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On the Compatibility of Nonholonomic Systems and Related Variational Systems*

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Abstract: I consider the compatibility of the equations of motion which follow from d'Alembert's principle in the case of a general autonomous non-holonomic mechanical system in *N* dimensions, with those equations which follow for the same system by assuming the validity of a specific variational action principle, in which the nonholonomic conditions are implemented by means of the multiplication rule in the calculus of variations. The equations of motion which follow from the principle of d'Alembert are not identical in form to the equations which follow from the variational action principle. I describe a recent proof, according to which the solutions to the equations of motion which follow from d'Alembert's principle do not in general satisfy the equations which follow from the action principle with nonholonomic constraints. This means that the d'Alembertian and variational systems are not compatible, except in the case of N = 2. My interest in the compatibility of the d'Alembertian system and the variational systems in question has its origin in an analysis of Yang-Mills theory made ten years ago, in which a gauge was used which is a natural generalisation of the Abelian Coulomb gauge for non-Abelian Yang-Mills theory.

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1. Introduction

Quite some time ago I was intensely preoccupied with formulating the dynamics of semi-classical Yang-Mills theory [1] using unconventional gauge conditions. One such gauge condition was the following *generalised Coulomb gauge condition*, involving both the gauge potential $A\ \ (x)$ as well as its time-derivative $\ \ (x)$,

$$\sum_{k\ 1}^{3}\ {}_{k}(A)\ \ ^{k}(x) : \sum_{k\ 1}^{3}\ (\ \ _{k}\ \ ^{k}(x)\ \ ig[A_{k}(x),\ \ ^{k}(x)])\ \ 0. \tag{1}$$

The gauge potential $A\ \ (x)$ takes values in an appropriate matrix representation of the Lie algebra of the chosen semi-simple compact gauge group. For further explanations and notation I refer to the paper [2], where it was shown that the condition (1) is indeed a gauge condition, in the sense that there exists a gauge transform by means of which a general gauge potential which does not necessarily satisfy the condition (1), is transformed into a gauge potential which satisfies this condition.

In the terminology of *Classical Mechanics*, the condition (1) is a nonholonomic condition. In implementing the nonholonomic gauge condition (1) in Yang-Mills theory, I used a procedure which had been advocated in a paper by Berezin [3], entitled (in translation

*) *Dedicated to Professor Milan Noga on the occasion of his 70th birthday.*

into English) "Hamiltonian Formalism in the General Lagrange Problem". In this paper, which was proposed as an alternative to the constraint theory of Dirac [4] for certain types of constrained systems, it was advocated that the constraints should be implemented by a variational procedure involving the multiplication rule in the *Calculus of Variations* [5], both in the cases of holonomic and nonholonomic constraints.

When I followed the procedure advocated by Berezin in Yang-Mills theory with the gauge condition (1), the results [6] were not very encouraging. It then occurred to me that one ought to examine the validity of the variational procedure proposed by Berezin for general systems with a finite number of degrees of freedom, and with nonholonomic constraints.

In classical mechanics, there exists an alternative to variational arguments, namely the fundamental *Principle of d'Alembert*. This principle leads to equations of motion for systems with both holonomic and nonholonomic constraints. I soon discovered that the equations of motion for a general autonomous system with a finite number of degrees of freedom, and with nonholonomic constraints linear in the generalised velocities, were not the same as those equations that followed by an application of the variational procedure advocated by Berezin. For holonomic constraints both the principle of d'Alembert and the variational procedure in question gave identical equations of motion. I then concluded that one should not use the variational procedure with nonholonomic constraints, and lost gradually interest in the generalised Coulomb gauge (1).

In the years following the experience described above I noticed from time to time the existence of papers in the field of classical mechanics, in which one advocated the use of variational procedures involving the multiplication rule in the calculus of variations for systems with nonholonomic constraints.

A fairly recent paper by Flannery [7] considers anew the question of applying the variational principle involving the multiplication rule to systems with a finite number of degrees of freedom, and reaches the conclusion that the equations of motion obtained by the variational principle are not identical to the equations which follow from the principle of d'Alembert in the case of nonholonomic constraints. The problems discussed by Flannery are not new; they have been discussed in the literature at least since Hertz's text-book [8], in which the use of variational principles in mechanics was questioned. We refer in particular to an early paper by Hölder [9], in which essential differences between holonomic and non-holonomic systems were discussed. Two later papers published by Jeffreys [10] and Pars [11] considered again the Hamilton's principle for non-holonomic systems, and proposed rectification of previous papers in which the variational procedure (action principle) involving the multiplication rule had been proposed for systems with non-holonomic constraints. One should, in this context, also recall the paper by Berezin [3], which has been mentioned previously.

Even though the equations of motion following from the principle of d'Alembert and from the variational action principle with non-holonomic constraints are different in form, one may still argue that the equations in question may have the same solutions. It has only very recently been proved [12] that this is not the case; the solutions to the d'Alembertian equations of motion and the variational equations of motion referred to above, are in general not coincident when the constraints are nonholonomic. The proof, which will be briefly discussed below, is valid for a general autonomous system with a finite number of degrees of freedom, restricted only by reasonable smoothness conditions. For simplicity only the case of one nonholonomic constraint is considered. This constraint is taken to be linear and homogeneous in the generalised velocities.

2. The d'Alembertian and variational equations of motion

Consider an autonomous mechanical system with independent generalised co-ordinates $q = (q^1, \ldots, q^N)$, and velocities $\dot q = (\dot q^1, \ldots, \dot q^N)$. We denote the kinetic energy of the system by T, and the generalised applied forces on the system by Q_A, $A = 1, \ldots, N$. It is further assumed that the system is constrained by one nonholonomic condition, which is linear and homogeneous in the generalised velocities. Thus, the constraint is of the form

$$\sum_{A=1}^{N} a_A(q)\,\dot q^A = 0, \tag{2}$$

where the functions $a_A(q)$, $A = 1, \ldots, N$ are arbitrary functions of the variables $q = (q^1, \ldots, q^N)$, except for the condition that not all of the quantities $a_A(q)$ vanish identically. Naturally, it is also assumed that the functions $a_A(q)$, $A = 1, \ldots, N$ satisfy appropriate smoothness conditions.

The principle of d'Alembert (see e.g. the classical texts by Goldstein [13] or Whittaker [14]) then gives the following equations of motion for the system in question,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q^A} - \frac{\partial T}{\partial q^A} = Q_A + \lambda\, a_A(q), \quad A = 1, \ldots, N, \tag{3}$$

where λ is a multiplier to be determined.

The N equations of motion above, are consequences of the principle of d'Alembert. One should still add the equations of constraint (2) to the equations of motion above. There are thus altogether $N + 1$ equations for the determination of N quantities $q^A(t)$, $A = 1, \ldots, N$, and the multiplier $\lambda(t)$, when appropriate boundary conditions for the quantities $q^1, \ldots, q^N$ and $\dot q^1, \ldots, \dot q^N$ are given.

It is now assumed that the external applied forces can be derived from a generalised potential. In what follows we thus assume the existence of a potential V such that

$$Q_A = -\frac{\partial V}{\partial q^A} + \frac{d}{dt}\frac{\partial V}{\partial \dot q^A}, \quad A = 1, \ldots, N. \tag{4}$$

Using the notation

$$L_0(q, \dot q) = T - V, \tag{5}$$

we rewrite the equations (3) as follows,

$$\frac{d}{dt}\frac{\partial L_0(q, \dot q)}{\partial \dot q^A} - \frac{\partial L_0(q, \dot q)}{\partial q^A} = \lambda\, a_A(q), \quad A = 1, \ldots, N. \tag{6}$$

The equations (6) are the d'Alembertian equations of motion, when it is assumed that the generalised forces are expressible in terms of a generalised potential V, as in Eq. (4) above.

The quantity $L_0(q, \dot q)$ defined above in Eq. (5) depends on the kinetic energy and on the external applied forces of the system under consideration. In the absence of nonholonomic constraints the quantity $L_0(q, \dot q)$ would be the Lagrangian of the system. Naturally, it is assumed that the functions T and V, respectively, satisfy appropriate smoothness conditions.

It should be observed that the principle of d'Alembert is not a straightforward variational principle although it involves so-called virtual displacements δq^A, $A = 1, \ldots, N$. As a consequence of the nonholonomic constraint (2) these virtual displacements have to satisfy the condition

$$\omega := \sum_{A=1}^{N} a_A(q)\, \delta q^A = 0\,. \tag{7}$$

We then consider the variational action principle referred to previously for the system defined above. This action principle involves the multiplication rule in the calculus of variation.

Consider the following action functional,

$$S_0 := \int dt\, L_0(q, \dot{q})\,. \tag{8}$$

The action principle in question is simply the requirement that the action functional (8) be stationary when the non-holonomic constraint (2) is in force throughout a suitable region D in configuration space, i.e. when the condition

$$\sum_{A=1}^{N} a_A\, \dot{q}^A = 0, \quad (q^1, \ldots, q^N) \in D \tag{9}$$

is in force. Using the multiplication rule in the calculus of variations, the action principle formulated above becomes equivalent to the following free variational problem,

$$\delta \int dt \left(L_0(q, \dot{q}) + \lambda \sum_{A=1}^{N} a_A\, \dot{q}^A \right) = 0\,, \tag{10}$$

which involves the Lagrange multiplier λ.

The variational equations following from Eq. (10) are,

$$\frac{d}{dt}\left(\frac{\partial L_0(q, \dot{q})}{\partial \dot{q}^A}\right) - \frac{\partial L_0(q, \dot{q})}{\partial q^A} = -\dot{\lambda}\, a_A(q) + \lambda \sum_{B=1}^{N} M_{AB}\, \dot{q}^B, \quad A = 1, \ldots, N. \tag{11}$$

where

$$M_{AB}(q) := \frac{\partial a_A(q)}{\partial q^B} - \frac{\partial a_B(q)}{\partial q^A}, \qquad A, B = 1, \ldots, N. \tag{12}$$

The N equations (11) together with the condition (2) are supposed to determine the quantities $q^1, \ldots, q^N$ and the Lagrange multiplier λ, when appropriate boundary conditions for $q^1, \ldots, q^N$ and $\dot{q}^1, \ldots, \dot{q}^N$ are given.

The variational equations (11) are not identical to the d'Alembertian equations of motion (6). The assumptions underlying the variational equations and the d'Alembertian equations of motion, respectively, are also basically different. The non-holonomic constraint (2) is only supposed to be valid for the actual motion when one applies the principle of d'Alembert to derive the equations of motion, whereas the same non-holonomic condition is supposed to be valid throughout a whole appropriate region D when one considers the variational problem (10), as indicated in Eq. (9) above.

If the one-form ω occurring in Eq. (7) is integrable, then the system under consideration is holonomic. This is the case if the following conditions hold true,

$$M_{AB} = 0, \quad A, B = 1, \ldots, N. \tag{13}$$

It is readily seen that the d’Alembertian equations of motion (6) and the variational equations of motion (11) become identical in this case, upon a simple change of notation: .

One reaches the same conclusion when the one-form in Eq. (7) can be made integrable by multiplication with an integrating factor. In two space dimensions there always exists an integrating factor. In what follows we thus consider systems of dimension $N \quad 3$. The necessary and sufficient conditions for the existence of an integrating factor, when $N \quad 3$, are as follows (see *e.g.* Ref. [15]),

$$a_A(q)M_{BC}(q) \quad a_B(q)M_{CA}(q) \quad a_C(q)M_{AB}(q) \quad 0, \quad A, B, C \quad 1, \ldots, N. \tag{14}$$

The conclusion to be drawn from this discussion is that the d’Alembertian and variational equations of motion are always equivalent in two space dimensions. These equations are equivalent when $N \quad 3$, if the constraint is (2) integrable, and therefore holonomic, or can be made integrable by means of an integrating factor.

It remains to consider the compatibility of the d’Alembertian equations of motion (6) and the variational equations of motion (11) in the case when the constraint (2) is truly nonholonomic, *i.e.* neither integrable nor reducible to the integrable case by means of an integrating factor.

3. Incompability of the d’Alembertian and variational equations of motion in the nonholonomic case

We are now concerned with a truly nonholonomic constraint of the form (2), which means that neither the Eqs. (13) nor the Eqs. (14) are in force.

We will prove that the d’Alembertian equations of motion (6) including the nonholonomic constraint (2), and the variational equations of motion (11), with the same constraint (2) included, do not in general have coincident solutions. To begin with, it will be assumed that the equations in question in fact do have sufficiently smooth solutions, e.g. C^2-solutions in some appropriate time interval.

The proof uses an argument of *reductio in absurdum*, *i.e.*, it is assumed that the equations of motion (6) and (11), respectively, do have coincident solutions, which are specified by appropriate general boundary conditions. It is then shown that this assumption leads to contradictions.

Assume that the equations (6) and (11) have coincident solutions. By subtracting Eqs. (11) from Eqs. (6), one obtains the following equations,

$$(\quad)a_A \quad \overset{N}{\underset{B\ 1}{}} M_{AB}(q)\ {}^{B}, \qquad A \quad 1, \ldots, N, \tag{15}$$

which have to be satisfied by the general solutions $(q^1, \ldots, q^N)$ of the d’Alembertian equations (6).

Let us first note that one must necessarily have $\quad 0$ in the Equations (15) above, since otherwise one would have

$$a_A \quad 0, \qquad A \quad 1, \ldots, N. \tag{16}$$

The condition (16) above implies that $\quad 0$, which cannot be true in the case of a general non-holonomic constraint. Hence, $\quad 0$.

We introduce the notation

$$: \quad \overline{} \, . \tag{17}$$

The conditions (15) are then equivalent to the following equations,

$$\sum_{B \quad 1}^{N} M_{AB}(q) \quad {}^{B} \qquad a_A, \qquad A \quad 1, \ldots, N, \tag{18}$$

which are inescapable consequences of the assumption that the equations (6) and (11) have general coincident solutions. It should be noted that the matrix (M_{AB}) occurring in Eqs. (18) is anti-symmetric (skew) and has real-valued matrix elements,

$$M_{AB} \qquad M_{BA}, \qquad A, B \quad 1, 2, \ldots, N. \tag{19}$$

The analysis of Eq. (18) is particularly simple in the three-dimensional case $N \quad 3$. We consider this case separately, and return to the cases $N \quad 3$ subsequently.

The case $N = 3$:

The quantity in the Eqs. (18) is unknown, but either zero or nonzero. We consider first the case

$$0 \, . \tag{20}$$

In this case Eqs. (18) are the following homogeneous equations

$$\sum_{B \quad 1}^{3} M_{AB}(q) \, z^B \quad 0, \qquad A \quad 1, 2, 3, \tag{21}$$

which (for $M \quad 0$) have the following general non-trivial solutions,

$$(z^1, z^2, z^3) \qquad (\quad {}_{23}, \quad {}_{31}, \quad {}_{12}), \tag{22}$$

where is a non-zero constant. The solution $(\; {}^1, \; {}^2, \; {}^3) \quad (z^1, z^2, z^3)$ must satisfy the non-holonomic constraint (2), *i.e.*

$$0 \quad a_1 z^1 \quad a_2 z^2 \quad a_3 z^3 \qquad [a_1 \quad {}_{23} \quad a_2 \quad {}_{31} \quad a_3 \quad {}_{12}] \, . \tag{23}$$

However, the condition (23) is nothing but the three-dimensional version of the necessary and sufficient conditions (14) for the existence of an integrating factor for the constraint (2). But this is a contradiction, since the constraint (2) is supposed to be truly nonholonomic.

We then consider the case

$$0 \, . \tag{24}$$

The equation to be considered is then the following,

$$\sum_{B \quad 1}^{3} M_{AB}(q) \quad {}^{B} \qquad a_A, \qquad A \quad 1, 2, 3 \, . \tag{25}$$

In order that Eqs. (25) be solvable, it is necessary that the vector (a_1, a_2, a_3) be orthogonal to the solutions (22) of the homogeneous equations (21). But this condition is nothing but the condition (23), which again contradicts the assumption that the constraint (2) is truly nonholonomic.

It has thus been shown that the d'Alembertian equations of motion (6) together with the nonholonomic constraint (2) are not in general compatible with the variational equations of motion (11) including the same nonholonomic constraint (2), in the case of $N \quad 3$.

The same conclusions are obtained in the general N-dimensional case, as was shown in a recent paper[1] made in collaboration with T. Raita [12], which was already referred to in the Introduction.

In some of the papers by Milan Noga and myself, such as Ref. [16] entitled "First-order Lagrangians and the Hamiltonian formalism" or Ref. [17] entitled "Multi- Hamiltonian structure of Lotka-Volterra and quantum Volterra models", we developed and used techniques which rely on properties of anti-symmetric matrices. The basic properties of anti-symmetric matrices are well-known in the mathematical literature, cf. *e.g.* the textbook by W. H. Greub [18]. The techniques in question were also used in Ref. [12], in which the general proof in the N-dimensional cases was first given.

The proof in the general N-dimensional cases of the incompatibility of the d'Alembertian equations (6) and and the variational equations (11) requires a discussion of the boundary conditions used for the solutions of the equations of motion. We will now consider Equations (6) and (11) as initial value problems, with solutions specified by the following initial values for the co-ordinates q^A and the velocities $\ ^A,\ A \quad 1, \ldots, N$,

$$[q^A(t)]_{t \quad t_0} \quad q_0^A, \quad [\ ^A(t)]_{t \quad t_0} \quad {}_0^A, \quad A \quad 1, \ldots, N. \tag{26}$$

The initial values $q^A(t)$ and $\ ^A(t)$ at $t \quad t_0$ are free parameters within an appropriate region of the configuration- and velocity space, except for the restriction

$$\sum_{A \quad 1}^{N} a_A(q_0) \ {}_0^A \quad 0. \tag{27}$$

The condition (27) is a consequence of the non-holonomic constraint (2).

We now return to the conditions (18) in the general N-dimensional cases. The solutions $\ ^A,\ A \quad 1, \ldots, N$, of the algebraic equations (18) can be obtained in a fairly explicit form, by using known properties of anti-symmetric matrices, cf. *e.g.* [16], [17] or [18]. In particular, it is known that any anti-symmetric matrix M with real-valued matrix elements is equivalent to a matrix of the following normal form,

$$M \sim \begin{array}{cccccccc} 0 & {}_1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ {}_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & 0 & {}_p & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & {}_p & 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{array}, \tag{28}$$

where the quantities $\ ,\ \quad 1, 2, \ldots, p$, are positive, and where p is a certain integer (which depends on the matrix M) in the range

$$2 \quad 2p \quad N. \tag{29}$$

The even integer $2p$ is the rank of the matrix M.

The matrix M has the normal form (28) in a basis $\{b \ \}_1^N$, which satisfies the following equations,

[1]This paper regrettably contains a few misprints. The equations (17), (26) and (34) in Ref. [12] contain the redundant symbols = 0 at the end of the equations.

$$\sum_{B\ 1}^{N} M_{AB} b_{2\ \ 1,B} \qquad b_{2\ ,} \ ; \qquad \sum_{B\ 1}^{N} M_{AB} b_{2\ ,B} \qquad b_{2\ \ 1,} \ , \qquad 1, \ldots, p, \tag{30}$$

and, if $2p < N$,

$$\sum_{B\ 1}^{N} M_{AB} b_{\ ,B} \quad 0, \qquad 2p \quad 1, \ldots, N. \tag{31}$$

The basis vectors are ortho-normalized in the following inner product,

$$(b\ , b\ ): \sum_{A\ 1}^{N} b_{\ ,} \ b_{\ ,} \ : \begin{matrix} 1 & \text{if} \\ 0 & \text{otherwise}. \end{matrix} \tag{32}$$

We now return to the Equation (18), which was shown above to be a necessary consequence of the assumption that Eqs. (6) and (11), respectively, have coincident general solutions. It is again convenient to consider separately the case 0 and 0.

The case 0:

When 0, Eqs. (18) read as follows,

$$\sum_{B\ 1}^{N} M_{AB}(q) \ ^{B} \quad 0, \qquad A \quad 1, \ldots, N, \tag{33}$$

We first consider the case when the matrix M is regular, *i.e.* $\det(M_{AB}(q))$ 0. This can happen only if the rank $2p$ of M equals N, in which case N is an even integer. The only solutions of Eqs. (33) are then the following,

$$^{A} \quad 0, \qquad A \quad 1, \ldots, N. \tag{34}$$

However, the solutions (34) are not possible, since they are not consistent with the general initial value conditions (26) and (27).

Let then the rank $2p$ of M be less than N. The general solution of Eqs. (33) is in this case a linear combination of the N $2p$ basis vectors b , $2p$ 1, …, N, *i.e.*,

$$^{A} \quad \sum_{2p\ \ 1}^{N} b_{\ ,A}, \quad A \quad 1, \ldots N, \tag{35}$$

where the quantities , $2p$ 1, …, N, are free parameters.

It then follows from the orthogonality conditions (32) that the solutions (35) are orthogonal to the basis vectors b when 1, …, $2p$, *i.e.*,

$$(b\ , \ ) \quad \sum_{A\ 1}^{N} b \quad ^{A} \quad 0, \qquad 1, \ldots, 2p. \tag{36}$$

The conditions (36) must in particular also be satisfied for the initial values of q^{A} and ^{A}, A 1, …, N, at t t_0, *i.e.*,

$$\sum_{A\ 1}^{N} [b \quad ^{A}]_{t\ t_0} \quad 0, \qquad 1, \ldots, 2p. \tag{37}$$

Since the number $2p$ is in the range (2, …, N 1), the number of conditions in (37) is at most N 1, but at least 2. Thus the solutions (35) are not possible, since they involve at least two special conditions of the form (37) on the $2N$ initial values q_0^A and $\ _0^A$, $A\ \ 1, \ldots, N$. This is not consistent with the general initial value conditions (26) and (27).

We have now shown that the case $\ \ 0$ leads to contradictions in any N-dimensional configuration space of dimension $N\ \ 3$. It remains to consider the case $\ \ 0$.

The case $\ \ 0$:

We return to the Equations (18), where $\ \ 0$,

$$\sum_{B\ 1}^{N} M_{AB}(q)\ ^{B} \qquad a_A, \qquad 0, \quad A \quad 1, \ldots, N. \tag{38}$$

The general solution $(\ ^1, \ldots, \ ^N)$ of Eqs. (38) can be written as a linear combination of the basis vectors $b\ $, $\ \ 1, \ldots, N$,

$$^A \quad \sum_{1}^{p} \ b_{2\ \ 1,A} \quad \sum_{1}^{p} \ b_{2\ ,A} \quad \sum_{2p\ 1}^{N} \ b_{\ ,A}, \quad A \quad 1, \ldots, N, \tag{39}$$

with the understanding that the last sum in Eq. (39) is absent if $N = 2p$, in which case the matrix M is regular.

If $2p < N$, it is necessary for the existence of solutions of Eqs. (38) that the quantity $(a_1, \ldots, a_N)$ be orthogonal to all the solutions of the homogeneous equations (31), *i.e.*,

$$(b\ , a) \quad 0, \qquad 2p \quad 1, \ldots, N. \tag{40}$$

The quantities $\ $ and $\ $, $\ \ 1, \ldots, p$, are determined by multiplying the equations (38) from the left with the basis vector components $b_{2\ \ ,}$ and $b_{2\ \ 1,}$, respectively, where $\ $ is the fixed index in the range $(1, \ldots, p)$, and then by summing over A in the range $(1, \ldots, N)$.

Using the anti-symmetry of the matrix elements M_{AB} and Equations (30), one obtains

$$(b_{2\ \ 1},\ ) \qquad ^1(b_{2\ }, a), \qquad 1, \ldots, p. \tag{41}$$

and

$$(b_{2\ },\ ) \qquad ^1(b_{2\ \ 1}, a), \qquad 1, \ldots, p. \tag{42}$$

Inserting the expressions (41) for the quantities $\ $ and the expressions (42) for the quantities $\ $ in the expression (39) one obtains

$$^A \quad \sum_{1}^{p} \ ^1[(b_{2\ }, a) b_{2\ \ 1,} \quad (b_{2\ \ 1}, a) b_{2\ ,}\ ] \quad \sum_{2p\ 1}^{N} \ b_{\ ,A}, \quad A \quad 1, \ldots, N. \tag{43}$$

It has already been shown above that the d'Alembertian equations of motion (6), together with the nonholonomic constraint (2), are not in general compatible with the variational equations of motion (11) including the same nonholonomic constraint (2), in the case of $N = 3$. Thus, we consider in what follows the cases $N\ \ 4$. It is convenient to consider the cases $p\ \ 1$ and $p\ \ 2$ separately.

The cases $N\ \ 4$, $p\ \ 2$:

We consider the expressions (43), which in particular are also valid at the initial time $t\ \ t_0$,

$$[\ ^{A}]_{t\ t_0}\quad (t_0)\ \sum_{1}^{p}\ ^{1}[(b_{2}\ ,a)b_{2\ \ 1,}\quad (b_{2\ \ 1},a)b_{2\ \ ,}\ ]_{t\ t_0}\quad \sum_{2p\ \ 1}^{N}\ [b_{\ ,A}]_{t\ t_0},\ A\ \ 1,\ldots,N. \quad (44)$$

The quantities $[\quad]_{t\ t_0}$, the quantities $[a_A]_{t\ t_0}$, and the quantities $[b_{\ ,A}]_{t\ t_0}$, are all determined by the initial values q_0^A of the co-ordinates $q^A(t)$, $A\ \ 1,\ldots,N$, at the time $t\ \ t_0$. Thus there are $N\ \ 2p$ free parameters (t_0), $2p\ \ 1,\ldots,N$, and one free parameter (t_0), in the expressions (44), *i.e.* all in all $N\ \ 2p\ \ 1$ free parameters. Since $p\ \ 2$, there are at most $N\ \ 3$ free parameters in the expressions (44), for the initial velocities $^A(t_0)$, $A\ \ 1,\ldots,N$. However, the initial velocities $^A(t_0)$, $A\ \ 1,\ldots,N$, are supposed to be specified by $N\ \ 1$ freely chosen initia values, *i.e.* the N initial velocity components $^A(t_0)$, $A\ \ 1,\ldots,N$, restricted by one condition, namely the condition (27), which is a consequence of the nonholonomic constraint (2). This is in contradiction with the fact that there are at most $N\ \ 3$ free parameters available in the expression (44).

It has thus been shown that the d'Alembertian equations of motion (6) together with the nonholonomic constraint (2) and the variational equations of motion (11) including the same nonholonomic constraint (2), do not have coincident solutions in general in the cases $N\ \ 4$, $p\ \ 2$.

It remains to consider the situation, in which $p = 1$.

The exceptional cases $N\ \ 4$, $p\ \ 1$:

For $p = 1$, the expressions (44) become as follows,

$$^{A}\quad ^{1}[(b_2,a)b_{1,}\quad (b_1,a)b_{2,}\ ]\quad \sum_{3}^{N}\ b_{\ ,A},\quad A\ \ 1,\ldots,N. \qquad (45)$$

The solutions (45) are valid only if the following conditions hold true,

$$(b\ ,a)\ \ 0,\qquad 3,\ldots,N. \qquad (46)$$

Thus, in the basis $b\ $, $\ 1,\ldots,N$, there are only two non-vanishing components of the quantity $(a_1,\ldots,a_N)$, which defines the nonholonomic constraint (2). Likewise, for $p\ \ 1$, the matrix M given by the matrix elements (M_{AB}) defined in Eqs. (12), becomes similar to an essentially 2 2 matrix,

$$M \sim \begin{matrix} 0 & _{1} & \cdots & \cdots & \cdots \\ _{1} & 0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & 0 & \vdots & \vdots \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 0 \end{matrix}\ . \qquad (47)$$

Thus, for $p\ \ 1$ the problem is essentially two-dimensional. Now the number of free parameters in the expression (45) at $t\ \ t_0$ coincides with the number of freely available initial conditions $^A(t_0)$. Hence the expression (45) does not lead to any contradiction concerning the choices of initial values for the velocity components ^{A}, $A\ \ 1,\ldots,N$.

It has already been noted above, that the d'Alembertian equations of motion (6) together with the nonholonomic constraint (2) are compatible with the variational equations of motion (11) including the same nonholonomic constraint (2), in the strictly two-dimensional case $N\;\;2$. In the present case with $p\;\;1$, the problem becomes essentially two-dimensional, as indicated by Equations (46) and (47).

It is conjectured that the essentially two-dimensional cases with $p\;\;1$ and $N\;\;4$ can be shown to be equivalent to the strictly two-dimensional case $N\;\;2$, for which the d'Alembertian and variational equations under consideration always can be made compatible, by using an appropriate integrating factor. Naturally, I will do my best to verify this conjecture in the near future.

4. Summary and discussion

I have presented a somewhat simplified and improved version of the proof in Ref. [12], which demonstrates that the d'Alembertian equations of motion for fairly general systems with non-holonomic constraints, in three or more dimensions, are not compatible with the corresponding variational equations with the constraints implemented by the multiplication rule in the calculus of variations. The proof of incompatibility breaks down in certain exceptional cases, which however, are essentially two-dimensional.

The incompatibility (or non-equivalence) means two things, namely that the equations of motion in question are not identical in form, and also that they do not have coincident general solutions.

The variational action principle discussed here, which has been proposed several times in the literature for systems with non-holonomic constraints, as an extension of the similar action principle which is valid for systems with holonomic constraints, is thus not in general consistent with the principle of d'Alembert, when the constraints are non-holonomic.

The fact that the variational equations of motion discussed here, for a fairly general system with unrestricted nonholonomic constraints, are not compatible with the corresponding d'Alembertian equations of motion describing the same constrained system, should hopefully put an end to any new attempts at proposing such variational equations for nonholonomic systems which have been described above, in which the constraints are implemented by means of the multiplication rule in the calculus of variation.

The proof discussed here may still seem a bit involved, but at least the analysis of the three-dimensional case is very transparent and simple, and as such suitable in any undergraduate course on analytical mechanics, for instance on the level of the well known classical text-book by Goldstein [13].

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