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Shape of Cheap Superconducting Vortices*

R. Hlubina

Department of Experimental Physics, Comenius University, Mlynská Dolina F2, 842 48 Bratislava, Slovakia

Abstract: We study a phenomenological model of a superconducting vortex whose core is assumed to be made of a non-superconducting phase of unspecified origin. If the core state is competitive in energy with the superconducting state, the vortex energy is lowered with respect to the usual case with a normal-state core. Such vortices have been called ″cheap″ in the literature on the pseudogap state of the cuprates. Within the London theory, we present a variational argument which shows that sufficiently cheap vortices generically exhibit a broken rotational symmetry. We argue that, within the generalized Ginzburg-Landau theory, the stability of asymmetric vortices depends on the precise form of the energy functional.

1. Introduction

One of the interpretations of the pseudogap state in the high-temperature superconductors postulates the presence of an incoherent liquid of singlet electron pairs on the bonds of the square lattice [1]. A more conventional picture views the pseudogap state as a disordered superconducting state with destroyed phase coherence. The major open problem in the latter line of thinking is to explain the very large difference between the small superconducting transition temperature and the large pseudogap temperature. It has been argued that, in order to destroy the phase ordering and to stabilize the pseudogap state, the presence of the so-called cheap vortices, i.e. of vortices with low energy, is required [2, 3]. In this paper we study a simple phenomenological model of a superconducting vortex with a vortex core made of a non-superconducting phase of unspecified origin. We assume that the non-superconducting core state is competitive in energy with the superconducting state. Several candidate states for the vortex core have been considered in the literature, such as the antiferromagnetic state [4], the stripe phase [5], and the staggered flux state [6].

In Section 2 we discuss the vortex energetics within the London theory. First we discuss the usual rotationally symmetric vortex and we show that the vortex core size of a cheap vortex is larger than the coherence length . Then we present a variational argument which shows that sufficiently cheap vortices generically do not have a rotationally symmetric shape.

In Section 3 we construct generalized Ginzburg-Landau theories, in which both the superconducting order parameter and a scalar order parameter m describing the vortex core state are taken into account. We study how the stability of asymmetric vortex solutions depends on the form of the Ginzburg-Landau functional.

*) *Dedicated to Professor Viktor Bezák on the occasion of his 70th birthday.*

2. The London Theory

A. Symmetric solution

Let us start by considering a usual vortex along the z axis. We assume that the magnetic field is $\mathbf{B} = (0, 0, B(x, y))$, which trivially solves the Maxwell equation $\mathbf{B} = 0$. The usual vortex solution of the London equation $^2B \quad B/ \quad ^2$, where is the penetration depth read as $B(r) \quad B_0 K_0 \frac{r}{}$, where K_0 is the modified Bessel function of imaginary argument. We shall assume that the above solution holds for distances $r \quad a$ from the vortex center, whereas in the region $r \quad a$ there exists the non-superconducting vortex core of unspecified origin, pierced by the magnetic field $B \quad B(a)$. We shall assume that the vortex core size satisfies the inequalities a . One can show easily that the flux quantization condition $d^2\mathbf{r}B \quad _0$ implies that the magnetic field amplitude for a simply reads $B_0 \quad _0/(2 \quad ^2)$.

The optimal vortex core size will be determined by minimization of the vortex energy (a) per unit length of the vortex with respect to a. To this end, let us split , to be called in what follows the vortex line energy, into two parts , corresponding to $r \quad a$ and $r \quad a$, respectively. For $r \quad a$, we obtain

$$\frac{1}{2 \quad _0} \quad d^2\mathbf{r}[\mathbf{B}^2 \quad (\quad _0 \quad)^2 \mathbf{j}^2] \quad \frac{1}{2 \quad _0} \quad d^2\mathbf{r}[\mathbf{B}^2 \quad ^2(\quad \mathbf{B})^2].$$

Simple transformations show that

$$d^2\mathbf{r}(\quad \mathbf{B})^2 \quad \circ d\mathbf{S} \; \mathbf{B} \quad (\quad \mathbf{B}) \quad d^2\mathbf{r}\mathbf{B} \; [\quad (\quad \mathbf{B})],$$

where the first integral on the right side is taken over the surface of the superconducting region and $d\mathbf{S}$ is the surface element pointing out the superconductor in the normal direction. Making use of $(\quad \mathbf{B}) \quad \mathbf{B}/ \quad ^2$ we thus find

$$\frac{^2}{2 \quad _0} \circ d\mathbf{S} \; \mathbf{B} \quad (\quad \mathbf{B}) \quad \frac{^2 a}{_0} B(a)B(a) \quad K_0 \frac{a}{},$$

where $/(4 \quad _0 \quad ^2)$. On the other hand, the contribution of the vortex core reads $a^2/(2 \quad _0)[B^2 \quad B_c^2]$, where the first term is due to the finite field inside the core and the second term describes the condensation energy loss. Within the Ginzburg-Landau theory, the magnitude of the thermodynamic critical field B_c is related to and as follows:

$$B_c \quad \frac{1}{2 \quad \sqrt{2}} \frac{_0}{} . \tag{1}$$

The dimensionless factor in the expression for describes the renormalization of the condensation energy. In a BCS superconductor 1, but in a superconductor with a competing phase in the vortex core we can have 1. In the limit a we can neglect the contribution of B^2 to with respect to and the total vortex line energy reads

$$\frac{(a)}{} \quad K_0 \frac{a}{} \quad \frac{^2}{4} \frac{a}{}^2 \quad \ln \frac{a}{} \quad 0.12 \quad \frac{^2}{4} \frac{a}{}^2, \tag{2}$$

where / and we have used the asymptotics of the function $K_0(x)$ for x 0. Minimizing the function (a), we find that the optimal vortex core is a $\sqrt{2/}$. In the BCS case this yields the usual estimate a ~ , whereas for 1 we obtain a . The vortex line energy $_S$ of the optimal symmetric vortex is therefore

$$\frac{}{} \ln \quad 0.27 \quad \frac{1}{2}\ln\frac{1}{} . \qquad (3)$$

Note that for 1 the vortex line energy is substantially diminished with respect to its BCS value for 1 and the vortex becomes cheap.

B. Asymmetric solution

We assume again solution of the London equation in the form **B** = (0, 0, $B(x, y)$), but instead of a circular vortex core with radius a we assume a rectangular vortex core with dimensions $2a$ 2 and a . The magnetic field in the non-superconducting core region has to be constant. Since is the smallest length scale, we shall neglect its finite value and we shall seek the magnetic field outside the core by replacing the core region with a line between the endpoints (a, 0) and (a, 0). We will require that B = const along the "core line". We seek the solution in the form

$$B(x,y) \quad \int_{a}^{a} d \; f(\;)K_0 \frac{\sqrt{(x \quad)^2 \quad y^2}}{} \quad ,$$

which obviously satisfies the London equation. The function $f(\;)$ has to be found from the integral equation $B(x, 0)$ = const valid for x in the interval $\langle \; a, a \rangle$. Replacing the Bessel function by its small-argument asymptotics, which is a good approximation as long as a which we assume, the integral equation is recognized as Carleman's equation [7]

$$\int_{a}^{a} d \; f(\;) \ln|x \quad | \quad \text{const} ,$$

which can be solved exactly. The final solution, which satisfies also the flux quantization condition d^2rB $_0$, reads as

$$B(x,y) \quad \frac{_0}{2 \; ^2 \; ^2} \int_{a}^{a} \frac{d}{\sqrt{a^2 \quad ^2}} K_0 \frac{\sqrt{(x \quad)^2 \quad y^2}}{} \quad . \qquad (4)$$

Note that, if the dimensions are measured in units of , the magnetic field distribution is up to a multiplicative factor determined by the dimensionless ratio $a/$. Making use of Eq. (4) and of the Maxwell equation **B** $_0$**j**, the current density **j** can be easily calculated. Thus, we obtain

$$B(x,y) \quad \frac{_0}{2 \quad ^2} \quad \frac{1}{} \quad d \; K_0 \frac{\sqrt{(x \quad a \sin \;)^2 \quad y^2}}{} \quad ,$$

$$j_x(x,y) \quad \frac{_0}{2 \quad _0 \quad ^3} \quad \frac{1}{} \quad d \; \frac{y}{\sqrt{(x \quad a \sin \;)^2 \quad y^2}} K_1 \frac{\sqrt{(x \quad a \sin \;)^2 \quad y^2}}{} \quad ,$$

$$j_y(x,y) \quad \frac{0}{2 \quad {}_0^{\;3}} \quad \frac{1}{\quad} \quad d \; \frac{x \quad a \sin}{\sqrt{(x \quad a \sin \;)^2 \quad y^2}} K_1 \; \frac{\sqrt{(x \quad a \sin \;)^2 \quad y^2}}{\quad} \; .$$

In the limit a the integrals can be explicitly evaluated along the line $y \quad 0$ and we obtain

$$j_x(x,0) \quad \frac{0}{2 \quad {}_0^{\;2}} \frac{\mathrm{sign}(y)}{\sqrt{a^2 \quad x^2}}; \quad |x| \quad a$$

$$j_y(x,0) \quad \frac{0}{2 \quad {}_0^{\;2}} \frac{\mathrm{sign}(x)}{\sqrt{x^2 \quad a^2}}; \quad |x| \quad a$$

and zero otherwise. The distribution of the magnetic field $B(x, y)$ and of the magnitude of the supercurrent $|\mathbf{j}|(x, y)$ is plotted in Fig. 1.

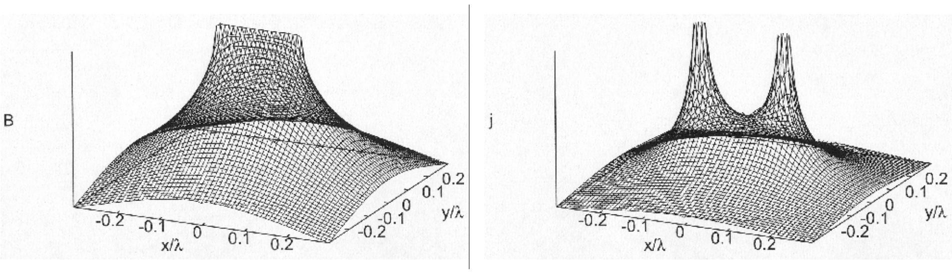

Fig. 1. Magnetic field (left panel) and the magnitude of the supercurrent (right panel) in an asymmetric vortex with $a\,/\quad\;0.1$.

Let us turn now to the evaluation of the vortex line energy of the asymmetric vortex. The contribution to the line energy of the asymmetric vortex can be written as

$$\frac{1}{2 \quad {}_0} \quad d^2r[B^2 \quad (\;{}_0\;)^2 \mathbf{j}^2] \quad F \; \frac{a}{\quad} \; .$$

Note that since the magnetic field in the asymmetric vortex is finite and the current exhibits only a weak divergence $j \quad r^{\;1/2}$, where r is the distance from the endpoints $(\quad a, 0)$ of the cut, the dimensionless function $F(a\,/\quad)$ is finite for finite values of a and no cut-offs need to be introduced by hand. We have calculated the function $F(x)$ numerically and the result is plotted in Fig. 2, together with the function $K_0(x)$. Note that $F(x) \quad K_0(x)$ for all values of x. In other words, for a given radius a, the energy is smaller in the symmetric solution. Fig. 2 shows that the function $F(x)$ can be fitted quite well by the simple formula $f(x) \quad \ln(1/x) \quad 0.8$.

The contribution of the core to the line energy of the vortex reads

$$(2a \;/\;\; {}_0)[B^2 \quad B_c^2],$$

where B is the field inside the core. The first term on the right hand side can be neglected for $a \qquad {}^2$ and the total line energy of an asymmetric vortex can be therefore written as

$$\frac{(a)}{\quad} \quad F \; \frac{a}{\quad} \quad \frac{a}{\quad} \quad \ln \frac{\quad}{a} \quad 0.8 \quad \frac{a}{\quad} \qquad (5)$$

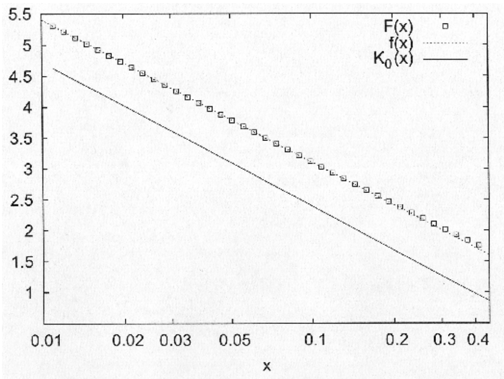

Fig. 2. Plot of the functions $F(x)$ and $K_0(x)$. Note that $F(x)$ can be fitted well by the simple form $f(x) \quad \ln(1/x) \quad 0.8$.

Note that the core energy scales only linearly with $a/$, whereas in the symmetric solution it scales with $a^2/\ ^2$. Minimizing the function (a) we find the optimal core size $a \quad /$ which is, for a given 1, much larger than the core size in the symmetric case. Therefore the vortex line energy of the optimal asymmetric vortex is

$$\frac{}{A} \ \ln \quad 0.66 \quad \ln\frac{1}{}. \tag{6}$$

Note that in the BCS case 1 the asymmetric vortex has a higher energy than the symmetric vortex. However, with decreasing the energy of the asymmetric vortex decreases faster than that of the symmetric vortex and for $_c$ the asymmetric vortex becomes energetically favourable. The critical value $_c$ can be determined from the equation $_S(\ _c) \quad _A(\ _c)$. We find $_c \quad 0.46$.

Our results are further illustrated by Fig. 3, where we plot the functions Eq. 2 and Eq. 5 for 100, which is of the correct order of magnitude for the cuprates, and for several representative values of .

3. The Ginzburg-Landau approach

Within the usual Ginzburg-Landau approach, the difference $\mathcal{F}$ between the free energy densities of the normal and of the superconducting states can be written as

$$\mathcal{F} \quad \frac{1}{2\ _0}\mathbf{B}^2 \quad (\ _0\ )^2\frac{\mathbf{j}^2}{|f|^2} \quad \frac{B_c^2}{_0} \quad |f|^2 \quad \frac{1}{2}|f|^4 \quad ^2(\ |f|)^2 \ , \tag{7}$$

where we have defined the complex order parameter function $f \quad |f|e^{i}$ normalized in such a way that in a homogeneous piece of a superconductor in the equilibrium we have $|f| \quad 1$. B_c is the thermodynamic critical field and is the coherence length. The superconducting current density is given by

$$\mathbf{j} \quad \frac{1}{_0\ ^2}|f|^2 \ \mathbf{A} \quad \frac{\hbar}{2e} \qquad .$$

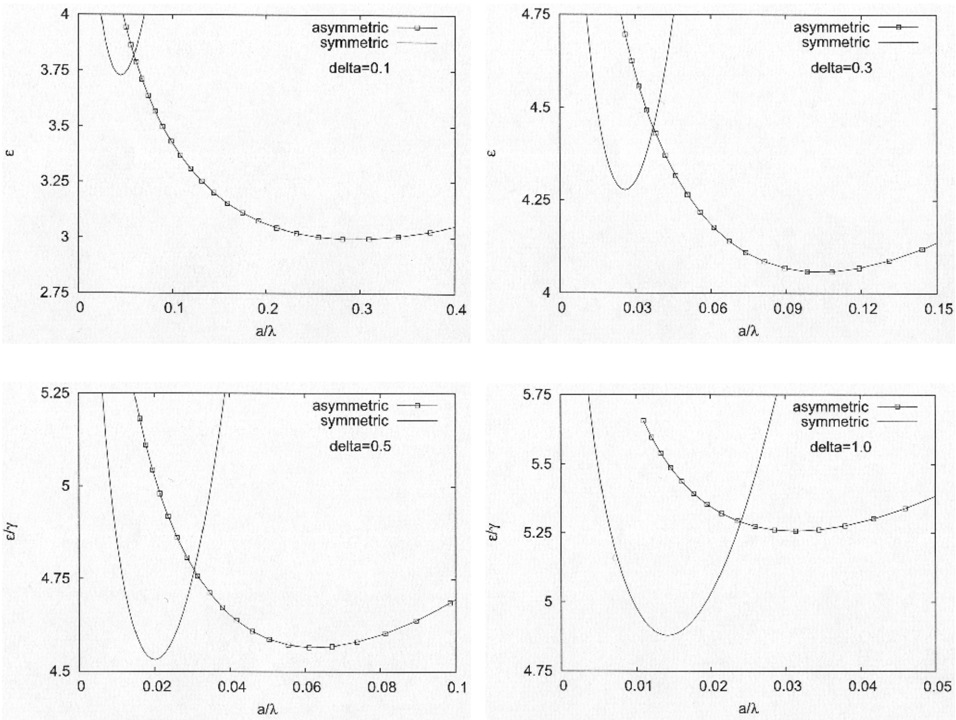

Fig. 3. The energies per unit length of the symmetric and asymmetric vortices as functions of a / for 0.1, 0.3, 0.5, and 1.0. All results were calculated using 100, as appropriate for the cuprates.

Within the usual Ginzburg-Landau approach, the penetration depth is not an independent quantity, but it rather depends on B_c and , as can be seen from Eq. 1. However, it should be pointed out that in a superconductor with strong fluctuations, the true penetration depth might be much larger than can be estimated from Eq. 1. In other words, the renormalization of and due to fluctuations can be very different. It is usually assumed that the stiffness of phase fluctuations is renormalized much more than the amplitude stiffness . In any case, it can be shown that the renormalization of does not change our conclusions and it therefore shall not be considered.

Within the London theory, we have assumed that $|f|$ 1 outside the vortex core and we have assumed that $|f|$ 0 inside the core. In absence of a competing order parameter in the core, the condensation energy density loss would be $B_c^2/(2\ _0)$. In presence of a competing order parameter, we need to generalize the second term in $\mathscr{F} = \mathscr{F}_{\text{mag}} + \mathscr{F}_{\text{cond}}$ by assuming the finite value of the vortex core order parameter m. For the sake of simplicity, we will assume that m is a scalar quantity. In what follows we construct generalized Ginzburg-Landau theories, in which both the uperconducting order parameter and a scalar order parameter m describing the vortex core state are taken into account.

If, in analogy with the SO(5) theory of superconductivity [8], we assume that the real and imaginary parts f_1 and f_2 of the complex superconducting field f f_1 if_2, together with the order parameter m, form three components of a vector order parameter $X \quad (f_1, f_2, m)$, then it is natural to write

$$\mathcal{F}_{\text{cond}} \quad \frac{B_c^2}{\quad_0} \quad |X|^2 \quad \frac{1}{2}|X|^4 \quad {}^2(\quad |X|)^2 \quad \frac{}{2}m^2 \quad , \tag{8}$$

where $|X|^2 \quad f_1^2 \quad f_2^2 \quad m^2$ and we have assumed a small (of the order) anisotropy in the X space, slightly favoring the superconducting solution. In this case we can choose $|f| \quad 1, \quad m^2 \quad 0$ outside and $|f| \quad 0, \quad m^2 \quad 1 \quad /2$ inside the vortex core. In other words, as one moves towards the vortex core, the order parameter X slightly contracts and rotates from the xy plane outside to the z direction inside the core. Note that for $\quad 1$ and if X is smooth on the scale , the gradient term can be safely neglected and the condensation energy density loss is only $\mathcal{F}_{\text{cond}} \quad B_c^2/(2 \quad_0) \quad B_c^2/(2 \quad_0)$, and our London-type analysis is qualitatively correct.

In the general case $\mathcal{F}_{\text{cond}}$ depends on many more parameters and lacks the symmetry of Eq. 8. In order to keep the number of independent coefficients as small as possible we assume that the only difference with respect to Eq. 8 has to do with the gradient terms. If we study the opposite extreme case

$$\mathcal{F}_{\text{cond}} \quad \frac{B_c^2}{\quad_0} \quad |X|^2 \quad \frac{1}{2}|X|^4 \quad {}^2(\quad |f|)^2 \quad {}^2(\quad m)^2 \quad \frac{}{2}m^2 \quad , \tag{9}$$

then the conclusions of our London-type analysis are not valid any more. In fact, let us consider the rectangular vortex core and let us assume that its dimensions are $2a \quad 2b$ where $b \quad a$. The optimal size b in the y direction has to be determined by minimization of . As an order of magnitude estimate we then obtain

$$2a \quad dy \quad \mathcal{F}_{cond} \quad \frac{B_c^2}{2 \quad_0} \sim \frac{2aB_c^2}{\quad_0} \quad b \quad \frac{\quad^2}{b} \quad .$$

The core energy is seen to be minimized for $b \sim \quad /\sqrt{\quad}$, much larger than $b \sim$ assumed in the London theory. If this correction is taken into account, then the asymmetric solution never becomes stable.

4. Conclusions

In conclusion, we have studied the shape of a superconducting vortex. We have assumed that the energy difference between the superconducting state and the non-superconducting state inside the vortex core is renormalized by a parameter . Within the London theory, we have shown that for $\quad_c \quad 0.46$, i.e. for sufficiently cheap vortices, the optimal superconducting vortex cores spontaneously break the rotational symmetry. It is worth pointing out that asymmetric vortices may have already been observed in STM studies of the vortex cores in the cuprates by the Geneva group [9, 10].

We have further studied the asymmetric vortices within the generalized Ginzburg-Landau theory, in which both the superconducting order parameter and a scalar order parameter m describing the vortex core state have been taken into account. We have shown that the stability of asymmetric vortices depends on the form of the gradient term: they are stable for the Ginzburg-Landau theory Eq. 8, whereas they are unstable for the Ginzburg-Landau theory Eq. 9 and the rotational symmetry of the vortices is restored.

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