

2. On Position Operators

In quantum physics, just as in classical physics, observables $\mathcal{A}$ determine the measurements available on the system. They form an algebra. That is, if , are observables, we have a multiplication map m from $\mathcal{A}$ $\mathcal{A}$ to $\mathcal{A}$:

$$m: \qquad m(\qquad , \tag{2.1}$$

which is linear in each entry. The algebra $\mathcal{A}$ in quantum physics is non-commutative, whereas the corresponding algebra $\mathcal{A}_c$ is commutative in classical physics.

The classical algebra $\mathcal{A}_c$ can be realised as real functions f on the phase space T^*Q with (local) coordinates $(q_1,\ldots,q_N;p_1,\ldots,p_N)$: (q,p):

$$f: \; T^*Q \quad \mathbb{R}$$
$$\overline{f(q,p)} \quad f(q,p)\,. \tag{2.2}$$

The property which corresponds to reality in quantum physics is that there is a star operation or "hermitean conjugation" * defined on $\mathcal{A}$:

$$\mathcal{A} \qquad \mathcal{A}\,. \tag{2.3}$$

The outcome of experiments in classical physics is given by a probability distribution $_c$ on $\mathcal{A}_c$. It has the following basic properties:

- $_c(q,p) \quad 0\,.$
- $d \;(q,p) \;\; _c(q,p) \quad 1\,.$
- Mean value of a_c $\mathcal{A}_c$: $\langle a_c \rangle \quad d \;(q,p) \;\; _c(q,p)a_c(q,p) \quad \mathbb{R}\,.$

Here d is the Liouville volume form on T^*Q: $d \;(q,p) \quad dq_1 \;\ldots\; dq_N dp_1 \;\ldots\; dp_N$.

Correspondingly, in quantum physics, we have a *state* on $\mathcal{A}$. It is a linear map with (a) giving the mean value of a $\mathcal{A}$. It has the properties:

- $(a^*a) \quad 0\,.$
- $(\mathcal{I}) = 1.$
- $(a^*) \quad \overline{(a)}\,.$

Here, the first two properties adapt all the properties of $_c$ before, while the last property preserves the * of $\mathcal{A}$ as complex conjugation on $\mathbb{C}$.

It is a theorem of Gelfan'd, Naimark and Segal (GNS) that, given a state on $\mathcal{A}$, there exists a Hilbert space $\mathcal{H}$ on which $\mathcal{A}$ is realised as an algebra of operators, still denoted by $\mathcal{A}$ by us. Also, the $*$-operator becomes the hermitean adjoint †. Finally, can be represented by a density matrix :

$$|\;\; _i\rangle\langle \;\; _i|\,, \qquad |\;\; _i\rangle \;\; \mathcal{H}, \qquad \mathrm{Tr} \;\; =1\,. \tag{2.4}$$

From this abstract formulation, the wave function in non-relativistic quantum mechanics is recovered as follows. The algebra $\mathcal{A}$ has an operator $\hat{x}$, called the position operator, with commuting components. If $|\vec{x}\rangle$ is the eigenstate of $\hat{x}$,

$$\hat{x}_i|\vec{x}\rangle \quad x_i|\vec{x}\rangle\,, \qquad \langle\vec{x}\,|\vec{x}\rangle \quad ^d(x \quad x)\,, \tag{2.5}$$

where the spatial dimension d is 3 for $\mathbb{R}^3$, we can write

$$(\vec{x}) \quad \langle\vec{x}| \;\; \rangle \tag{2.6}$$

for a vector $|\;\; \rangle \;\; \mathcal{H}$ of norm 1: $\langle \;\; | \;\; \rangle \;\; 1$, associated with the rank 1 (pure) state $|\;\; \rangle\langle \;\; |$. Then, gives the wave function of non-relativistic physics, subject to Born's interpretation: $|\;\; (\vec{x})|^2$ is the probability density for finding the system at $\vec{x}$.

It is important that $\hat{x}$ transforms correctly under the Galilei group which is the governing group of non-relativistic physics. Thus, it is a rotational vector and under a spatial translation $\vec{a}$, it changes to $\hat{x} \quad \vec{a}$.

For a special relativistic particle, the Galilei group is changed to the Poincaré group $\mathcal{P}$, which has the Lorentz group $\mathcal{L}$ as a subgroup. The Lorentz group $\mathcal{L}$ transforms the spacetime point $x \quad (x_0, \vec{x})$ to $x \quad (\quad x \ )$. It transforms time and the new time $(\quad x)_0$ depends on the old spatial coordinate $\vec{x}$. This fact leads to the disturbing result that a covariant position operator $\hat{x} \quad (\hat{x}^0, \hat{\vec{x}})$ does not exist for interacting relativistic particles. This basic result is due to Curry, Jordan and Sudarshan [5]. The requirement of a covariant position operator is also called the world line condition and discussed in Sudarshan and Mukunda [5].

In relativistic quantum field theory, a similar situation prevails. The NewtonWigner position operator [6] is not covariant and unsuitable for discussion of, say, causality.

The conclusion is that Bohr's interpretation of quantum mechanics cannot be adapted to relativistic systems.

But we need the notion of spacetime localisation. It is a central element in formulating causality: this is the requirement that if spacetime regions K_1 and K_2 are spacelike separated, the corresponding observables commute. It is also needed to interpret the statement that measurements are done on observables localised in a spacetime region K. Such a notion, called "modular localisation", will be described below.

We now informally indicate the reason why the covariant position operator does not exist in a relativistic theory in the presence of interactions.

2.1. On Covariant Position Operators

We illustrate the problem by considering the case of N point particles with masses m_i. If $Z(\tau^{(i)}) \quad Z \ (\tau^{(i)}) \quad \mathbb{R}^4$ are the trajectories of the particles labelled by the parameters $\tau^{(i)} \quad (\quad , \ )$ and if the particles are non-interacting, we can describe them by the following Lagrangian:

$$\mathfrak{L} \quad {}_i \quad \mathfrak{L}^{(i)}, \quad \mathfrak{L}^{(i)} \quad m_i \sqrt{\frac{dZ(\tau^{(i)})}{d\tau^{(i)}}}^{\ 2} . \tag{2.7}$$

The corresponding action is

$$S \quad {}_i \quad S^{(i)}, \quad S^{(i)} \quad d\tau^{(i)} \mathfrak{L}^{(i)} . \tag{2.8}$$

This action is perfectly compatible with Poincaré invariance. It can be quantised [7]. Each $S^{(i)}$ gives the unitary irreducible representation (UIRR) of the Poincaré group for mass m_i and spin 0. If $\mathcal{H}^{(i)}$ is the Hilbert space carrying the UIRR for the i-th particle, the full Hilbert space is $\mathcal{H} = \quad {}_i \ \mathcal{H}^{(i)}$. Thus we get the tensor product of N UIRR's.

Now suppose that we wish to put in interactions. They will couple different $Z(\tau^{(i)})$'s. That will involve the identification of different $\tau^{(i)}$'s, that is, effectively of different time coordinates, in some fashion.

But there is no consistent manner to do so since, as remarked above, Lorentz transformations change time in a manner which involves spatial coordinates.

In the literature, there are many attempts to overcome this "no-interaction theorem", but none of them have led to a satisfactory approach, compatible with causality and Poincaré invariance.

In a quantum field theory (QFT), the position operator has to be constructed using the quantum field , so that it is covariant. Early attempts to construct a position operator by Newton and Wigner [6] and others did not succeed in finding such a four-vector.

References from which literature after [5, 6] can be traced are [8] and recent papers involving Schroer.

2.2. The Two Concepts of Localisation

Earlier, it was emphasised that both classical and quantum physics are formulated using the concepts of both states and observables. Thus, we can study the localisation of either states or observables (or perhaps both).

In non-relativistic physics, it so happens that either localisation implies the other. We can informally explain why that is so. If K is a bounded spatial region and

$$P_K \quad {}_K d^d x |\vec{x}\rangle\langle\vec{x}| \tag{2.9}$$

is the projection operator which projects vectors in the Hilbert space $\mathcal{H}$ to vectors with support in K,

$$_K \quad P_K| \ \rangle \quad {}_K d^d x|\vec{x}\rangle \quad \vec{x} \ , \tag{2.10}$$

then, for two such vectors $_K$, $_K$,

$$\langle \ _K|a| \quad \rangle \quad d^3x\, d^3y\langle \ |P_K a P_K| \ \rangle . \tag{2.11}$$

This shows that we can restrict wave functions to K, or equivalently restrict obsservables a to K by considering $P_K a P_K$.

But this reciprocity between localised states and localised observables fails in relativistic theories. We cannot localise states as discussed above. But we *can* localise observables. It is this localisation that we discuss below.

3. Localisation in QFT

3.1. Preliminaries

We consider only free fields. We also restrict attention for now to a relativistic free field of spin zero so that

$$(x) \quad \frac{d^3k}{2k_0(2 \ )^3} \quad a_k e^{\ ikx} \quad a_k^\dagger e^{ikx} \ ,$$

$$[a_k, a_k^\dagger] \quad 2k_0(2 \ )^3 \ {}^3(k \quad k), \qquad k_0 \quad \sqrt{|\vec{k}|^2 \quad m^2} \ . \tag{3.1}$$

(The metric is $(1, -1, -1, -1)_{\text{diagonal}}$.)

The associated Hilbert space carries a (anti-)unitary irreducible representation of the Poincaré group including total spacetime reflection (which is here identified with CPT).

The commutator D of at x and y is the causal function D:

$$|\ (x),\ (y)| \quad D(x \quad y)\,, \tag{3.2}$$

$$D(x) \quad \frac{d^3k}{2k_0(2\ \ )^3}(e^{\ ikx} \quad e^{ikx})\,. \tag{3.3}$$

3.2. The Weyl algebra $\mathcal{W}$

Let f be a real test function for . That means that if f has support in a spacetime region K,

$$(f) \quad d^4x\, f(x)\ \ (x) \tag{3.4}$$

is the field localised in K.

In the absence of a good notion yet of localisation, this remark needs clarification. It will emerge later. For now, we use it to derive the Weyl algebra.

Following Weyl, we replace the unbounded operator (f) by the unitary operator

$$W(f) \quad e^{(i/\sqrt{2})\ \ (f)}\,. \tag{3.5}$$

As a consequence of (3.3), W's fulfill

$$W(f)W(g) \quad W(f \quad g)e^{\frac{i}{4}\ (f,g)}\,, \qquad (f,g) \quad \mathrm{Im}\ \ d^4x\, d^4y\, f(x)D(x \quad y)g(y)\,. \tag{3.6}$$

Here, $(f,g) \qquad (g,f)$. Also, since

$$(\Box + m^2)D(x) = 0\,, \tag{3.7}$$

we have

$$((\Box + m^2) \quad g) = 0 \tag{3.8}$$

for functions of compact support (say). Modulo such functions, can be shown to be a symplectic form on test functions.

Let us introduce a scalar product on f's using Fourier transform:

$$\tilde{f}(k) \quad d^4x\, f(x)e^{ikx}, \qquad k_0 \quad \sqrt{\vec{k}^2 \quad m^2}\,. \tag{3.9}$$

$$(f,g) \quad \frac{d^3k}{2k_0(2\ \ )^3}\bar{\tilde{f}}(k)\tilde{g}(k)\,. \tag{3.10}$$

Then, we can write

$$W(f)W(g) \quad W(f \quad g)e^{\frac{i}{2}\,\mathrm{Im}(f,\,g)}\,, \tag{3.11}$$

In addition, we have the $*$-relation

$$W(f)^* \quad W(\ \ f)\,. \tag{3.12}$$

Equations (3.11) and (3.12) are the defining relations for the Weyl algebra $\mathcal{W}$. The quantisation of the free field can be recovered from the quantisation of $\mathcal{W}$.

3.3. Quantisation of $\mathcal{W}$: the Fock Space

We now specialise to a real scalar field so that, from the vacuum, it creates an irreducible representation space of the Poincaré group. So the field in (3.2) is "hermitean".

The quantisation of $\mathcal{W}$ as it emerges from the Fock space quantisation of is the following. Let $\mathcal{H}$ be the Hilbert space with the scalar product introduced above. Then, the bosonic Fock space $\mathcal{F}(\mathcal{H})$ is

$$\exp(\mathcal{H}) := \mathbb{C}e_0 \quad \mathcal{H} \quad \mathcal{H} \quad {}_S\mathcal{H} \quad \ldots , \tag{3.13}$$

where e_0 is the vacuum state with norm 1 and ${}_S$ denotes symmetrised tensor product. Then,

$$W(f)e_0 \quad e^{\|f\|^2/4} e^{\frac{i}{\sqrt{2}}f} \quad \mathcal{F}(\mathcal{H}) . \tag{3.14}$$

3.4. An Abstract Definition of Weyl Algebra

We can now state this result for a real scalar eld in a more convenient and abstract manner. Let $\mathcal{H}$ be a complex Hilbert space. Let Re $\mathcal{H}$ be a "real" subspace of $\mathcal{H}$ so that it is closed only for real linear combination of its vectors. Then, consider operators $W(h)$ labelled by h Re $\mathcal{H}$ and fullling the algebraic relations

$$W(h_1)W(h_2) \quad W(h_1 \quad h_2)e^{\frac{i}{2}\mathrm{Im}(h_1, h_2)}, \qquad W^*(h) \quad W(\quad h) . \tag{3.15}$$

The algebra generated by the W's is the Weyl algebra $\mathcal{W}(\mathrm{Re}\,\mathcal{H})$.

We can find a representation of $\mathcal{W}_{\mathrm{Re}\,\mathcal{H}}$ following (3.13) and (3.14).

3.5. Remarks

The way we pick Re $\mathcal{H}$ in further developments is by constructing an anti-linear involution S:

$$S^2 = \mathcal{I}. \tag{3.16}$$

Then,

$$S \qquad \text{if} \qquad \mathrm{Re}\,\mathcal{H} . \tag{3.17}$$

The subspace Re $\mathcal{H}$ is said to be "standard" if

$$\overline{\mathrm{Re}\,\mathcal{H} \quad i\,\mathrm{Re}\,\mathcal{H}} = \mathcal{H}, \qquad \mathrm{Re}\,\mathcal{H} \cap i\,\mathrm{Re}\,\mathcal{H} = \{0\}. \tag{3.18}$$

The bar means closure in the Hilbert space norm.

In this case we can unambiguously decompose a vector $\mathcal{H}$ into its "real" and "imaginary" parts Re and Im :

$$\mathrm{Re} \quad -(\mathcal{I}+S) \quad , \qquad \mathrm{Im} \quad \frac{i}{} (\mathcal{I}-S) \quad ,$$

$$\mathrm{Re} \quad i\mathrm{Im} \quad , \qquad S \quad \mathrm{Re} \quad i\mathrm{Im} \quad . \tag{3.19}$$

If an anti-linear involution S gives a "standard" decomposition of $\mathcal{H}$ into Re $\mathcal{H}$ i Re $\mathcal{H}$ on using (3.17), S is said to be the Tomita-Takesaki operator (in its real version).

4. Quantum Field Theory: Requirements on Localisation

As alluded to before, we will localise the algebra of observables, that is the Weyl algebra. The localised algebras will be presented abstractly in terms of real subspaces

defined using Tomita-Takesaki involutions. Their interpretation as algebras localised in spacetime regions will subsequently emerge.

We will *not* try to localise states. We *cannot* do that.

Consider a spacetime region K. Then, let K denote its causal complement, so that if $x \quad K$, $x \quad K$, then x and x are spacelike separated.

The given region K is said to be *causally complete* if $K \quad K$.

We will index a family of Weyl algebras by causally complete K, writing $\mathcal{W}(K)$ for the indexed algebra. But to physically interpret $\mathcal{W}(K)$ as the algebra of observables localised in K, it must have the following properties, which are physically well motivated:

Covariance: Let $\mathcal{P}_+$ denote the Poincaré group including the total reflection R: $R \triangleright (x_0, x_1, x_2, \ldots) \quad (x_0, \quad x_1, \quad x_2, \ldots)$. Let $g \quad \mathcal{P}_+$. It acts on K: $K \quad gK$. We require that there is a representation of $\mathcal{P}_+$ on $\mathcal{H}$ where (g) is unitary if $g \quad \mathcal{P}$ and anti-unitary if $g \quad R\mathcal{P}$, such that

$$\mathcal{W}(gK) \quad (g)\mathcal{W}(K) \quad {}^{1}(g). \tag{4.1}$$

The operator (R) will be denoted by . It is anti-unitary.

Haag duality which implies causality: Let $\mathcal{W}(K)$ denote the commutant of $\mathcal{W}(K)$. Then $\mathcal{W}(K) \quad \mathcal{W}(K)$.

Isotony: If $K_1 \quad K_2$, then $\mathcal{W}(K_1) \quad \mathcal{W}(K_2)$.

5. The Construction of $\mathcal{W}(K)$

As stated above, we can assume that we are given a representation of of $\mathcal{P}_+$ on $\mathcal{H}$. We assume it to be (anti-)unitary, irreducible (UIRR) and of *positive energy*, $p_0 > 0$. For now, we consider the spin zero representation.

The net of local algebras emerges just from the UIRR's of $\mathcal{P}_+$, that is from Wigner's original research. It does not appeal to classical concepts like Lagrangians and actions. This is a remarkable fact.

Fix a wedge W, say

$$W \quad \{x \quad M^4 : x_1 \quad |x_0|\}. \tag{5.1}$$

It is used as a device to label the Weyl algebras, even the existence of spacetime need not enter in its conception. We classify them below.

Then, the Lorentz boosts

$$_W(t) \quad \begin{matrix} \cosh t & -\sinh t & 0 & 0 \\ -\sinh t & \cosh t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \tag{5.2}$$

leave W invariant:

$$_W(t)W \quad W. \tag{5.3}$$

It is contained in the stability group of W. The full stability group is generated by these Lorentz boosts and rotations and translations of the $x_2 \quad x_3$ plane.

Consider the $x_0 \quad x_1$ reflection j_W:

$$j_W \triangleright (x_0, x_1, x_r) \quad (\quad x_0, \quad x_1, x_r), \tag{5.4}$$

where x_r denotes the remaining spatial coordinates. It maps W to its causal complement W . The Figure 1 shows W, W and j_W for (1 + 1) spacetime.

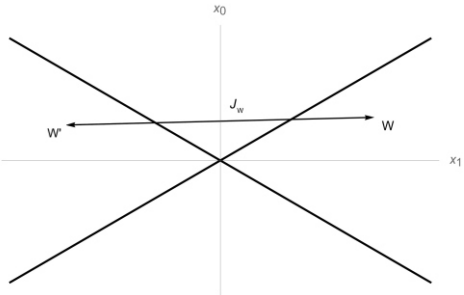

Fig. 1. W, W and j_W in (1 + 1)-dimensional spacetime.

An important property of j_W is that it commutes with $_W(t)$:

$$j_W \quad _W(t) \quad _W(t) j_W \ . \tag{5.5}$$

Under the representation map , $_W(t)$ becomes

$$_W(t) \quad e^{itK_W}, \tag{5.6}$$

while

$$J_W : \quad (j_W) \qquad (\quad \text{rotation around 1-axis}). \tag{5.7}$$

The expression for J_W in terms of assumes that the spacetime is four-dimensional and follows from

$$(R \circ \quad \text{rotation around 1-axis}) \triangleright (x_0, x_1, x_r)$$

$$R \triangleright (x_0, x_1, \quad x_r) \quad (\quad x_0, \quad x_1, x_r) \quad j_W \triangleright (x_0, x_1, x_r) \ . \tag{5.8}$$

By (5.5),

$$J_W e^{itK_W} \quad e^{itK_W} J_W \ . \tag{5.9}$$

But J_W is anti-unitary. Hence,

$$J_W K_W \quad K_W J_W \ . \tag{5.10}$$

We now come to the anti-linear involutions S_W and $S_W \quad J_W S_W J_W^{1} \quad J_W S_W J_W$. They pick out the real subspaces $\mathcal{H}_{W,W}$ and the associated Weyl algebras $\mathcal{W}(W), \mathcal{W}(W \ )$. They commute, as required by causality, and as shown below.

Consider

$$e^{\ K_W} \quad _W^{1/2} \ . \tag{5.11}$$

This operator is defined by the analytic continuation of e^{itK_W} to the strip

$$0 \quad \mathrm{Im} t \quad . \tag{5.12}$$

We will see that this continuation is possible.

The operator S_W, the Tomita-Takesaki operator, is given by

$$S_W \quad J_W \ _W^{1/2} \ . \tag{5.13}$$

Since

$$J_W \quad {}^{1/2}_{W} \qquad {}^{1/2}_{W} J_W \tag{5.14}$$

by (5.10),

$$S_W S_W \quad J_W \; {}^{1/2}_{W} J_W \; {}^{1/2}_{W} \quad J_W^2 \quad \mathcal{I}\,, \tag{5.15}$$

so that S_W is an anti-linear involution. But it is *not* anti-unitary, since

$$S_W^* S_W \qquad {}_W\,. \tag{5.16}$$

${}_W$ is self-adjoint, but not unitary. The operator K_W has neither upper nor lower bound. Hence, ${}_W$ is not bounded above, just like the Hamiltonian.

The real subspace Re $\mathcal{H}$ for W, which we denote by Re $\mathcal{H}(W)$, is determined by S_W:

$${}_W \quad \mathrm{Re}\,\mathcal{H}(W) \qquad S_W \; {}_W \qquad {}_W\,. \tag{5.17}$$

As for W , by covariance,

$$S_W \quad J_W S_W J_W \qquad {}^{1/2}_{W} J_W \quad J_W \; {}^{1/2}_{W}\,, \tag{5.18}$$

so that

$${}_W \quad \mathrm{Re}\,\mathcal{H}(W\,) \qquad S_W \; {}_W \qquad {}_W\,. \tag{5.19}$$

We now come to the crucial result.

5.1. Causality

This requires the proof that the Weyl algebras $\mathcal{W}_{W,W}$ are commutants of each other:

$$\mathcal{W}_W \quad \mathcal{W}_W\,, \tag{5.20}$$

the superscript prime denoting commutant. There is also a change of notation: the spacetime region labeling the Weyl algebra is being put as a subscript.

Here, we will only prove that

$$\mathcal{W}(\;{}_W)\mathcal{W}(\;{}_W) = \mathcal{W}(\;{}_W)\mathcal{W}(\;{}_W)\,. \tag{5.21}$$

Since

$$\mathcal{W}(\;{}_W)\mathcal{W}(\;{}_W) = \mathcal{W}(\;{}_W \qquad {}_W)\, e^{\frac{i}{2}\mathrm{Im}(\;{}_W \;\; {}_W)}\,, \tag{5.22}$$

we must verify that

$$(\;{}_W\,,\;{}_W) \quad \mathbb{R}\,. \tag{5.23}$$

For this purpose, we need the identity

$$(J_W \quad , J_W \qquad\qquad , \qquad , \qquad \mathcal{H}\,, \tag{5.24}$$

since J_W is anti-unitary.

Now

$$(\;{}_W\,,\;{}_W) \quad (S_W \;\; {}_W\,, S_W \;\; {}_W) \quad (J_W \; {}^{1/2}_{W} \;\; {}_W\,, J_W \; {}^{1/2}_{W} \;\; {}_W)\,. \tag{5.25}$$

By (5.13) and (5.18),

$$(\;{}_W\,,\;{}_W) \quad (\;{}^{1/2}_{W} \;\; {}_W,\; {}^{1/2}_{W} \;\; {}_W)\,. \tag{5.26}$$

By (5.24),

$$(\;{}_W\,,\;{}_W) \quad (\;{}_W \quad {}_W)\,, \tag{5.27}$$

since ${}^{1/2}_{W}$ are self-adjoint.

Hence, $(\;{}_W\,,\;{}_W)$ is real and causality is established.

It is important to note that the causal complement $\mathcal{W}_W$ of $\mathcal{W}_W$ is its "symplectic complement". Also, nowhere have we tried to localise states.

5.2. On the Tomita-Takesaki Operator

Let us denote the representation of the Weyl algebra $\mathcal{W}_W$ by the same symbol.
The Fock space representation of $\mathcal{W}_W$ is built from the vacuum state e_0. It has the following important properties: it is *cyclic* and *separating*.

"Cyclic" means that the Weyl algebra (complex linear combinations of all $\mathcal{W}(h)$) acting on e_0 gives the full Hilbert space $\mathcal{H}$ on closure.

"Separating" means that if $\mathcal{W}_W$ annihilates e_0, then 0:

$$e_0 \quad 0 \qquad 0 . \tag{5.28}$$

The implications of the remarkable results of Tomita-Takesaki theory are as follows. Since e_0 is cyclic and separating for the C^*– algebra $\mathcal{W}_W$ there exists a unique anti-linear involution $\tilde{S}_W$,

$$\tilde{S}_W^2 \quad \mathcal{I} \tag{5.29}$$

with the property

$$\tilde{S}_W \quad e_0 \quad {}^{*} e_0 . \tag{5.30}$$

Our S_W fulfills (5.30) since by (3.14),

$$S_W \mathcal{W}(h) e_0 \quad S_W e^{\|h\|^2/4} e^{ih/\sqrt{2}} e_0 \quad e^{\|h\|^2/4} e^{ih/4} e_0 \quad \mathcal{W}^*(h) e_0 , \tag{5.31}$$

where we used $S_W e_0 \quad e_0$.

Hence,

$$\tilde{S}_W \quad S_W . \tag{5.32}$$

The polar decomposition of S_W is just

$$S_W \quad J_W \; {}_W^{1/2} . \tag{5.33}$$

The unitary group

$$U(t) \quad {}_W^{it} , \tag{5.34}$$

which leaves the vacuum invariant, generates the "modular automorphism" of $\mathcal{W}_W$.
Since

$${}_W^{it} \quad e^{itK_W} , \tag{5.35}$$

where W is a wedge in this case, the boost group is the modular automorphism group and K_W can be called the modular Hamiltonian.

But below, when we sharpen the localisation from wedges to smaller regions $\mathcal{O}$, the corresponding operators $S_{\mathcal{O}} \quad J_{\mathcal{O}} \; {}_{\mathcal{O}}$ with $S_{\mathcal{O}}^2 \quad \mathcal{I}$ and ${}_{\mathcal{O}}^{it}$ all exist, but in general do not have geometric interpretation.

5.3. Remarks on the Real Subspaces of $S_{\mathcal{O}}$

As before, we can define the real subspace Re $\mathcal{H}(\mathcal{O})$ of $\mathcal{H}$ using $S_{\mathcal{O}}$. It is also standard:

$$\overline{\mathrm{Re}\,\mathcal{H}(\mathcal{O}) + \mathrm{iRe}\,\mathcal{H}(\mathcal{O})} = \mathcal{H}\,. \tag{5.36}$$

The converse is also true: if $S_{\mathcal{O}}$ leads to a standard real subspace Re $\mathcal{H}(\mathcal{O})$ of $\mathcal{H}$, it fulfills (5.31).

6. Operators localised in *W*

We need a simple definition of operators $\mathcal{W}_W(h)$ when h Re $\mathcal{H}(W)$. We can obtain it by first recalling an elementary result in Fourier transforms.

Consider the Fourier transform smooth compactly supported f of a function $\tilde{f}(x)$ which is supported on the half-line:

$$f(\ ) \quad {}_0 dx\, \tilde{f}(x) e^{i\ x}\,. \tag{6.1}$$

This integral converges if is continued into a complex variable with Im > 0. It is holomorphic if Im > 0.

The elements of the real subspace Re $\mathcal{H}(W)$ can be constructed in a similar manner. We can find them by starting with

$$_W(p) \quad {}_0 dx\ dx\ \tilde{}_W(x\ ,x\ ) e^{i(p^0 x_0\ \ p^1 x_1)}, \quad x \quad x_1 \quad x_0 \tag{6.2}$$

where we have suppressed the variables $x_r (r \quad 2, 3)$. In W, $x \quad 0$ so that the integral is over W. The representation of $\mathcal{P}_+$ can clearly be realised using the complex function $_W$ of momentum p. We now argue that for positive energy representations,

$$p_0 \quad p_1 \quad 0\,, \tag{6.3}$$

S_W can be applied on $_W$. The requirement

$$S_W\ \ _W \quad {}_W \tag{6.4}$$

then implies that

$$\tilde{}_W(x\ ,x\ ) \quad \mathbb{R}\,. \tag{6.5}$$

The real subspace Re$\mathcal{H}(W)$ is thus spanned by functions $_W$ with real Fourier transforms which are supported in W.

Let us show this result. With $p \quad p^0 \quad p^1 \quad 0$, as is the case in positive energy representations,

$$_W(p) \quad {}_0 dx\ dx\ \tilde{}_W(x\ ,x\ ) e^{i(p\ x\ \ \ p\ x\ )/2}\,. \tag{6.6}$$

Under the boost transformation

$$_W(t) \quad \begin{matrix} \cosh t & -\sinh t & 0 & 0 \\ -\sinh t & \cosh t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}\,, \tag{6.7}$$

we have

$$(\ _W(t)) \quad e^{itK_W} \tag{6.8}$$

and

$$(e^{itK_W} \quad {}_W)(p) \quad {}_W(\ {}^{1}_{W}(t)p) \quad {}_0 dx\ dx\ \tilde{}_W(x\ ,x\ )e^{i(e^t p\ x \quad e^{\ t}p\ x\ )}. \tag{6.9}$$

For

$$t \quad i\ , \quad 0 \qquad , \tag{6.10}$$

we get

$$(e^{\ K_W} \quad {}_W)(p) \quad {}_0 dx\ dx\ \tilde{}_W(x\ ,x\ )e^{i\cos\ (p\ x \quad p\ x\ )}e^{\ \sin\ (p\ x \quad p\ x\ )}. \tag{6.11}$$

The first exponential has modulus 1, while the second is a damping factor in the interval (6.10), since $p\ ,x \quad 0$. Thus, (6.9) is the boundary value $\quad 0$ of a holomorphic function in the strip

$$0 \quad \mathrm{Im}\, t \quad . \tag{6.12}$$

When

$$\mathrm{Im}\, t \quad , \tag{6.13}$$

we get

$$(e^{\ K_W} \quad {}_W)(p) \quad (\ {}^{1/2}_{W} \quad {}_W)(p) \quad {}_0 dx\ dx\ \tilde{}_W(x\ ,x\ )e^{\ i(p\ x \quad p\ x\ )}. \tag{6.14}$$

Hence,

$$(S_W \quad {}_W)(p) \quad (J_W \ {}^{1/2}_{W}\ )(p) \quad {}_0 dx\ dx\ \overline{\tilde{}}_W(x\ ,x\ )e^{i(p\ x \quad ip\ x\ )} \tag{6.15}$$

and the condition $S_W \quad {}_W \qquad {}_W$ implies that $\tilde{}_W(x\ ,x\ ) \quad \mathbb{R}$, as claimed.

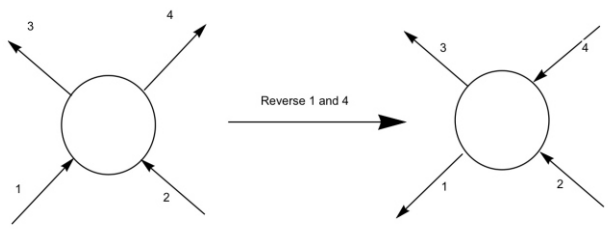

Fig. 2. ${}^{1/2}_{w}$ is seen to reverse the sign of energy.

6.1. Remarks

The above analyticity and hence the existence of S_W and localisation can be established only for "positive energy representations", where $(p_0 \quad |\vec{p}|) \quad 0$. From (6.14), ${}^{1/2}_{w}$ is seen to reverse the sign of energy. In Feynman's language, it converts an outgoing particle line into an incoming anti-particle line in a scattering diagram. We illustrate this interpretation in Figure 2.
Thus, as Fassarella and Schroer [3] have emphasised, ${}^{1/2}_{w}$ seems related to crossing symmetry.

All this means in particular that localisation requires anti-particles (which may be the same as particles).

7. Sharpening Localisation

Wedge localisation is rather weak as a wedge is not even compact. One would like localisation in spacetime regions $\mathcal{O}$ of arbitrary small size.

For this purpose, first consider the intersection of two wedges W_1 and W_2 producing the "causal diamond" (shown in Figure 3). We can then consider the associated real Hilbert space:

$$\mathrm{Re}\,\mathcal{H}(W_1 \bigcap W_2): \quad \mathrm{Re}\,\mathcal{H}(W_1) \bigcap \mathrm{Re}\,\mathcal{H}(W_2)\,. \tag{7.1}$$

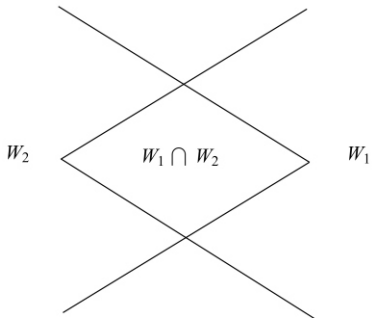

Fig. 3. Causal Diamond.

One then shows that $\mathrm{Re}\,\mathcal{H}(W_1 \bigcap W_2)$ is standard:

$$\mathcal{H} \quad \mathrm{Re}\,\mathcal{H}(W_1 \bigcap W_2) \quad i\,\mathrm{Re}\,\mathcal{H}(W_1 \bigcap W_2)\,,$$

$$\mathrm{Re}\,\mathcal{H}(W_1 \bigcap W_2) \bigcap i\,\mathrm{Re}\,\mathcal{H}(W_1 \bigcap W_2) \quad \{0\} \tag{7.2}$$

(where taking closure is understood).

Now (7.2) is enough to define the modular operator and show causality.

Thus, if $\mathcal{H}$, we have the unique decomposition

$$\mathrm{Re}_{W_1\ W_2} \quad i\mathrm{Im}_{W_1\ W_2}\,, \tag{7.3}$$

where the first term is in $\mathrm{Re}\,\mathcal{H}_{W_1\ W_2}$ and the second in $i\,\mathrm{Re}\,\mathcal{H}_{W_1\ W_2}$.

The modular involution $S_{W_1\ W_2}$ is then defined by

$$S_{W_1\ W_2} \quad \mathrm{Re}_{W_1\ W_2} \quad i\mathrm{Im}_{W_1\ W_2}\,. \tag{7.4}$$

The definition of polar decomposition of $S_{W_1\ W_2}$,

$$S_{W_1\ W_2} \quad J_{W_1\ W_2}\ {}^{1/2}_{W_1\ W_2}\,, \tag{7.5}$$

shows that

$$_{W} = S^{\dagger}_{W_1\ W_2} S_{W_1\ W_2} \tag{7.6}$$

where the RHS can be calculated from (7.4). Then, from (7.5), we have $J_{W_1\ W_2}$.

Just as before as in the case of S_W, one shows that the modular involution

$$J_{W_1\ W_2} S_{W_1\ W_2} J_{W_1\ W_2} \quad {}^{1/2}_{W_1\ W_2} J_{W_1\ W_2} \tag{7.7}$$

determines the causal complement $\mathcal{W}_{W_1\ W_2}$ of $\mathcal{W}_{W_1\ W_2}$.

In this way, we have the algebra $\mathcal{W}_{W_1\ W_2}$ localised in $W_1 \cap W_2$.

We can even characterise the elements $_{W_1\ W_2}$ in $\mathrm{Re}\,\mathcal{H}_{W_1\ W_2}$: we use (6.2), but with real functions $\tilde{}_{W_1\ W_2}$ supported in $W_1 \cap W_2$.

7.1. Further Sharpening of Localisation

A spacetime region $\mathcal{O}$ is said to be *causally complete* if the following condition is satisfied: Let $\mathcal{O}$ denote the causal complement of $\mathcal{O}$ so that points of $\mathcal{O}$ are spacelike separated from $\mathcal{O}$. Let $\mathcal{O}$ be the causal complement of $\mathcal{O}$. Then, $\mathcal{O}$ is causally complete if $\mathcal{O} = \mathcal{O}$. The diamond in the last Figure above is causally complete.

A causally complete region $\mathcal{O}$ is known to be the intersection of wedges. As wedges are mutually related by the action of $\mathcal{P}_+$, causally complete regions are invariant under the action of $\mathcal{P}_+$. They form a covariant net.

Given this net, we can localise the Weyl algebra to $\mathcal{O}$, to obtain $\mathcal{W}_{\mathcal{O}}$.

We now show how to explicitly construct $\mathcal{W}_{\mathcal{O}}$. It involves the construction of the standard real subspace Re $\mathcal{H}_{\mathcal{O}}$ $\mathcal{H}$.

We can describe the elements of Re $\mathcal{H}_{\mathcal{O}}$ using (6.2), but with the real functions $\tilde{}_{\mathcal{O}}$ now supported in $\mathcal{O}$.

The Weyl algebra is then constructed from its elements as described earlier.

It is important to know also the modular involution $S_{\mathcal{O}}$ and the causal complement of $\mathcal{W}_{\mathcal{O}}$.

As for $S_{\mathcal{O}}$, we follow (7.3)–(7.5), but with $W_1 \cap W_2$ replaced by $\mathcal{O}$. That gives us $J_{\mathcal{O}}$ and ${}^{1/2}_{\mathcal{O}}$ in $S_{\mathcal{O}}$ J_W ${}^{1/2}_{\mathcal{O}}$.

The causal complement $\mathcal{W}_{\mathcal{O}}$ of $\mathcal{W}_{\mathcal{O}}$ is then

$$J_{\mathcal{O}}\mathcal{W}_{\mathcal{O}}J_{\mathcal{O}}^{1} \quad J_{\mathcal{O}}\mathcal{W}_{\mathcal{O}}J_{\mathcal{O}} . \tag{7.8}$$

7.2. Remarks

We can show that the vacuum state $|0\rangle$: $|e_0\rangle$ restricted to the observables in the Rindler wedge W is mixed: it is a thermal or KMS state. This is Unruh's result.The proof is as follows.

First recall the KMS condition. In terms of a density matrix exp(H), it is

$$(AB): \; Tr(\;AB)/(Tr\;) \qquad (B\exp(\;\;H)A\exp(\;H) \qquad (BU_i(A)) \tag{7.9}$$

where U_i (A) A evolved for imaginary time i .

A state is KMS if it fulfills

$$(AB) \qquad (BU_i(A)) \tag{7.10}$$

even if this state does not come from a density matrix.

It is not dificult to show that

$$S_W V(h)|e_0\rangle \quad V(h)^*|e_0\rangle . \tag{7.11}$$

Hence

$$\langle e_0|AB|0\rangle \quad \langle A^* e_0|Be_0\rangle \quad \langle J_W\sqrt{\;_W}Ae_0|J_W\sqrt{\;_W}B^* e_0\rangle \tag{7.12}$$

$$\langle\sqrt{\;_W}B^* e_0|\sqrt{\;_W}Ae_0\rangle \qquad \text{(as } J_W \text{ is anti-unitary)} \tag{7.13}$$

$$\langle e_0|B\;\;_W A|e_0\rangle \qquad \text{(is } \sqrt{\;_W} \text{ is self-adjoint)} \tag{7.14}$$

$$\langle e_0|B\;\;_W A\;\;{}^{1}_{W}|e_0\rangle \qquad \text{(as } |e_0\rangle \text{ is invariant under } {}^{1}_{W}) . \tag{7.15}$$

Thus $|e_0\rangle\langle e_0|$ is a mixed KMS state for the algebra $\mathcal{W}_{\mathrm{W}}$ and for the 'Hamiltonian' $H \quad 2\;K_W\;/\;$.

But since the spectrum of K_W is unbounded above and below, $_W$ is not of trace class and we cannot construct a density matrix like above for this state.

It is known that

$$\mathrm{Re}\,\mathcal{H}_{\mathcal{O}} \quad \bigcap_{W \quad \mathcal{O}} \mathrm{Re}\,\mathcal{H}_{\mathrm{W}}, \tag{7.16}$$

$$\mathcal{W}_{\mathcal{O}} \quad \bigcap_{W \quad \mathcal{O}} \mathcal{W}_{\mathrm{W}} . \tag{7.17}$$

We can show as before that the vacuum defines a KMS state for the Hamiltonian $2 \;\; K_{\mathcal{O}} / \;$, where $_{\mathcal{O}} \quad e^{\;2\;K_{\mathcal{O}}}$.

But when $\mathcal{O}$ is not a wedge, not even when it is a causal diamond, $K_{\mathcal{O}}$ has no known geometrical meaning. It is a boost generator of $\mathcal{P}_+$ only when $\mathcal{O}$ is a wedge.

The theory shows that $_{\mathcal{O}} e_0 \quad e_0$ or that

$$e^{itK_{\mathcal{O}}} e_0 \quad e_0 \tag{7.18}$$

for every causally complete $\mathcal{O}$. Thus we get an infinite number of localised boost groups $_{\mathcal{O}} \quad \{e^{itK_{\mathcal{O}}}\}$ labelled by the causally complete net, all of which leave the vacuum invariant, just like $_W$. Their localisation reminds us of gauge groups, but the latter either act trivially on *all* quantum states or define superselection sectors. Neither is the case with $_{\mathcal{O}}$.

The physical meaning of $_{\mathcal{O}}$'s has not been understood.

8. Introducing Spin

In these notes, we have not treated the construction of the UIRR's of $\mathcal{P}_+$ using Wigner's approach. For this reason, we will treat only the spin 1/2 case, assumming familiarity with the construction of its UIRR. We refer to [9] for example for further details.

For relativistic particles with spin, the transformation properties of a state vector with definite momentum involves the Wigner boost and Wigner rotation. Their presence spoils the analyticity property of e^{itK_W} in the strip $0 \quad \mathrm{Im}\,t \quad$. Localisation for such representations involves additional considerations.

We illustrate the situation for a UIRR of $\mathcal{P}_+$ with spin 1/2 and mass $m > 0$.

8.1. Massive Particle of Spin 1/2

8.1.1. Preliminaries

Let

$$\hat{p} \quad (m, \vec{0}) \tag{8.1}$$

be the standard momentum. A basic ingredient in setting up the UIRR is the choice of the Wigner boost $L_p \quad \mathcal{P}_+$ which transforms $\hat{p}$ to momentum p (see [9]):

$$L_p \hat{p} \quad p . \tag{8.2}$$

A convenient choice of L_p uses the $2 \;\; 2$ representation of p:

$$p \qquad p \quad (\quad p) \;, \qquad (\;^0 \quad \mathcal{I}, \;^i \quad \text{Pauli matrices}). \tag{8.3}$$

In this representation, $\mathcal{P}_+$ acts by elements $g \quad SL(2,\mathbb{C})$:

$$p \quad g \quad pg^{\dagger}. \tag{8.4}$$

For rotations, $g \quad SU(2)$, so that $g^{\dagger} \quad g^{\ 1}$. For boosts, $g \quad g^{\dagger}$. Thus, since

$$\widehat{p} \quad m\mathcal{I} \tag{8.5}$$

and

$$p \quad 0, \tag{8.6}$$

that is, its eigenvalues are positive, as may be verified, we can choose for the boosts,

$$g(p) \quad \frac{p}{m}^{1/2} \quad SL(2,\mathbb{C}), \tag{8.7}$$

where the square root is the positive one. Thus,

$$(L_p\,\widehat{p}) \quad g(p)(\quad \widehat{p})g^{\dagger}(p). \tag{8.8}$$

Here $g(p)$ is self-adjoint.

The transformations in $\mathcal{P}_+$ leaving $\widehat{p}$ invariant is $SO(3)$. In the $2 \quad SL \quad \mathbb{C}$ representation, it becomes the UIRR $D^{1/2}$ of $SU(2)$ for angular momentum 1/2. In the Wigner approach, we first introduce the vectors $|\widehat{p},\ \rangle$. If the UIRR is U and $h \quad SU(2)$, we set

$$U(h)|\widehat{p},\ \rangle \quad |\widehat{p},\ \rangle D^{1/2}(h), \quad U(L_p)|\widehat{p},\ \rangle \quad |p,\ \rangle. \tag{8.9}$$

Using the $2 \quad 2$ matrix rotation, we can change notation as follows:

$$\begin{aligned} &|\widehat{p},\ \rangle \quad |\quad \widehat{p},\ \rangle, \\ &U(h)|\quad \widehat{p},\ \rangle \quad |\quad \widehat{p},\ \rangle D^{1/2}(h), \\ &U(L_p)|\quad \widehat{p},\ \rangle \quad |(\tfrac{p}{m})^{1/2} \quad \widehat{p}(\tfrac{p}{m})^{1/2},\ \rangle \quad |\quad p,\ \rangle. \end{aligned} \tag{8.10}$$

It is a consequence of (8.10) that if $g \quad SL \quad \mathbb{C}$,

$$U(g)|\quad p,\ \rangle \quad |\quad (g)p,\ \rangle D^{1/2}(h(p,g)), \tag{8.11}$$

where (g) the Lorentz transformation associated with g,

$$g \quad pg^{\dagger} \quad (g)p \tag{8.12}$$

and $h(p,g) \quad SU(2)$ is called the Wigner rotation:

$$h(p,g) \quad \frac{(g)p}{m}^{1/2} g \frac{p}{m}^{1/2}. \tag{8.13}$$

For further details, see [9].

Thus, if g is the boost $e^{\ itK_W}$, which becomes $e^{t\ \ 1/2}$ in the $2 \quad 2\ SL \quad \mathbb{C}$ representation,

$$h(p,e^{t\ \ 1/2}) \quad \frac{e^{itK_W}p}{m}^{1/2} e^{t\ \ 1/2} \frac{p}{m}^{1/2}. \tag{8.14}$$

8.1.2. Analyticity

We need the analyticity of (8.14) in the strip $0 \quad \mathrm{Im}t \quad$ (See (6.12)). But for $p_r(r \quad 2) \quad 0$, this requirement is not met, leading to an obstruction to localisation.

The way around it is as follows. Let us imbed the $D^{1/2}$ UIRR of $SU(2)$ in the $D^{(1/2,0)}$ IRR of $SL(2, \mathbb{C})$. Then, we can write

$$D^{(1/2,0)}h(p,g)) \quad D^{(1/2,0)} \quad \frac{(g)p}{m}^{1/2} \quad {}^{1} \quad D^{(1/2,0)}(g)D^{(1/2,0)} \quad \frac{p}{m}^{1/2} . \tag{8.15}$$

Using this decomposition, let us define.

$$| \quad p, \quad \rangle^{*} \quad | \quad p, \quad \rangle D^{(1/2,0)} \quad \frac{p}{m}^{1/2} \quad {}^{1} \tag{8.16}$$

It follows from (8.11) and (8.15) that

$$U(g)| \quad p, \quad \rangle^{*} \quad | \quad (g)p, \quad \rangle^{*} D^{(1/2,0)}(g) . \tag{8.17}$$

Thus, by working with functions f of p (or p) with the transformation (8.17), we can remove the obstruction to analyticity encountered above.

8.1.3. Causality

For the spin 0 case we treated above, the quantum field which emerges *commutes* for spacelike separations. We can see this as follows.

In the notation of (6.2), let

$$(\tilde{}_{W}) \quad a^{*}(\ _{W}) \quad a(S_{W} \ _{W}) , \tag{8.18}$$

where $\tilde{}_{W}$ is supported in W and is real, as before. So $S_{W} \quad {}_{W} \quad {}_{W}$, but we put in S_{W} for later convenience. We also set

$$[a(S_{W} \ _{W}), a^{*}(\ _{W})] \quad (S_{W} \ _{W}, \ _{W}) , \tag{8.19}$$

where $\tilde{}_{W}$ also has support W. All other commutators involving a and a^{*} vanish as usual.

Similarly,

$$(\tilde{}_{W}) \quad a^{*}(\ _{W}) \quad a(S_{W} \ _{W}) , \tag{8.20}$$

where

$$S_{W} \quad J_{W} S_{W} J_{W} \quad {}^{1/2}_{W} J_{W} . \tag{8.21}$$

Hence,

$$[\ (\tilde{}_{W}), \ {}^{*}(\tilde{}_{W})] \quad (S_{W} \ _{W}, \ _{W}) \quad (S_{W} \ _{W}, \ _{W})$$
$$(S_{W} \ _{W}, \ _{W}) \quad (\ {}^{1/2}_{W} J_{W} \ _{W}, \ _{W}) \quad 0 , \tag{8.22}$$

where the anti-unitarity of J_{W} has been used.

Thus, spacelike separated ’s commute.

We now extend this analysis to spin 1/2. Fields of spin 1/2 must *anti-commute* for spacelike separation, whereas the modular involution S_{W} leads to a commutation relation. Therefore, in the definition of a spin 1/2 field , we change S_{W} to

$$\hat{S}_{W} \quad i S_{W} \tag{8.23}$$

It too has the property

$$\hat{S}_{W}^{2} \quad I . \tag{8.24}$$

Then, for a spin 1/2 field ,

$$(\tilde{\ }_W): \quad a^*(\ \ _W) \quad a(\hat{S}_W \ \ _W), \tag{8.25}$$

where we set

$$[a(\hat{S}_W \ \ _W), a^*(\ \ _W)] \quad (\hat{S}_W \ \ _W, \ _W), \tag{8.26}$$

with zero for the other anti-commutators.

For W , by covariance,

$$\hat{S}_W \quad J_W \hat{S}_W J_W \quad iJ_W (J_W \ \ ^{1/2}_W) J_W \quad i \ ^{1/2}_W J_W \ . \tag{8.27}$$

Therefore,

$$\begin{aligned}(\tilde{\ }_W), \ (\tilde{\ }_W) \quad & (\hat{S}_W \ \ _W, \ _W) \ (\hat{S}_W \ \ _W, \ _W) \\ & i(J_W \ \ ^{1/2}_W \ \ _W, \ _W) \ \ i(\ ^{1/2}_W J_W \ _W \ \ _W) \quad 0,\end{aligned} \tag{8.28}$$

where the last line follows from anti-linearity of J_W.

The i is the "statistical" factor which corrects the commutator to anti-commutator. Its square being -1, which corresponds to 2 rotation being -1, it accounts for the spin-statistics theorem.

8.2. Final Remarks

The Poincaré group has two "exceptional" classes of positive energy UIRR's.

One occurs in 3 + 1 dimensions for massless particles where the little or stability group in general is $\overline{E}(2)$, the two-fold covering group of the Euclidean group. For particles like photons with two helicities, the translation part of $\overline{E}(2)$ is represented trivially, by identity operators.

But there are UIRR's where the translations of $\overline{E}(2)$ are represented non-trivially. In these UIRR's, helicity takes on all half-integral values for fermions and all integral values for bosons. Particles characterised by such UIRR's are said to have continuous spin.

The second class of "exceptional" UIRR's occurs in 2 + 1 dimensional spacetime. They are the anyons. For anyons, 2 -rotation is neither (+1) nor (−1). Further, they obey braid statistics. The latter is based on the braid group [10] and not on the permutation group. Such particles, which can occur as excitations in two-dimensional lattices of spins, are thought to be important for "topological quantum computations".

If we exclude these exceptional UIRR's, for all other UIRR's of the Poincaré group, localisation in the manner we have described works. Familiar local fields can also be constructed, as in [3, 4].

But that is not the case for the exceptional UIRR's [3, 4]. For such UIRR's, standard local fields, such as or above, do not exist. The best-localised fields are localised on "strings". Thus, such a field integral helicity say, is labelled by a spacetime position x and a spacelike direction e:

$$e \ e \quad 1 \ . \tag{8.29}$$

Both x and e transform under Lorentz transformations:

$$: \quad (x, e) \quad (\ ^1x, \ ^1e) \ . \tag{8.30}$$

As for causality, the condition is novel. Let $\mathbb{R}$

$$x + \mathbb{R}^+ e: \quad x \quad e: \ 0 \qquad . \tag{8.31}$$

Thus, $x + \mathbb{R}^+ e$ is a spacelike string from x to . Then, causality is expressed by

$$[\ _1(x, e),\ \ _2(x, e)] \quad 0 \tag{8.32}$$

if $x + \mathbb{R}^+ e$ is spacelike to $x + \mathbb{R}^+ e$, that is, each point p of the former, $p \quad x \quad \mathbb{R}^+ e$ is spacelike to each point p of the latter, $p \quad x \quad \mathbb{R}^+ e$.

The two-point function for such fields has been worked out.

Incidentally, such string-localised fields exist even for non-exceptional UIRR's [3, 4]. They have better ultra-violet behaviour.

8.3. Remarks

Dirac [11] had long ago considered fields dependent on a spacelike direction in the context of gauge theories. Thus, for a $U(1)$ gauge theory with a charged field and electromagnetic connection A, he had defined the field

$$\widehat{\ }(x, e) = \ P\exp\ i\ \overset{x}{A}\ (x)dx \qquad (x), \tag{8.33}$$

where the integral is along the line $x \quad \tau e$ as τ increases from ( ) to 0.
The field $\widehat{\ }$ is invariant under the gauge transformation

$$(x) \quad e^{i\ (x)}\ (x), \quad A\ (x \quad \tau e) \quad A\ (x \quad \tau e) \quad (\quad)(x \quad \tau e), \tag{8.34}$$

with the usual condition $(\quad)(x \quad \tau e) \quad 0$ as τ .
The field $\widehat{\ }$ of Dirac does not seem to be the string-localised field considered above.
The latter is a free field and not coupled to a gauge field.

It is a striking and important result that string-localised fields do not admit a Lagrangian description. They seem to have no classical counterpart of a familiar sort.

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