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Arrival Time and Uncertainty Relation between Time and Energy

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Abstract: We present the derivation of the uncertainty relation between the arrival time registered by a detector and the part of the mean energy of a particle corresponding to the detector volume.

1. Introduction

In our previous paper [1] we proposed an expression for the calculation of the arrival time. We considered a (non-relativistic) particle which was created at time t_0 and described by the wave packet $(\vec{x} \quad \vec{x}_0)$. Subsequently we looked for the quantity $P_D(\langle t_0, t\rangle)$ which stands for the probability that particle in question entered the detector (having volume V_D and defining the space angle $_D$ with respect to the point $\vec{x}_0$) during the time interval $\langle t_0, t\rangle$. We considered two events E_1, E_2. The event E_1 means that a particle in question has momentum $\vec{p} \quad _D$ (it expresses that the half-line $\vec{x} \quad \vec{x}_0 \quad k\vec{p}, k \quad (0, \quad)$ passes through the detector). The event E_2 can be formally written as

$$E_2 \quad \bigcup_{t \quad \langle t_0, t\rangle} E_2(t),$$

where $E_2(t \quad)$ means that a particle was situated in the detector at time t. Now the probability $P_D(\langle t_0, t\rangle)$ that a particle entered the detector during the time interval $\langle t_0, t\rangle$ is given by

$$P_D(\langle t_0, t\rangle) \quad P(E_2 \cap E_1) \quad P(E_2|E_1)P(E_1). \tag{1}$$

The probability $P(E_1)$ means that a particle has momentum $\vec{p} \quad _D$ and we can simply write

$$P(E_1) \quad _D d \quad (\vec{n} \quad _0 p^2 dp\,|C(p\vec{n})|^2. \tag{2}$$

Where $C(p\vec{n}) \quad C(\vec{p}), \quad \vec{n} \quad \frac{\vec{p}}{|\vec{p}|}$ is given by

$$(x \quad x_0 \quad \frac{d^3p}{(2 \quad)^{3/2}} C(\vec{p}) e^{i\vec{p} \quad \vec{x} \quad \vec{x}_0}.$$

We proposed the following expression for $P(E_2|E_1)$:

$$P(E_2|E_1) \quad \frac{{}^{t}_{t_0} dt \quad {}_{V_D} d^3\vec{x}| \quad {}_D(\vec{x},t)|^2}{{}_{t_0} dt \quad {}_{V_D} d^3\vec{x}| \quad {}_D(\vec{x},t)|^2}, \tag{3}$$

Where

$${}_D(\vec{x},t) \quad {}_D d \quad (\vec{n}) \quad {}^{\vec{n}}(\vec{x},t), \tag{4}$$

and $\quad {}^{\vec{n}}(\vec{x},t)$ is defined by

$$\vec{x},t \quad e^{iH \ t \ t_0} \quad \vec{x} \quad \vec{x}_0 \qquad \frac{d^3\vec{p}}{(2 \ )^{3/2}} C(\vec{p}) e^{iE \ t \ t_0 \quad i\vec{p} \ \vec{x} \ \vec{x}_0}$$

$$d \quad (\vec{n}) \ {}_0 \frac{p^2 dp}{(2 \ )^{3/2}} C(p\vec{n}) e^{iE \ t \ t_0 \quad ip\vec{n} \ \vec{x} \ \vec{x}_0} \qquad d \quad (\vec{n}) \quad {}^{\vec{n}}(\vec{x},t).$$

The quantity $P(E_2|E_1)$ means that if E_1 set in then also E_2 set in with the probability $P(E_2|E_1)$. On account of that we expressed $P(E_2|E_1)$ through the $\quad {}_D$ because if E_1 set in, then the wave function of a considered particle is $\quad {}_D$.

If we reduce the detector to the point, say overrightarrow $\vec{x}_D$, then

$$P(E_2|E_1) \quad \frac{{}^{t}_{t_0} dt \left| \quad {}^{\vec{n}_D}(\vec{x}_D,t)\right|^2}{{}_{t_0} dt \left| \quad {}^{\vec{n}_D}(\vec{x}_D,t)\right|^2}, \tag{5}$$

where

$$\vec{n}_D \quad \frac{\vec{x}_D \quad \vec{x}_0}{|\vec{x}_D \quad \vec{x}_0|}$$

The quantity

$$\frac{dP_D(\langle t_0,t\rangle)}{dt} \sim \left| \quad {}^{\vec{n}_D}(\vec{x}_D,t)\right|^2 \tag{6}$$

can be interpreted as the density of probability (with respect to t) that the considered particle entered the detector at time t.

Now we can define the averaged value of the time widehat $\hat{T} \quad t \quad t_0$ (arrival-time) by

$$\langle T\rangle_D \quad \frac{{}_{t_0} dt(t \quad t_0)\left| \quad {}^{\vec{n}_D}(\vec{x}_D,t)\right|^2}{{}_{t_0} dt\left| \quad {}^{\vec{n}_D}(\vec{x}_D,t)\right|^2}. \tag{7}$$

The formula (6) defines the probability distribution for the arrival time T and offers the possibility to derive the uncertainty relations between $\hat{T}$ and $\hat{E}$, where $\hat{E}$ is an operator satisfying $[\hat{T},\hat{E}] \quad i$. One intuitively feels that $\hat{E}$ is the operator of the energy. But what energy? We shall analyse this question in the next section.

2. The Derivation of the Uncertainty Relation between Time and Energy

Let us now consider the quantity

$$\langle E\rangle_D = \frac{\frac{1}{2}\int_{t_0}^{\infty} dt \int_{V_D} d^3\vec{x}\,[\psi_D^*(\hat{H}\psi_D) + (\hat{H}\psi_D)^*\psi_D]}{\int_{t_0}^{\infty} dt \int_{V_D} d^3\vec{x}\,|\psi_D|^2}.$$

This quantity can be interpreted as averaged value of the energy contained in the detector (we would like to note that it is averaged over the time, too). Using the equation

$$i\partial_t\psi_D = \hat{H}\psi_D,$$

then

$$\langle E\rangle_D = \frac{1}{2A}\int_{t_0}^{\infty} dt \int_{V_D} d^3\vec{x}\,\left(2\psi_D^* i\partial_t\psi_D\right) + \frac{i}{2}[\psi_D^*\psi_D]_{t_0},$$

where

$$A = \int_{t_0}^{\infty} dt \int_{V_D} d^3\vec{x}\,|\psi_D|^2.$$

The wave packet $\psi(\vec{x}-\vec{x}_0)$ is being smudged and for $t\to\infty$ the function ψ_D will become equal to zero in V_D. Taking into consideration the following relations:

$$\int_{t_0}^{\infty} dt\,\delta(t-t_0)f(t) = \int_{-\infty}^{\infty} dt\,\delta(t-t_0)\,\theta(t-t_0)f(t) = \frac{1}{2}\int_{-\infty}^{\infty} dt\,f(t)\frac{d}{dt}\theta^2(t-t_0) = \frac{1}{2}f(t_0),$$

where $\theta(t-t_0)$ is the step function, then one receives for averaged energy

$$\langle E\rangle_D = \frac{1}{A}\int_{t_0}^{\infty} dt\,f(t)\int_{V_D} d^3\vec{x}\,\psi_D^*[i\partial_t + i\delta(t-t_0)]\psi_D. \tag{8}$$

At the limit $V_D\to 0$ the right hand side of (8) reduces to

$$\langle E\rangle_D = \frac{\int_{t_0}^{\infty} dt\,\psi^{(\vec{n}_D)^*}(\vec{x}_D,t)\hat{E}\psi^{(\vec{n}_D)}(\vec{x}_D,t)}{\int_{t_0}^{\infty} dt\left|\psi^{(\vec{n}_D)}(\vec{x}_D,t)\right|^2}, \tag{9}$$

where the operator $\hat{E} = i\frac{d}{dt} + i\delta(t-t_0)$ and the commutator between $\hat{T}$ and $\hat{E}$ is $[\hat{T},\hat{E}] = -i$.

Now we can write $\Delta T\Delta E \geq 1/2$, where

$$\Delta T = \sqrt{\langle\hat{T}^2\rangle_D - \langle\hat{T}\rangle_D^2}$$

$$\Delta E = \sqrt{\langle\hat{E}^2\rangle_D - \langle\hat{E}\rangle_D^2}$$

3. Concluding Remarks

There exist a lot of approaches to the problem (see e.g. [2 5]). None of them seems to be generally accepted [5]. As for our considerations on the theme it is evident that our proposal of the solution of the above mentioned problem is related to the very specific case and difficulties may appear in other realistic cases. We also feel that some ideas and approaches used in our paper require more detailed and precise formulations and additional critical analyses.

References

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