Contextualized Knowledge Repositories for the Semantic Web

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Abstract. We propose *Contextualized Knowledge Repository* (CKR): an adaptation of one of the well studied logics of context in AI for the Semantic Web. A CKR is composed of a set of OWL 2 knowledge bases, which are embedded in a *context* by a set of qualifying attributes (time, space, topic, etc.) specifying the boundaries within which the knowledge base is assumed to be true. Contexts of a CKR are organized by a hierarchical *coverage* relation, which enables an effective representation of knowledge and a flexible method for its reuse between the contexts. The paper defines syntax and semantics for CKR; shows that concept satisfiability and subsumption are decidable with the complexity upper bound of 2NEXPTIME, and finally it provides a sound and complete Natural Deduction calculus for CKR.

1 Introduction

Despite most of the information available in the Semantic Web (SW) in form of OWL ontologies or RDF linked data is context dependent, there is a lack of a widely accepted mechanism to qualify knowledge with the context in which it is supposed to hold. In the current practice, contextual information is crafted in the ontology identifier or in attributes like rdfs:comment, owl: AnnotationProperty which do not affect reasoning. Extensions of SW languages with specific qualification mechanisms were proposed, e.g., [8] propose a way to qualify knowledge with its provenance; [17] propose qualification w.r.t. time and events. Also [10, 21, 14] suggest some directions of investigation. However, none of these works provide a comprehensive, well developed and sufficiently general approach. In order to clarify the representational requirements for a contextual representation framework for the SW let us consider the following scenario.

Suppose we want to represent knowledge about Football (FB), FIFA world cups (FWC), national football leagues (NFL), worldwide news (WN), and national news (NN). Suppose also that all the information about FWC and NFL should be included in FB, and that for each nation, all the facts about its NFL should be included in its NN, and also all the information about FWC should be included in WN. On the other hand, only a part of information about NFL should be included in WN (only that of worldwide interest). Suppose also that we are interested in qualifying all the facts with the time and geographical region. The contextual formalisms we are designing should support the following representational requirements:

contextually bounded facts: in each context we should be able to state facts with local effect that do not necessarily propagate everywhere, e.g., an axiom like "a player is a member of only one team" should be true in some contexts (e.g., FWC, NFL, for each year) but not in more general contexts like FB;

reuse/lifting of facts: be able to include "automatically" all the information contained in more specific contexts. For example, facts in FWC should be lifted up into the WN, and FB. This lifting should be done without spoiling locality of knowledge;

overlapping and varying domains: objects can be present in multiple contexts, but not necessarily in all contexts, e.g., a player can exist in both the FWC context and in the NFL contexts, but many players present in NFL will not be present in FWC;

inconsistency tolerance: two contexts may possibly contain contradicting facts. For instance NN of Italy could assert that "Cassano is the best player of the world", while at the same time the world news report that "Messi is the best player of the world", without making the whole system inconsistent;

complexity invariance: the qualification of knowledge by context should not increase the complexity.

Instead of extending the current SW languages, we propose a shift of approach: to adapt one of the well studied AI theories of context [18, 9] to the current SW setting. We propose the notion of *Contextualized Knowledge Repository* (CKR), a logical framework that allows to qualify knowledge represented in OWL 2 [24] with contextual information. A CKR is composed of a set of OWL 2 knowledge bases, called *contexts*, which are qualified by a set of attributes (e.g., time, space, topic, etc.) that specify the boundaries within which the knowledge base is assumed to be true. Contexts are organized by a hierarchical *coverage* relation that regulates the propagation of knowledge between them. The paper defines syntax and semantics for CKR; shows that concept satisfiability and subsumption are decidable with the complexity upper bound of 2NEXPTIME (i.e., same as for OWL 2); and finally it provides a sound and complete Natural Deduction calculus for CKR.

2 Preliminaries

The CKR framework is built on top of the $\mathcal{SROIQ}$ DL [11] which us used as the local languages of contexts. This language constitutes the logical foundation of OWL 2 [24] and it is currently the most expressive language relevant to the Semantic Web. In this section, we briefly introduce the necessary DL preliminaries. For more details the reader is referred to [11, 1]. Although semantically we are able to handle full $\mathcal{SROIQ}$ DL inside CKR, in order to achieve decidability of reasoning we will slightly limit its expressive power as we shall see below.

A DL vocabulary $\Sigma = N_{\mathrm{C}} \uplus N_{\mathrm{R}} \uplus N_{\mathrm{I}}$ is a set of symbols composed of three mutually disjoint countably infinite subsets: the set N_{C} of atomic concepts including the top concept $\top$ and the bottom concept $\bot$, the set N_{R} of atomic roles including the universal role U and the identity role I, and the set N_{I} of individuals.

Complex concepts (complex roles) are recursively defined as the smallest sets containing all concepts and roles that can be inductively constructed using the concept (role) constructors in

Table 1, where A is any atomic concept, C and D are any concepts, R is any atomic role, S and Q are any (possibly complex) roles, a and b are any individuals, and n stands for any positive integer.

A $\mathcal{SROIQ}$ knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \rangle$ consists of a TBox $\mathcal{T}$ which contains GCI axioms, an RBox $\mathcal{R}$ which contains RIA axioms, and an ABox $\mathcal{A}$ which contains assertions. Syntax of all axioms is shown at the bottom of Table 1. The closure $\sqsubseteq^*_{\mathcal{R}}$ of the RBox $\mathcal{R}$ is defined as follows: if $R \sqsubseteq Q \in \mathcal{R}$ and $Q_1 \circ Q \circ Q_2 \sqsubseteq S \in \mathcal{R}$ then $Q_1 \circ R \circ Q_2 \sqsubseteq_{\mathcal{R}} S$; and $\sqsubseteq^*_R$ is transitive and reflexive closure on $\sqsubseteq_{\mathcal{R}}$.

Concept constructors	Syntax	Semantics
atomic concept	A	$A^{\mathcal{I}}$
complement	$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
intersection	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
existential restriction	$\exists R.C$	$\{x \in \Delta^{\mathcal{I}} \mid \exists y \, \langle x,y \rangle \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$
self restriction	$\exists R.\mathrm{Self}$	$\{x \in \Delta^{\mathcal{I}} \mid \langle x,x \rangle \in R^{\mathcal{I}}\}$
min cardinality restriction	$\geqslant n R.C$	$\{x \in \Delta^{\mathcal{I}} \mid \sharp\{y \vert \, \langle x,y \rangle \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\} \geq n\}$
nominal	{a}	$\{a^{\mathcal{I}}\}$
Role constructors	Syntax	Semantics
atomic role	R	$R^{\mathcal{I}}$
inverse role	R^-	$\{\langle y,x \rangle \mid \langle x,y \rangle \in R^{\mathcal{I}}\}$
role composition	$S \circ Q$	$S^{\mathcal{I}} \circ Q^{\mathcal{I}}$
Axioms	Syntax	Semantics
concept inclusion (GCI)	$C \sqsubseteq D$	$C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$
role inclusion (RIA)	$S \sqsubseteq R$	$S^{\mathcal{I}} \subseteq R^{\mathcal{I}}$
reflexivity assertion	$\mathrm{Ref}(R)$	$R^{\mathcal{I}}$ is reflexive
role disjointness	$\mathrm{Dis}(R,S)$	$R^{\mathcal{I}} \cap S^{\mathcal{I}} = \emptyset$
concept assertion	$C(a)$	$a^{\mathcal{I}} \in C^{\mathcal{I}}$
role assertion	$R(a,b)$	$\langle a^{\mathcal{I}}, b^{I} \rangle \in R^{\mathcal{I}}$
negated role assertion	$\neg R(a,b)$	$\langle a^{\mathcal{I}}, b^{I} \rangle \notin R^{\mathcal{I}}$

Table 1. Syntax and Semantics of $\mathcal{SROIQ}$

A DL interpretation is a pair $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ where $\Delta^{\mathcal{I}}$ is a set called interpretation domain and $\cdot^{\mathcal{I}}$ is the interpretation function which provides denotations for individuals, concepts and roles. In $\mathcal{SROIQ}$, as much as in any classical DL, $\Delta^{\mathcal{I}}$ is required to be non-empty. We will see later on that in CKR we will relax from this requirement. The interpretation function $\cdot^{\mathcal{I}}$ assigns an element $a^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$ to each individual a, a subset $C^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$ to each concept C, and a subset $R^{\mathcal{I}}$ of the product $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ to each role R. In addition for any complex concept and role the respective semantic constraint listed in Table 1 must be satisfied by $\cdot^{\mathcal{I}}$, plus $\top^{\mathcal{I}} = \Delta^{\mathcal{I}}$, $\bot^{\mathcal{I}} = \emptyset$, $U^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, and $I^{\mathcal{I}} = \{\langle x,x \rangle \mid x \in \Delta^{\mathcal{I}}\}$[3]. An axiom ϕ is satisfied by an interpretation $\mathcal{I}$ (denoted $\mathcal{I} \models_{\mathrm{DL}} \phi$) if $\mathcal{I}$ satisfies the respective semantic constraint listed in Table 1. An interpretation $\mathcal{I}$ is a model of $\mathcal{K}$ (denoted $\mathcal{I} \models_{\mathrm{DL}} \mathcal{K}$) if it satisfies all axioms of $\mathcal{K}$.

Only simple roles are allowed in the min cardinality restriction, in the self restriction constructors, as well as in the reflexivity and the disjointness axioms. Simple roles are defined recursively

[3] The identity role I is not originally part of $\mathcal{SROIQ}$ [11], however it can be easily introduced as syntactic sugar by adding the axioms $\top \sqsubseteq \exists I.\mathrm{Self}$ and $\top \sqsubseteq \neg \geqslant 2 I.\top$.

as follows: a) atomic role is simple if it does not occur on the right-hand side of a RIA in $\mathcal{R}$; b) an inverse role R^- is simple if R is simple; c) if R occurs on the right-hand side of a RIA in $\mathcal{R}$ and each such RIA is of the from $S \sqsubseteq R$ where S is a simple role, than R is also simple. Also the universal role U is not allowed on the left-hand side of RIA axioms.

There are additional $\mathcal{SROIQ}$ constructors and axioms [11]. Specifically, concept constructors $C \sqcup D$, $\forall R.C$, $\leqslant nR.C$ and $=nR.C$, and RBox axioms $\mathrm{Sym}(R)$, $\mathrm{Tra}(R)$, $\mathrm{Irr}(R)$. Although we occasionally use some of them to simplify the notation, they are all fully reducible[4] into the core constructs listed in Table 1 which allows us to leave them out when laying out the theoretical foundation of CKR. Note that also $\mathrm{Ref}(R)$ is reducible, but only in cases when R is simple (i.e., by replacing it with $\top \sqsubseteq \exists R.\mathrm{Self}$).

Two basic reasoning tasks for $\mathcal{SROIQ}$ and for any DL are concept satisfiability, the task to decide for a given (possibly complex) concept C whether there is a model $\mathcal{I}$ of $\mathcal{K}$ such that $C^{\mathcal{I}}$ is nonempty; and entailment, the task to decide for some axiom ϕ whether $\mathcal{I} \models_{\mathrm{DL}} \phi$ for all models $\mathcal{I}$ of $\mathcal{K}$ (which is then denoted by $\mathcal{K} \models_{\mathrm{DL}} \phi$). These two tasks are known to be inter-reducible [1, 11]. A tableaux algorithm that decides these two tasks for $\mathcal{SROIQ}$ was given in [11]. The algorithm, however, requires further syntactic restrictions on the language, as discussed below.

In addition, it is required that the RBox is regular, defined as follows [11]. A regular order on roles $\prec$ is a strict partial order (transitive and irreflexive binary relation) on roles such that $R \prec S$ iff $R^- \prec S$. Given a regular order on roles $\prec$, a RIA is $\prec$-regular if it has one of the following forms: a) $R \circ R \sqsubseteq R$; b) $R^- \sqsubseteq R$; c) $S_1 \circ \cdots \circ S_n \sqsubseteq R$ and $S_i \prec R$ for $1 \leq i \leq n$; d) $R \circ S_1 \circ \cdots \circ S_n \sqsubseteq R$ and $S_i \prec R$ for $1 \leq i \leq n$; e) $S_1 \circ \cdots \circ S_n \circ R \sqsubseteq R$ and $S_i \prec R$ for $1 \leq i \leq n$. An RBox $\mathcal{R}$ is regular if there exists a regular order on its roles $\prec$ such that all RIA of $\mathcal{R}$ are $\prec$-regular.

$\mathcal{SROIQ}$ is decidable for knowledge bases with a regular RBox. This is because the restriction assures an existence of a regular automaton corresponding to the language generated by the RBox, which is then used by the tableaux algorithm [11]. More recently an alternative condition called stratified RBox was introduced [13]. Given a preorder (a transitive and reflexive binary relation) $\precsim$ on roles, let $R \approx S$ if $R \precsim S$ and $S \precsim R$, and let $R \prec S$ if $R \precsim S$ and not $R \approx S$. A RIA $R_1 \circ \cdots \circ R_n \sqsubseteq R$ is $\precsim$-admissible if $R_i \precsim R$ for $1 \leq i \leq n$. Give a set of $\precsim$-admissible RIA $\mathcal{R}$ a RIA $Q \sqsubseteq S$ is $\precsim$-stratified in $\mathcal{R}$, if for every R such that $R \approx S$ and $Q = Q_1 \circ R \circ Q_2$ there exits P such that $Q_1 \circ R \sqsubseteq^*_{\mathcal{R}} P$ and $P \circ Q_2 \sqsubseteq^*_{\mathcal{R}} S$. Finally, $\mathcal{R}$ is stratified if every RIA of $\mathcal{R}$ is admissible for some preorder $\precsim$ on the roles of $\mathcal{R}$ and if every RIA $R \sqsubseteq S$ such that $R \sqsubseteq^*_{\mathcal{R}} S$ is $\precsim$-stratified in $\mathcal{R}$.

The stratification condition is less restrictive then regularity, i.e., every regular RBox $\mathcal{R}$ can be extended into a stratified RBox $\mathcal{R}'$ modulo-$\mathcal{R}$-equivalent to $\mathcal{R}$, but not vice versa [13]. As also showed by [13], for deciding satisfiability and entailment with respect to a $\mathcal{SROIQ}$ knowledge base with stratified RBox it is possible to use the same tableaux algorithm introduced by [11]. This is due to the existence of the regular automaton required by the algorithm is also assured for stratified RBoxes. In order to show decidability of reasoning in CKR we will rely on this result. In addition, we will slightly limit the expressive power of $\mathcal{SROIQ}$ as the local language:

[4] $C \sqcup D$ reduces into $\neg(\neg C \sqcap \neg D)$; $\forall R.C$ reduces into $\neg\exists R.\neg C$; $\leqslant nR.C$ reduces into $\neg{\geqslant}n{+}1R.C$; $=nR.C$ reduces into ${\geqslant}nR.C \sqcap \neg{\geqslant}n{+}1R.C$; $\mathrm{Sym}(R)$ reduces into $R^- \sqsubseteq R$; $\mathrm{Tra}(R)$ reduces into $R \circ R \sqsubseteq R$; $\mathrm{Irr}(R)$ reduces into $\exists R.\mathrm{Self} \sqsubseteq \bot$.

disjointness axioms $\mathrm{Dis}(R, S)$ will be excluded from CKR and role reflexivity axioms $\mathrm{Ref}(R)$ will only be allowed if R is simple. Therefore the DL on which CKR is built can be described as almost full $\mathcal{SROIQ}$. We deem this to be a reasonable sacrifice in the expressivity of the local language that allows us to achieve the contextualized representation of knowledge that is enabled by CKR.

3 Contextualized Knowledge Repository

A CKR is composed by a set of *contexts*. Following the "context as a box" metaphor [4], a context contains a set of logical statements and it is qualified by a set of contextual attributes, also called *dimensions*. An example of this type of representation is shown in Fig. 1 where an excerpt from a context representing the Italian national football league in 2010 is depicted.

time = 2010, location = Italy, topic = NFL

$\mathsf{Team} \sqsubseteq {=}22\mathsf{has\_player.Player}$
$\mathsf{Player} \sqsubseteq {\leqslant}1\mathsf{plays\_for.Team}$
$\mathsf{Team(Milan)}$
$\mathsf{plays\_for(Cassano, Milan)}$
$\dots$

Fig. 1. Italian national football league under the context as a box metaphor

To specify a context, we use two languages. Contextual attributes are specified in the meta-language, with vocabulary called called the meta-vocabulary. The content of the context is specified in the object-language, with vocabulary called object-vocabulary. In our case both these languages are DL. The meta-vocabulary contains specific set of symbols in order to identify contexts and to assign dimensional values to contexts. It contains a distinguished set of individuals that will be used as context identifiers. For each dimension it contains a dedicated role A that will be used to assign dimensional values to the contexts, a set of admissible dimensional values D_A which are individuals, and a role $\prec_A$ which will be used to model the cover relation between dimensional values.

Definition 1 (Meta-vocabulary). *A meta-vocabulary Γ is a DL vocabulary that contains:*

1. *a set of individuals called* context identifiers*;*
2. *a finite set of roles* $\mathbf{A}$ *called* dimensions*;*
3. *a set of individuals D_A called* dimensional values*, for every dimension $A \in \mathbf{A}$;*
4. *a role $\prec_A$, called* coverage relation*, for every dimension $A \in \mathbf{A}$.*

Meta vocabulary allows us to construct dimensional vectors of the form $\{A_{i_1}{=}d_1, \dots, A_{i_n}{=}d_n\}$ which are composed of attribute-value declarations such that each A_{i_k} is a dimension of Γ and each d_k is a value from $D_{A_{i_k}}$. In accordance with the context as a box vision, dimensional vectors will be used to address a context by a specific set of dimensional values. They are either full, if a value for each dimension in $\mathbf{A}$ is given, or partial, if some dimensions are missing. The set of all full dimensional vectors of Γ forms a dimensional space in which context will be located.

Definition 2 (Dimensional space). *Given a meta-vocabulary Γ with n dimensions $\mathbf{A} = \{A_1, \ldots, A_n\}$, let us define:*

1. *a full dimensional vector in Γ is a set of attribute-value declarations $\mathbf{d} = \{A_1{=}d_1, \ldots, A_n{=}d_n\}$ such that $d_k \in D_{A_k}$ for every k with $1 \leq k \leq n$;*
2. *a partial dimensional vector in Γ is a set of attribute-value declarations $\mathbf{d_B} = \{A_{i_1}{=}d_1, \ldots, A_{i_m}{=}d_m\}$ such that $0 \leq m \leq n$, $d_k \in D_{A_{i_k}}$ for every k with $1 \leq k \leq m$, and $\mathbf{B} = \{A_{i_1}, \ldots, A_{i_m}\} \subset \mathbf{A}$;*
3. *$\mathfrak{D}_\Gamma$, the dimensional space respective to Γ, is the set of all full dimensional vectors in Γ;*
4. *$\mathbf{d_B}{+}\mathbf{e_C}$, the completion of $\mathbf{d_B}$ w.r.t. $\mathbf{e_C}$, given two partial dimensional vectors $\mathbf{d_B}$ and $\mathbf{e_C}$, is equal to $\mathbf{d_B} \cup \{(A_{i_k}{=}d_k) \in \mathbf{e_C} \mid A_{i_k} \notin \mathbf{B}\}$.*

We use bold Latin letters $\mathbf{d}, \mathbf{e}, \mathbf{f}$, etc. to denote dimensional vectors. Given a dimensional vector $\mathbf{d}$ and $A \in \mathbf{A}$, we denote by d_A the value assigned to A in $\mathbf{d}$ (i.e., such that $(A{=}d_A) \in \mathbf{d}$). If $\mathbf{d}$ is partial and it does not contain a value for A, then d_A is undefined. Observe that in fact for any full dimensional vector $\mathbf{d}$, and any subset of dimensions $\mathbf{B} \subseteq \mathbf{A}$, a partial dimensional vector $\mathbf{d_B}$ is obtained by projection of $\mathbf{d}$ with respect to the dimensions in $\mathbf{B}$ (i.e., $\mathbf{d_B} = \{B = d_B \mid B \in \mathbf{B}\}$). By definition $\mathbf{d_A} = \mathbf{d}$. Note that the empty dimensional vector $\{\}$ is also a partial dimensional vector is Γ.

Inside the contexts, knowledge is encoded using the object-vocabulary. This is again a DL-vocabulary. While the object-vocabulary is shared between all contexts in a CKR knowledge base, the symbols may have different interpretation in different contexts. This is very natural when modeling contextualized information. For instance, in the context of FIFA WC 2010 the concept Finalist represents the finalist teams of the FIFA WC 2010, while in the context of FIFA WC 2006, the same concept represents the finalists of the 2006 edition of FIFA WC. Locality, however, does not imply opacity. When information propagates across contexts, we need a way to refer to the specific interpretation of a symbol in a remote context. To be able to do this, we introduce so called *qualified symbols* into the object-vocabulary. These are symbols with a dimensional vector in subscript which indicates with respect to which context the symbol should be interpreted.

Definition 3 (Object-vocabulary). *Let Γ be a meta-vocabulary. Given any DL-vocabulary Σ_B, the object-vocabulary Σ is an extension of Σ_B such that for every concept/role symbol X in Σ_B (including $\top$, U, I, but excluding $\bot$), and for every dimensional vector $\mathbf{d}$ (full or partial), Σ contains the concept/role symbol $X_\mathbf{d}$.*

An object-vocabulary Σ is possibly constructed on top of any DL-vocabulary Σ_B. In such a case, Σ_B is called the base-vocabulary of Σ. Any symbol from $\Sigma \setminus \Sigma_\mathrm{B}$ is called qualified symbol. Concepts and roles of Σ_B are non-qualified, but they can also be perceived as qualified with respect to the empty dimensional vector $\{\}$. If no ambiguity arises, we skip the brackets and the name of the attributes, so instead of $X_{\{\mathsf{location}=\mathsf{Italy},\mathsf{time}=\mathsf{2010}\}}$ we will write $X_{\mathsf{Italy},\mathsf{2010}}$, etc. Qualified symbols will be given a special interpretation by the CKR semantics, but they are used just like any other concept/role symbols. For instance, in the context of football 2005–2010 one would like to define the concept TopTeam as the set of teams that reached the final phase in at least one edition of the FIFA WC in the last 5 years (note that FIFA WC is run every 4 years). Given the dimensional value

FWC for the topic FIFA World Cup and 2006, 2010, etc. for years, the concept TopTeam can be defined with the following axiom:

$$\mathsf{TopTeam} \equiv \mathsf{Finalist}_{\mathsf{FWC,2010}} \sqcup \mathsf{Finalist}_{\mathsf{FWC,2006}}$$

Such an approach is reminiscent of the knowledge qualification and unqualification operations (also context *push* and *pop*) as known from the literature [4]. These operations allow for a statement to be popped out of the context, preserving its meaning, by modifying it to make the contextual parameters explicit. Or in the opposite direction, a qualified statement can be pushed inside a context and some of its qualifying parameters stripped. Later on we will formalize these operations in the CKR framework using a special operator called the @ operator.

A context is a unit of knowledge, from which a CKR knowledge base is composed. Each context has an identifier, a set of dimensional attributes, one for each dimension, which are respective to some meta-vocabulary Γ, and it features a DL knowledge base over some object-vocabulary Σ.

Definition 4 (Context). *Given a meta-vocabulary Γ and an object-vocabulary Σ, a* context *on (Γ, Σ) is a triple $\langle \mathcal{C}, \dim(\mathcal{C}), \mathrm{K}(\mathcal{C})\rangle$ where:*

1. *$\mathcal{C}$ is a* context identifier *of Γ;*
2. *$\dim(\mathcal{C})$ is a full dimensional vector of $\mathfrak{D}_\Gamma$;*
3. *$\mathrm{K}(\mathcal{C})$ is a $\mathcal{SROIQ}$ knowledge base over Σ.*

Note that while symbols appearing inside contexts can possibly be qualified with partial dimensional vectors, $\dim(\mathcal{C})$, the dimensional vector on which the context $\mathcal{C}$ resides, is always a full dimensional vector in $\mathfrak{D}_\Gamma$. We use the notation $\mathcal{C}_\mathbf{d}$ to denote a context with $\dim(\mathcal{C}) = \mathbf{d}$.

Finally, a CKR knowledge base is composed of a collection of contexts and an additional DL knowledge base over the meta-vocabulary which will be called metakonwledge. Metaknowledge assigns the dimensional values to each context and it also asserts a hierarchical organization of contexts, which will be called context coverage. This hierarchy is recorded by asserting a strict partial order on the dimensional values of each dimensional attribute $A \in \mathbf{A}$ using the role $\prec_A$.

Definition 5 (Contextualized Knowledge Repository). *Let Γ be a meta-vocabulary and let Σ be an object-vocabulary. A* Contextualized Knowledge Repository (CKR) *on (Γ, Σ) is a pair $\mathfrak{K} = \langle \mathfrak{M}, \mathfrak{C}\rangle$ such that:*

1. *$\mathfrak{C}$ is a set of contexts on (Γ, Σ);*
2. *$\mathfrak{M}$, called metaknowledge, is a DL knowledge base on Γ such that:*
   - *(a) every $A \in \mathbf{A}$ is declared a functional role;*
   - *(b) for every $\mathcal{C} \in \mathfrak{C}$ with $\dim(\mathcal{C}) = \mathbf{d}$ and for every $A \in \mathbf{A}$ we have $\mathfrak{M} \models A(\mathcal{C}, d_A)$;*
   - *(c) for every $A \in \mathbf{A}$, the relation $\{d \prec_A d' \mid \mathfrak{M} \models \prec_A(d, d')\}$ is a strict partial order on D_A.*

The coverage relation between the dimensional values of each dimension is encoded in the metaknowledge. This provides base to the coverage between dimensional vectors and contexts. One dimensional vector (or context) covers another, if its dimensional values cover the values of the other, one by one. That is, the coverage between dimensional vectors ($\prec$) is a product of the dimensional order relations $\prec_A$. One context covers another if same holds for their associated dimensional vectors. We will also introduce a handy notation for coverage with respect to a subset of dimensions only ($\prec_\mathbf{B}$).

Definition 6 (Coverage). *Given a CKR $\mathfrak{K}$ on (Γ, Σ) with dimensions $\mathbf{A}$, give any dimension $A \in \mathbf{A}$ and any two dimensional values $d, d' \in D_A$, given any two dimensional vectors $\mathbf{d}$ and $\mathbf{e}$ (full or partial), $\mathbf{B} \subseteq \mathbf{A}$, and give any two contexts $\mathcal{C}$ and $\mathcal{C}'$ we say that:*

1. *d covers d' w.r.t. A (denoted $d \prec_A d'$) if $\mathfrak{M} \models \prec_A(d, d')$;*
2. *$\mathbf{e}$ covers $\mathbf{d}$ w.r.t. $\mathbf{B}$ (denoted $\mathbf{d} \prec_{\mathbf{B}} \mathbf{e}$) if $d_B \prec_B e_B$ for all $B \in \mathbf{B}$;*
3. *$\mathbf{e}$ covers $\mathbf{d}$ (denoted $\mathbf{d} \prec \mathbf{e}$) if $\mathbf{d} \prec_{\mathbf{A}} \mathbf{e}$;*
4. *$\mathcal{C}'$ covers $\mathcal{C}$ (denoted $\mathcal{C} \prec \mathcal{C}'$) if $\dim(\mathcal{C}) \prec \dim(\mathcal{C}')$.*

Note that $\mathbf{d} \prec \mathbf{e}$ implies that $\mathbf{d}$ and $\mathbf{e}$ are defined on the same set of dimensions. If $\mathbf{d} \prec_{\mathbf{B}} \mathbf{e}$, then $\mathbf{d}$ and $\mathbf{e}$ may be defined on a different set of dimensions but both must be defined on all dimensions of $\mathbf{B} \subseteq \mathbf{A}$.

To give an example, let us now formally model the coverage relation for the Football contexts mentioned in the introduction. We will have the topic dimension with the following values in D_{topic}: FB (football), FWC (FIFA World Cup), NFL (National football league), WN (world news), NN (national news). The space dimension will have the values world, africa and italy in D_{space}. The time dimension will have only one value 2010. The following coverage between the dimensional values will be asserted in the ABox of $\mathfrak{M}$:

$$
\begin{array}{lll}
\mathsf{FWC} \prec_{\mathsf{topic}} \mathsf{WN} & \mathsf{NFL} \prec_{\mathsf{topic}} \mathsf{FB} & \mathsf{africa} \prec_{\mathsf{space}} \mathsf{world} \\
\mathsf{FWC} \prec_{\mathsf{topic}} \mathsf{FB} & \mathsf{NFL} \prec_{\mathsf{topic}} \mathsf{NN} & \mathsf{italy} \prec_{\mathsf{space}} \mathsf{world}
\end{array}
$$

Consequently, the context coverage relation shown in Figure 2 is generated on the dimensional vectors.

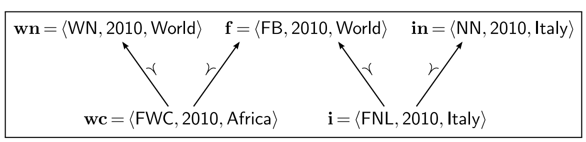

Fig. 2. Coverage relation between contexts

The semantics of CKR relies on the DL semantics inside each context (local semantics), while the relations between contexts are handled by some additional semantic conditions. Local interpretation and local models are like standard DL-interpretations and models with two notable exceptions: empty domains are allowed; and, while all contexts in a CKR share a common object-vocabulary Σ, not every symbol of Σ need to be interpreted by the local interpretation. This will be especially true in case of individuals which may but also may not be meaningful in a given context.

Definition 7 (Local Interpretation). *Given a CKR $\mathfrak{K}$ over (Γ, Σ) with $\Sigma = N_{\mathrm{C}} \uplus N_{\mathrm{R}} \uplus N_{\mathrm{I}}$, and a context $\mathcal{C}_{\mathbf{d}}$ of $\mathfrak{K}$, a pair $\mathcal{I}_{\mathbf{d}} = \left\langle \Delta_{\mathbf{d}}, \cdot^{\mathcal{I}_{\mathbf{d}}} \right\rangle$ is a local interpretation of $\mathcal{C}_{\mathbf{d}}$ if:*

1. *either $\Delta_{\mathbf{d}} = \emptyset$;*
2. *or there exists $N_{\mathrm{I}}' \subseteq N_{\mathrm{I}}$ s.t. $\mathcal{I}_{\mathbf{d}}$ is a DL-interpretation over $\Sigma' = N_{\mathrm{C}} \uplus N_{\mathrm{R}} \uplus N_{\mathrm{I}}'$.*

Note that for any complex concept or role X, $X^{\mathcal{I}_\mathbf{d}}$ is defined only if it is defined also for every individual occurring in X. In the following, whenever we write $X^{\mathcal{I}_\mathbf{d}}$ then we also mean that $\mathcal{I}_\mathbf{d}$ is defined for X. Observe in the definition below, that in a local model $\mathcal{I}_\mathbf{d}$ of $\mathcal{C}_\mathbf{d}$, $\mathcal{I}_\mathbf{d}$ is necessarily defined on every individual actually occurring in $\mathcal{C}_\mathbf{d}$ (apart from the case when $\Delta_\mathbf{d}$ is empty). It may be defined on some individuals in addition due to the semantic relations between contexts.

Definition 8 (Local Model). *Given a CKR $\mathfrak{K}$, a context $\mathcal{C}_\mathbf{d}$ of $\mathfrak{K}$, a local interpretation $\mathcal{I}_\mathbf{d}$ is a* local model *of $\mathcal{C}_\mathbf{d}$ (denoted $\mathcal{I}_\mathbf{d} \models_{\mathrm{DL}} \mathcal{C}_\mathbf{d}$) either if $\Delta_\mathbf{d} = \emptyset$ or if $\mathcal{I}_\mathbf{d} \models_{\mathrm{DL}} \phi$ for every axiom $\phi \in \mathcal{C}_\mathbf{d}$.*

A model of a CKR knowledge base is a collection of local models, one for each context, which are bound together by further semantic conditions in order to take into account relations between contexts. In a CKR model, local domains may possibly overlap, reflecting the fact that the context may possibly describe same things from a different perspective. Local domains will be organized in accordance with the coverage hierarchy. In addition special attention is given to individuals, which are interpreted equally if they occurs in two contexts that share a common super-context, and the meaning for the qualified concepts and roles is provided.

Definition 9 (CKR Model). *A model of a CKR $\mathfrak{K}$ is a family $\mathfrak{I} = \{\mathcal{I}_\mathbf{d}\}_{\mathbf{d}\in\mathfrak{D}_\Gamma}$ of local models such that for all $\mathbf{d}$, $\mathbf{e}$, and $\mathbf{f}$, for every atomic concept A, atomic role R, atomic concept/role X and individual a:*

1. $(\top_\mathbf{d})^{\mathcal{I}_\mathbf{f}} \subseteq (\top_\mathbf{e})^{\mathcal{I}_\mathbf{f}}$ *if* $\mathbf{d} \prec \mathbf{e}$
2. $(A_\mathbf{f})^{\mathcal{I}_\mathbf{d}} \subseteq (\top_\mathbf{f})^{\mathcal{I}_\mathbf{d}}$
3. $(R_\mathbf{f})^{\mathcal{I}_\mathbf{d}} \subseteq (\top_\mathbf{f})^{\mathcal{I}_\mathbf{d}} \times (\top_\mathbf{f})^{\mathcal{I}_\mathbf{d}}$
4. $a^{\mathcal{I}_\mathbf{e}} = a^{\mathcal{I}_\mathbf{d}}$ *either if* $\mathbf{d} \prec \mathbf{e}$ *and* $a^{\mathcal{I}_\mathbf{d}}$ *is defined*
   or if $\mathbf{e} \prec \mathbf{d}$, $a^{\mathcal{I}_\mathbf{e}}$ *is defined, and* $a^{\mathcal{I}_\mathbf{e}} \in \Delta_\mathbf{d}$
5. $(X_{\mathbf{d}_\mathbf{B}})^{\mathcal{I}_\mathbf{e}} = (X_{\mathbf{d}_\mathbf{B}+\mathbf{e}})^{\mathcal{I}_\mathbf{e}}$
6. $(X_\mathbf{d})^{\mathcal{I}_\mathbf{e}} = (X_\mathbf{d})^{\mathcal{I}_\mathbf{d}}$ *if* $\mathbf{d} \prec \mathbf{e}$
7. $(A_\mathbf{f})^{\mathcal{I}_\mathbf{d}} = (A_\mathbf{f})^{\mathcal{I}_\mathbf{e}} \cap \Delta_\mathbf{d}$ *if* $\mathbf{d} \prec \mathbf{e}$
8. $(R_\mathbf{f})^{\mathcal{I}_\mathbf{d}} = (R_\mathbf{f})^{\mathcal{I}_\mathbf{e}} \cap (\Delta_\mathbf{d} \times \Delta_\mathbf{d})$ *if* $\mathbf{d} \prec \mathbf{e}$
9. $\mathcal{I}_\mathbf{d} \models_{\mathrm{DL}} \mathcal{C}_\mathbf{d}$

Let us now explain the semantic constraints imposed in CKR models passing through the conditions of the definition one by one.

Condition 1 Given $\mathcal{C}_\mathbf{d} \prec \mathcal{C}_\mathbf{e}$, the perspective of $\mathcal{C}_\mathbf{d}$ is narrower than of $\mathcal{C}_\mathbf{e}$ and vice versa the perspective of $\mathcal{C}_\mathbf{e}$ is broader than then of $\mathcal{C}_\mathbf{d}$. In order to reflect this, $\Delta_\mathbf{d}$ is required to be a subset of $\Delta_\mathbf{e}$ in any CKR model. This is a basic premise in order to make the knowledge of $\mathcal{C}_\mathbf{d}$ accessible to $\mathcal{C}_\mathbf{e}$ by the latter constraints.

Conditions 2 and 3 take care that in every context $\mathcal{C}_\mathbf{d}$ the interpretations of symbols qualified with some $\mathbf{f}$ is roofed under the concept $\top_\mathbf{f}$ which thus represents the top of $\mathcal{C}_\mathbf{f}$ as viewed inside $\mathcal{C}_\mathbf{d}$ and in a CKR model $\top_\mathbf{f}^{\mathcal{I}_\mathbf{d}}$ represents the image that $\mathcal{I}_\mathbf{d}$ keeps of $\Delta_\mathbf{f}$. As implied by further conditions the image of $\mathcal{C}_\mathbf{f}$ in $\mathcal{C}_\mathbf{d}$ may be but as well may not be entirely precise, depending on how $\mathcal{C}_\mathbf{d}$ and $\mathcal{C}_\mathbf{f}$ are related by the coverage.

Condition 4 is responsible for the semantic treatment of individuals in CKR. If a narrower context $\mathcal{C}_{\mathbf{d}}$ is covered by a broader context $\mathcal{C}_{\mathbf{e}}$, and an individual a is defined in both of these contexts, then the interpretation of a must be equal in both of these contexts. This is assured by propagating the semantics of a from $\mathcal{C}_{\mathbf{d}}$ into $\mathcal{C}_{\mathbf{e}}$, but not necessarily the other way around: if $a^{\mathcal{I}_{\mathbf{d}}}$ is defined then $a^{\mathcal{I}_{\mathbf{e}}}$ must be defined and must be equal to $\mathcal{I}_{\mathbf{e}}$; on the other hand, if $a^{\mathcal{I}_{\mathbf{e}}}$ is defined then $a^{\mathcal{I}_{\mathbf{d}}}$ must be defined to the same value only if $a^{\mathcal{I}_{\mathbf{e}}}$ is part of the $\mathcal{C}_{\mathbf{e}}$'s image of $\top_{\mathbf{d}}$.

One practical consequence of this treatment is that if the same individual a occurs in two context which share at least one common super-context, it has the same interpretation. Another consequence is that it allows to predicate about (non)existence of objects in a context from a broader context. For instance if $\top_{\{}\mathsf{FWC}, 2010, \mathsf{Africa}\}(\mathsf{England})$ and $\neg\top_{\{}\mathsf{FWC}, 2010, \mathsf{Africa}\}(\mathsf{Egypt})$ are stated in a context broader than $\mathcal{C}_{\{\mathsf{FWC},2010,\mathsf{Africa}\}}$ (e.g., say in $\mathcal{C}_{\{\mathsf{FB},2010,\mathsf{World}\}}$), as a consequence it is implied that England participates in the last FIFA WC while Egypt does not, on the semantic level the individual England is always defined in this context while the individual Egypt is always undefined in it.

Condition 5 provides meaning for partially qualified symbols. It assures that the values for attributes which are not specified are always taken from the current context in which the expression appears. Therefore in the end all symbols even those partially qualified are treated as fully qualified by the semantics. It is important to understand that also symbols with no qualifying vectors are viewed as qualified symbols, they are qualified with empty dimensional vector . Their qualification is taken from the context in which they appear and they are thenceforth treated as fully qualified by the semantics.

Due to this kind of treatment, partially qualified symbols are in fact some syntactic sugar added to the framework, For instance, instead of $\mathsf{Coach}_{\{\mathsf{FB},\mathsf{World}\}}$ we can equivalently use $\mathsf{Coach}_{\{\mathsf{FB},\mathsf{World},2010\}}$ inside $\mathcal{C}_{\{\mathsf{FWC},2010,\mathsf{Africa}\}}$ and instead of playsFor we can equivalently use $\mathsf{playsFor}_{\{\mathsf{FWC},2010,\mathsf{Africa}\}}$ in the very same context. On the other hand, we consider partially qualified symbols necessary in order to achieve practical usability of the framework.

Condition 6 and the two consecutive conditions provide semantics for qualified concepts, where it is ensured that the meaning of a concept $X_{\mathbf{d}}$ is based on its interpretation in $\mathcal{C}_{\mathbf{d}}$ as much as the partially overlapping domains allow. The meaning of qualified concepts is also retained as much as possible whenever going up and down in the coverage hierarchy.

Condition 6 states that the interpretation $X_{\mathbf{d}}$ is strictly bound to $X^{\mathcal{I}_{\mathbf{d}}}$ in all contexts that cover $\mathcal{C}_{\mathbf{d}}$. This is indeed possible due to the fact that $\Delta_{\mathbf{d}}$ is totally contained the interpretation domains of all such contexts, which is assured by Condition 1. For example, the interpretation of $\mathsf{Team}_{\mathsf{FWC},2010,\mathsf{Africa}}$ in $\mathcal{C}_{\langle\mathsf{FB},2010,\mathsf{World}\rangle}$ is the same as the interpretation of Team in the context $\mathcal{C}_{\langle\mathsf{FWC},2010,\mathsf{Africa}\rangle}$.

Condition 7 and 8 assure that given $\mathcal{C}_{\mathbf{d}} \prec \mathcal{C}_{\mathbf{e}}$ and a symbol $X_{\mathbf{f}}$, where $\mathbf{f}$ is not necessarily related to $\mathbf{d}$ of $\mathbf{e}$, the two interpretations of $X_{\mathbf{f}}$ in $\mathcal{I}_{\mathbf{d}}$ and $\mathcal{I}_{\mathbf{e}}$ are equal modulo the interpretation domain of the narrower context $\Delta_{\mathbf{d}}$. This especially implies that if a particular individual (or pair of individuals if X is a role) occurs in both contexts $\mathcal{C}_{\mathbf{d}}$ and $\mathcal{C}_{\mathbf{e}}$, then it ether belongs to both $X_{\mathbf{f}}^{\mathcal{I}_{\mathbf{d}}}$ and $X_{\mathbf{f}}^{\mathcal{I}_{\mathbf{e}}}$ or it belongs to none of them.

Condition 9 states that in a CKR model, the local interpretation $\mathcal{I}_{\mathbf{d}}$ of each context $\mathcal{C}_{\mathbf{d}}$ is also a model of $\mathcal{C}_{\mathbf{d}}$ according to the local semantics, i.e., that of DL.

The two classic reasoning tasks for DL are satisfiability of concepts entailment (especially of subsumption formulae) with respect to a knowledge base. In a CKR model, a formula may be satisfied in one context but in another it may be unsatisfied. In addition, for some contexts in a knowledge base, all admissible CKR models may have a local model with empty domain wheres for other contexts there may be CKR models with non-empty local domain. Therefore given a CKR $\mathfrak{K}$ one has to specify with respect to which context the reasoning task in question is to be evaluated. Such reasoning tasks will be called $\mathbf{d}$-satisfiability of concepts and $\mathbf{d}$-entailment.

Definition 10 ($\mathbf{d}$-satisfiability of concepts). *Given a CKR knowledge base $\mathfrak{K}$ over (Σ, Γ) with $\mathbf{d} \in \mathfrak{D}_\Gamma$ and a concept C over Σ, we say that C is $\mathbf{d}$-satisfiable w.r.t. $\mathfrak{K}$ if there exists a CKR model $\mathfrak{I} = \{\mathcal{I}_\mathbf{d}\}_{\mathbf{d}\in\mathfrak{D}_\Gamma}$ of $\mathfrak{K}$ such that $C^{\mathcal{I}_\mathbf{d}} \neq \emptyset$.*

Definition 11 ($\mathbf{d}$-entailment). *Given a CKR knowledge base $\mathfrak{K}$ over (Σ, Γ) with $\mathbf{d} \in \mathfrak{D}_\Gamma$ and any formula ϕ over Σ with syntax listed in Table 1 under Axioms, we say that ϕ is $\mathbf{d}$-entailed by $\mathfrak{K}$ (denoted by $\mathfrak{K} \models \mathbf{d} : \phi$) if for every CKR model $\mathfrak{I} = \{\mathcal{I}_\mathbf{d}\}_{\mathbf{d}\in\mathfrak{D}_\Gamma}$ of $\mathfrak{K}$ we have $\mathcal{I}_\mathbf{d} \models_{\mathrm{DL}} \phi$.*

In addition, we consider satisfiability of the CKR knowledge base as a decision task. In this case it makes sense to define $\mathbf{d}$-satisfiability as well as global satisfiability.

Definition 12 ($\mathbf{d}$-satisfiability). *A CKR knowledge base $\mathfrak{K}$ over (Σ, Γ) with $\mathbf{d} \in \mathfrak{D}_\Gamma$ is said to be$\mathbf{d}$-satisfiable if there exists a CKR model $\mathfrak{I} = \{\mathcal{I}_\mathbf{d}\}_{\mathbf{d}\in\mathfrak{D}_\Gamma}$ of $\mathfrak{K}$ such that $\Delta_\mathbf{d} \neq \emptyset$.*

Definition 13 (Global satisfiability). *A CKR knowledge base $\mathfrak{K}$ over (Σ, Γ) is said to globally satisfiable if there exists a CKR model $\mathfrak{I} = \{\mathcal{I}_\mathbf{d}\}_{\mathbf{d}\in\mathfrak{D}_\Gamma}$ of $\mathfrak{K}$ such that for every $\mathbf{d} \in \mathfrak{D}_\Gamma$ we have $\Delta_\mathbf{d} \neq \emptyset$.*

As usual, the $\mathbf{d}$-entailment (of concept subsumption) and $\mathbf{d}$-(un)satisfiability are inter-reducible: $\mathfrak{K} \models \mathbf{d} : C \sqsubseteq D$ iff $\neg C \sqcup D$ is not $\mathbf{d}$-satisfiable w.r.t. $\mathfrak{K}$; on the other hand, C is $\mathbf{d}$-satisfiable w.r.t. $\mathfrak{K}$ iff $\mathfrak{K} \not\models \mathbf{d} : C \sqsubseteq \bot$. This follow from the fact that local models are all valid $\mathcal{SROIQ}$ models and from the fact that the reduction holds in $\mathcal{SROIQ}$.

4 Reasoning in CKR

In this section we provide a characterization of the entailment in CKR in terms of a natural deduction (ND) calculus [20]. CKR entailment is the product of two orthogonal semantic restrictions: *local entailment* and *cross-context entailment*, The former is induced by the local semantics and coincides with entailment in $\mathcal{SROIQ}$; the latter is induced by the constraints that Definition 9 imposes on each pair of contexts related via coverage. $\mathcal{SROIQ}$ entailment, i.e., $\phi_1, \ldots, \phi_n \models_{\mathsf{DL}} \phi$, is known to be decidable.[5] We therefore assume a black box decision procedure that checks if $\phi_1, \ldots, \phi_n \models_{\mathsf{DL}} \phi$.

[5] Decidability is guaranteed under the assumption that $\phi_1, \ldots, \phi_n$ is $\precsim$-stratified (see [13] for more details), and we have to impose this condition also for CKR. This point is discussed later.

Reasoning rules in the ND calculus allow to deduce conclusions in one of the contexts based on evidence from other contexts, they are therefore a kind of *bridge rules* [9]. As an example consider the following simple bridge rule:

$$\frac{\mathbf{d} : A \sqsubseteq B \quad \mathbf{d} \prec \mathbf{e}}{\mathbf{e} : A_{\mathbf{d}} \sqsubseteq B_{\mathbf{d}}} \tag{1}$$

The rule implies that whenever $A \sqsubseteq B$ is true in a context $C_{\mathbf{d}}$ such that $\mathbf{d} \prec \mathbf{e}$, then $A_{\mathbf{d}} \sqsubseteq B_{\mathbf{d}}$ should be true in $C_{\mathbf{e}}$. This is indeed sound thanks to conditions 5 and 6 of Definition 9 which together impose that in any CKR model $\mathfrak{I}$ the interpretation of A and B in $\mathcal{I}_{\mathbf{d}}$ coincide respectively with the interpretations of $A_{\mathbf{d}}$ and $B_{\mathbf{d}}$ in $\mathcal{I}_{\mathbf{e}}$.

The rationale of rule (1) is that a statement in a narrower context, namely $\mathcal{C}_{\mathbf{d}}$, can be embedded in a larger context, namely $\mathcal{C}_{\mathbf{e}}$, by applying a transformation that preserves semantics. We generalize this idea by introducing the notion of embedding between DL knowledge bases and by showing that in CKR such embedding preserves the meaning of $\mathcal{SROIQ}$ expressions.

A DL embedding is a mapping that embeds a DL knowledge base with a narrower perspective into another one with a broader perspective. The vocabulary of the context being embedded splits in two parts: Σ_c that contains symbols which are completely specified with respect to the embedded context, and Σ_e that contains the remaining symbols which are called external. For instance, the symbol $\mathsf{Player}_{\mathsf{sports}}$ is external in the context $\mathcal{C}$ with topic $= \mathsf{FB}$, given that $\mathsf{FB} \prec \mathsf{sports}$. If we state the axiom $\mathsf{Player}_{\mathsf{sports}} \sqsubseteq \exists.\mathsf{playsFor}.\mathsf{Team}$ inside $\mathrm{K}(\mathcal{C})$, it is only valid in this context where all the players are football players and football is a team sport. In other sports such as tennis players need not have to play for a team. Therefore, when embedding the axiom into the broader context of sports we need to take care to embed the proper meaning of the axiom it has in FB and so we need to pay attention to external symbols.

Definition 14 (DL embedding). *Let Σ and Σ' two DL alphabets, and let Σ be partitioned into two disjoint sets Σ_c and Σ_e with $\top \in \Sigma_c$. A* DL embedding *is a total function $f : \Sigma \rightarrow \Sigma'$ that maps individuals, atomic concepts, and atomic roles of Σ to individuals, atomic concepts, and atomic roles of Σ' respectively. The extension f^* of f that maps complex expressions and axioms over Σ into complex expressions and axioms over Σ' is defined as given in Table 2.*

The DL embedding is done on the syntactic level. On the semantic level, if one knowledge base is embedded into another, we should be able to embed models of the former knowledge base to the models of the latter. A pair of such models is said to be complying with the embedding.

Definition 15 (Embedding-complying intrpretations). *Two DL-interpretations $\mathcal{I}$ and $\mathcal{I}'$ of Σ and Σ' respectively comply with the DL embedding f if:*

1. $a^{\mathcal{I}} = f(a)^{\mathcal{I}'}$*, for each individual a of Σ;*
2. $X^{\mathcal{I}} = f(X)^{\mathcal{I}'}$*, for each concept/role $X \in \Sigma_c$;*
3. $A^{\mathcal{I}} = f(A)^{\mathcal{I}'} \cap f(\top)^{\mathcal{I}'}$*, for each concept $A \in \Sigma_e$;*
4. $R^{\mathcal{I}} = f(R)^{\mathcal{I}'} \cap f(\top)^{\mathcal{I}'} \times f(\top)^{\mathcal{I}'}$*, for each role $R \in \Sigma_e$.*

If two interpretations $\mathcal{I}$ and $\mathcal{I}'$ comply with the embedding f then $\mathcal{I}'$ is apparently an extension of $\mathcal{I}$. It contains the domain Δ of $\mathcal{I}$ as $\Delta = f(\top)^{\mathcal{I}'} \subseteq \Delta'$ and the f-images of all internal symbols

$$f^*(A) = \begin{cases} f(A) & \text{if } A \in \Sigma_c \\ f(\top) \sqcap f(A) & \text{if } A \in \Sigma_e \end{cases}$$
$$f^*(R) = \begin{cases} f(R) & \text{if } R \in \Sigma_c \\ f(I) \circ f(R) \circ f(I) & \text{if } R \in \Sigma_e \end{cases}$$
$$f^*(\exists R.C) = \begin{cases} \exists f(R).f^*(C) & \text{if } R \in \Sigma_c \\ f(\top) \sqcap \exists f(R).f^*(C) & \text{if } R \in \Sigma_e \end{cases}$$
$$f^*(\forall R.C) = \begin{cases} f(\top) \sqcap \forall f(R).f^*(C) & \text{if } R \in \Sigma_c \\ f(\top) \sqcap \forall f(R).(\neg f(\top) \sqcup f^*(C)) & \text{if } R \in \Sigma_e \end{cases}$$
$$f^*(\geqslant nR.C) = \begin{cases} \geqslant n f(R).f^*(C) & \text{if } R \in \Sigma_c \\ f(\top) \sqcap \geqslant n f(R).f^*(C) & \text{if } R \in \Sigma_e \end{cases}$$
$$f^*(\exists R.\text{Self}) = \begin{cases} \exists f(R).\text{Self} & \text{if } R \in \Sigma_c \\ f(\top) \sqcap \exists f(R).\text{Self} & \text{if } R \in \Sigma_e \end{cases}$$
$$f^*(\bot) = \bot$$
$$f^*(\neg C) = f(\top) \sqcap \neg f^*(C)$$
$$f^*(C \sqcap D) = f^*(C) \sqcap f^*(D)$$
$$f^*(C \sqcup D) = f^*(C) \sqcup f^*(D)$$
$$f^*(\{a\}) = \{f(a)\}$$
$$f^*(\leqslant nR.C) = f(\top) \sqcap \leqslant n f(R).f^*(C)$$
$$f^*(R^-) = (f(R))^-$$
$$f^*(R \circ S) = f^*(R) \circ f^*(S)$$
$$f^*(C(a)) = f^*(C)(f(a))$$
$$f^*(R(a,b)) = f(R)(f(a), f(b))$$
$$f^*(C \sqsubseteq D) = f^*(C) \sqsubseteq f^*(D)$$
$$f^*(R \sqsubseteq S) = f^*(R) \sqsubseteq f(S)$$
$$f^*(a = b) = f(a) = f(b)$$
$$f^*(a \neq b) = f(a) \neq f(b)$$

Table 2. DL-embedding on complex expressions and axioms

of Σ_c are interpreted inside $f(\top)^{\mathcal{I}'}$. The images of external symbols of Σ_e can possibly exceed $f(\top)^{\mathcal{I}'}$ when interpreted in $\mathcal{I}'$ but we can always obtain the corresponding interpretations of their pre-images by restriction to $f(\top)^{\mathcal{I}'}$. This corresponds to the fact that the symbols external to Σ are not completely specified in $\mathcal{I}$.

An important point is that the meaning of any symbol, internal or external, with respect to $\mathcal{I}$ can always be retained from $\mathcal{I}'$. The following lemma shows that this is also true for complex descriptions composed of a mixture of internal and external symbols and as a consequence also the meaning of axioms is preserved.

Lemma 1. *If two DL-interpretations $\mathcal{I}$ and $\mathcal{I}'$ comply with the embedding $f : \Sigma \to \Sigma'$, then, for every concept C, $C^{\mathcal{I}} = (f^*(C))^{\mathcal{I}'}$, for every role R, $R^{\mathcal{I}} = (f^*(R))^{\mathcal{I}'}$, and for every axiom ϕ, $\mathcal{I} \models \phi$ iff $\mathcal{I}' \models f^*(\phi)$.*

Armed with this result we will now construct a DL-embedding between each pair of contexts $\mathcal{C}_{\mathbf{d}}$ and $\mathcal{C}_{\mathbf{e}}$ with $\mathbf{d} \prec \mathbf{e}$ in any given CKR. In fact, given any $\mathbf{d}$ the embedding between $\mathcal{C}_{\mathbf{d}}$ and any other $\mathcal{C}_{\mathbf{e}}$ with $\mathbf{d} \prec \mathbf{e}$ is the same. For convenience we will call this embedding the $@\mathbf{d}$ operator. For any construct ϕ the embedded value will be $\phi@\mathbf{d}$. The operator will allow us to deal with partially qualified symbols and it will allow us to characterize the knowledge propagation in CKR in the two basic axes, from narrower to broader context and vice versa. Later on we will find another use for embeddings, when showing that CKR can be reduced into a regular DL knowledge base.

Definition 16 **($@\mathbf{d}$ operator).** *For every full dimensional vector* $\mathbf{d}$*, the operator* $(\cdot)@\mathbf{d}$ *is defined as* $f^*_{\mathbf{d}}(\cdot)$*, where* $f_{\mathbf{d}}$ *is an embedding from* Σ *into itself defined as follows:*

- $f_{\mathbf{d}}(a) = a$ *for every individual* a*;*
- $f_{\mathbf{d}}(X_{\mathbf{d}'_{\mathbf{B}}}) = X_{\mathbf{d}'_{\mathbf{B}}+\mathbf{d}}$ *for every concept/role* X*;*
- $\Sigma_c = \{X_{\mathbf{d}'_{\mathbf{B}}} \in \Sigma \mid \mathbf{d}'_{\mathbf{B}} \preceq \mathbf{d}_{\mathbf{B}}\}$*;* $\Sigma_e = \Sigma \setminus \Sigma_c$.

For instance if the concept Team occurs in $\mathcal{C}_{\mathbf{d}}$ with $\mathbf{d} = \langle \mathsf{FWC}, 2010, \mathsf{Africa}\rangle$, it belongs to Σ_c as $\mathbf{d}'_{\mathbf{B}} \preceq \mathbf{d}_{\mathbf{B}}$ for $\mathbf{B} = \emptyset$. Hence $\mathsf{Team}@\mathbf{d} = \mathsf{Team}_{\mathsf{FWC,2010,Africa}}$. This is natural, as in a context wider than $\mathcal{C}_{\langle \mathsf{FWC,2010,Africa}\rangle}$ the concept $\mathsf{Team}_{\mathsf{FWC,2010,Africa}}$ is fully defined by Team in $\mathcal{C}_{\langle \mathsf{FWC,2010,Africa}\rangle}$. But $\mathsf{NationalTeam}_{\mathsf{FB}} \notin \Sigma_c$ as $\mathsf{FB} \not\preceq \mathsf{FWC}$. Hence we have $\mathsf{NationalTeam}_{\mathsf{FB}}@\langle \mathsf{FWC}, 2010, \mathsf{Africa}\rangle$ = $\mathsf{NationalTeam}_{\mathsf{FB,2010,Africa}} \sqcap \top_{\mathsf{FWC,2010,Africa}}$. Intuitively, in order to embed $\mathsf{NationalTeam}_{\mathsf{FB}}$ from $\mathcal{C}_{\langle \mathsf{FWC,2010,Africa}\rangle}$ into a broader context one must restrict it to $\top_{\mathsf{FWC,2010,Africa}}$ because its interpretation in the broader context may be broader.

We now briefly introduce ND, for more details see [20]. A ND calculus is a set of inference rules of the form:

$$\frac{\alpha_1 \cdots \alpha_n \quad \begin{array}{c}[B_{n+1}]\\ \alpha_{n+1}\end{array} \quad \cdots \quad \begin{array}{c}[B_{n+m}]\\ \alpha_{n+m}\end{array}}{\alpha}\rho \qquad (2)$$

with $n, m \geq 0$, where α_i and α are formulae, B_i's are sets of formulae. The α_i's are the *premises* of ρ, α is the conclusion and the B_i's are the *assumptions discharged* by ρ. A *deduction* of α depending on a set of formulae Φ is a tree rooted in α inductively constructed starting from a set of assumptions including Φ by applying the inference rules. Formally deduction is defined by induction:

1. a formula α is a deduction of α depending on $\{\alpha\}$;
2. if for each $1 \leq i \leq n+m$, Π_i is a deduction of α_i depending on Φ_i and the calculus contains the rule (2), then
$$\frac{\Pi_1 \cdots \Pi_{n+m}}{\alpha}\rho$$
is a deduction of α depending on $\left(\bigcup_{i=1}^{n} \Phi_i\right) \cup \left(\bigcup_{i=n+1}^{n+m} (\Phi_i \setminus B_i)\right)$.

A formula α is derivable from Φ if there is a deduction of α depending on a subset of Φ. α is provable if it is derivable form the emtpy set.

A ND system for a CKR $\mathfrak{K} = \langle \mathfrak{C}, \mathfrak{M}\rangle$ over $\langle \Sigma, \Gamma\rangle$ is shown in Table 3. The premises of the ND rules of our caluculus are either object formulae of the form $\mathbf{d} : \phi$ where $\mathbf{d} \in \mathfrak{D}_\Gamma$ and ϕ is a DL formula over Σ, or meta formulare μ over Γ. Conlusions and dicharged assumptions are always object formulae.

Definition 17 **(Derivability in CKR).** *Given a CKR* $\mathfrak{K} = \langle \mathfrak{M}, \mathfrak{C}\rangle$ *over* $\langle \Sigma, \Gamma\rangle$ *and a set of obejct formulae* Φ*, an object formula* $\mathbf{d} : \phi$ *derivable from* $\mathfrak{K}$ *and* Φ *(denoted by* $\mathfrak{K}, \Phi \vdash \mathbf{d} : \phi$*) if it is defivable in the calculus given in Table 3 from* Ψ *for every* $\psi \in \Psi$ *one of the following is true:*

1. $\psi = \mathbf{e} : \chi$ *is an object formula such that* $\chi \in \mathcal{C}_{\mathbf{e}}$*;*
2. $\psi = \mu$ *is a meta formula such that* $\mathfrak{M} \models_{\mathrm{DL}} \mu$*;*

$$\frac{\mathbf{d}:\phi_1\ \dots\ \mathbf{d}:\phi_n\quad \{\phi_1\dots\phi_n\}\vdash_{\mathsf{DL}}\phi}{\mathbf{d}:\phi}\ \mathsf{LReas}$$

$$\frac{\mathbf{d}\preceq\mathbf{e}}{\mathbf{f}:C_{\mathbf{d}}\sqsubseteq\top_{\mathbf{e}}}\quad \frac{\mathbf{d}\preceq\mathbf{e}}{\mathbf{f}:\exists R_{\mathbf{d}}\top\sqsubseteq\top_{\mathbf{d}}}\quad \frac{\mathbf{d}\preceq\mathbf{e}}{\mathbf{f}:\top\sqsubseteq\forall R_{\mathbf{d}}\top_{\mathbf{d}}}\ \mathsf{Top}\qquad \frac{\mathbf{d}:\bot(a)}{\mathbf{e}:\top\sqsubseteq\bot}\ \mathsf{Bot}$$

$$\frac{\mathbf{e}:\phi@\mathbf{d}\quad \mathbf{e}:\top_{\mathbf{d}}(a_1)\ \cdots\ \mathbf{e}:\top_{\mathbf{d}}(a_n)\quad \mathbf{d}\preceq\mathbf{e}}{\mathbf{d}:\phi}\ \mathsf{Push}\qquad \frac{\mathbf{d}:\phi\quad \mathbf{d}\preceq\mathbf{e}}{\mathbf{e}:\phi@\mathbf{d}}\ \mathsf{Pop}$$

$$\frac{\mathbf{d}:A\sqcup B(x)\quad \begin{array}{c}[\mathbf{d}:A(x)]\\ \mathbf{e}:\phi\end{array}\quad \begin{array}{c}[\mathbf{d}:B(x)]\\ \mathbf{e}:\phi\end{array}}{\mathbf{e}:\phi}\ \sqcup\mathsf{E}$$

$$\frac{\mathbf{d}:\exists R.A(x)\quad \begin{array}{c}[\mathbf{d}:R(x,y),\ \mathbf{d}:A(y)]\\ \mathbf{e}:\phi\end{array}}{\mathbf{e}:\phi}\ \exists\mathsf{E}$$

$$\frac{\mathbf{d}:(\geq n)R.A(x)\quad \begin{array}{c}[\mathbf{d}:y_i\neq y_j]\quad [\mathbf{d}:R(x,y_i)]\\ [\mathbf{d}:A(y_i)]_{1\leq i\neq j\leq n}\\ \mathbf{e}:\phi\end{array}}{\mathbf{e}:\phi}\ (\geqslant n)\mathsf{E}$$

Restrictions: **1)** LReas can be applied if every individual occurring in ϕ occurs in a ϕ_i for some $1\leq i\leq n$; **2)** in the Push rule the individuals $a_1,\dots,a_n$ are assumed to be all individuals occuring in ϕ; **3)** $\exists$E can be applied if y does not occur in A, ϕ or any assumption different from $R(x,y)$ and $A(y)$ on which $\mathbf{e}:\phi$ depends; **4)** $(\geqslant n)$E can be applied if none of the constants y_i, $1\leq i\leq n$ occurs in A, ϕ or any assumption different from $R(x,y_i)$, $A(x)$, on which $\mathbf{e}:\phi$ depends.

Table 3. CKR inference rules

3. $\psi\in\Phi$.

Instead of $\mathfrak{K},\emptyset\vdash\mathbf{d}:\phi$ we simply write $\mathfrak{K}\vdash\mathbf{d}:\phi$. Even if ND derivations are formally defined as trees, we will often present them as a sequence of derivation steps. This can be naturally achieved, we only have to track the set of premises from which the resulting formula in each step is derived.

The first main result of this work is presented in Theorem 1 where the ND calculus of Table 3 is showed to be a sound and complete characterization of logical consequence in CKR. In other words, the calculus rules show us how logical consequence is propagated between contexts in a CKR knowledge base.

Theorem 1 (Soundness and Completeness). $\mathfrak{K}\vdash\mathbf{d}:\phi$ *if and only if* $\mathfrak{K}\models\mathbf{d}:\phi$.

Let us show some examples of deductions in the CKR. Example 1 shows how knowledge is propagated from $\mathcal{C}_{\mathbf{wc}}$ to $\mathcal{C}_{\mathbf{i}}$ via the common super-context $\mathcal{C}_{\mathbf{f}}$, and Example 2 shows how knowledge is propagated from $\mathcal{C}_{\mathbf{wn}}$ to $\mathcal{C}_{\mathbf{f}}$ via the common sub-context $\mathcal{C}_{\mathbf{wc}}$. Finally Example 3 shows how contradicting knowledge can coexist in different separated context.

Example 1. The following deduction shows how the subsumption $\mathbf{wc}:\mathsf{WChamp}\sqsubseteq\mathsf{Player}$ propagates from the FIFA WC context $\mathcal{C}_{\mathbf{wc}}$ to the Italian National League context $\mathcal{C}_{\mathbf{i}}$. Notice that the result of this deduction, i.e., $\mathbf{i}:\mathsf{WChamp}_{\mathbf{wc}}\sqsubseteq\mathsf{Player}_{\mathbf{wc}}$, in the context $\mathcal{C}_{\mathbf{i}}$ is weaker than the premise as it holds only on the set of players of the Italian National League. In other words, the knowledge shifting from $\mathcal{C}_{\mathbf{wc}}$ to $\mathcal{C}_{\mathbf{i}}$ is limited by the domain of interpretation of $\mathcal{C}_{\mathbf{i}}$.

(1)	$\mathbf{wc} : \mathsf{WChamp} \sqsubseteq \mathsf{Player}$	premise
(2)	$\mathbf{f} : (\mathsf{WChamp} \sqsubseteq \mathsf{Player})@\mathbf{wc}$	Pop, $\mathbf{wc} \preceq \mathbf{f}$
(3)	$\mathbf{f} : \mathsf{WChamp}_{\mathbf{wc}} \sqsubseteq \mathsf{Player}_{\mathbf{wc}}$	by @
(4)	$\mathbf{f} : \mathsf{WChamp}_{\mathbf{wc}} \sqcap \top_{\mathbf{i}} \sqsubseteq \mathsf{Player}_{\mathbf{wc}} \sqcap \top_{\mathbf{i}}$	LReas
(5)	$\mathbf{f} : (\mathsf{WChamp}_{\mathbf{wc}} \sqsubseteq \mathsf{Player}_{\mathbf{wc}})@\mathbf{i}$	by @
(6)	$\mathbf{i} : \mathsf{WChamp}_{\mathbf{wc}} \sqsubseteq \mathsf{Player}_{\mathbf{wc}}$	Push, $\mathbf{i} \preceq \mathbf{f}$

Example 2. The following deduction shows how $\mathbf{wn} : \mathsf{Player}_{\mathbf{f}} \sqsubseteq \mathsf{Pro}$ (i.e., every football player mentioned in the world news is a professional) propagates from $\mathcal{C}_{\mathbf{wn}}$ to $\mathcal{C}_{\mathbf{f}}$, trough the common sub-context $\mathcal{C}_{\mathbf{wc}}$.

(1)	$\mathbf{wn} : \mathsf{Player}_{\mathbf{f}} \sqsubseteq \mathsf{Pro}$	premise
(2)	$\mathbf{wn} : (\mathsf{Player}_{\mathbf{f}} \sqsubseteq \mathsf{Pro})@\mathbf{wn}$	Pop, $\mathbf{wn} \preceq \mathbf{wn}$
(3)	$\mathbf{wn} : \mathsf{Player}_{\mathbf{f}} \sqsubseteq \mathsf{Pro}_{\mathbf{wn}}$	by @
(4)	$\mathbf{wn} : \mathsf{Player}_{\mathbf{f}} \sqcap \top_{\mathbf{wc}} \sqsubseteq \mathsf{Pro}_{\mathbf{wn}} \sqcap \top_{\mathbf{wc}}$	by LReas
(5)	$\mathbf{wc} : \mathsf{Player}_{\mathbf{f}} \sqsubseteq \mathsf{Pro}_{\mathbf{wn}}$	Push, $\mathbf{wc} \preceq \mathbf{wn}$
(6)	$\mathbf{f} : \mathsf{Player}_{\mathbf{f}} \sqcap \top_{\mathbf{wc}} \sqsubseteq \mathsf{Pro}_{\mathbf{wn}} \sqcap \top_{\mathbf{wc}}$	Pop, $\mathbf{wc} \preceq \mathbf{f}$
(7)	$\mathbf{f} : \mathsf{Player}_{\mathbf{f}} \sqcap \top_{\mathbf{wc}} \sqsubseteq \mathsf{Pro}_{\mathbf{wn}}$	LReas

Notice that we did not infer that $\mathbf{f} : \mathsf{Player}_{\mathbf{f}} \sqsubseteq \mathsf{Pro}_{\mathbf{wn}}$, i.e., that every Player of football is a professional player in the world news, but the fact that this subsumption holds only on the players of the FIFA world cup domain.

Example 3. Suppose that the Italian News context $\mathcal{C}_{\mathbf{in}}$ contains the facts that Rooney does not take part to the Italian league in 2010, i.e., $\neg\top_{\mathbf{i}}(\mathsf{Rooney})$, and that he is not considered a good football player, i.e., $\neg\mathsf{GoodPlayer}_{\mathbf{f}}(\mathsf{Rooney})$. Suppose also that the world news context $\mathcal{C}_{\mathbf{wn}}$ contains the opposite evaluation, i.e. $\mathsf{GoodPlayer}_{\mathbf{f}}(\mathsf{Rooney})$. In the CKR of Figure 2, these two contradicting statements do not necessarily lead to inconsistency. Indeed, to derive inconsistency one has to find a context where to combine the two contradicting facts. However, to transfer the facts $\mathbf{wn} : \mathsf{GoodPlayer}_{\mathbf{f}}(\mathsf{Rooney})$ and $\mathbf{in} : \neg\mathsf{GoodPlayer}_{\mathbf{f}}(\mathsf{Rooney})$ into a common context, one have to pass through $\mathcal{C}_{\mathbf{i}}$. But the fact that Rooney is not an individual of $\mathcal{C}_{\mathbf{i}}$ disables any inference about Rooney in $\mathcal{C}_{\mathbf{i}}$. Model-theoretically we admit CKR models where $\mathsf{Rooney}^{\mathcal{I}_{\mathbf{wn}}} \neq \mathsf{Rooney}^{\mathcal{I}_{\mathbf{in}}}$.

5 Decidability and Complexity

Decidability of CKR entailment is proved indirectly by embedding a CKR into a single DL knowledge base. To do this we reuse the notion of embedding between DL knowledge bases as previously defined. First we need a vocabulary that is robust enough to keep track of all semantic relations inside a CKR knowledge base. Since each qualified symbol $X_{\mathbf{d}}$ may have different meaning in different contexts, we need to introduce one version $X_{\mathbf{d}}^{\mathbf{e}}$ of the symbol per each context $C_{\mathbf{e}}$. We know from the semantics that non-qualified concept and role symbols have the same meaning as if qualified with respect to the context where they appear. In addition, also constants may possibly have different meaning in different contexts, therefore for each constant a we introduce a version $a^{\mathbf{e}}$ for each context $\mathcal{C}_{\mathbf{e}}$.

Definition 18 (Transformed vocabulary $\#(\Gamma, \Sigma)$). *Given a meta-vocabulary Γ with the respective dimensional space $\mathfrak{D}_\Gamma$, and given an object-vocabulary $\Sigma = N_{\mathrm{C}} \uplus N_{\mathrm{R}} \uplus N_{\mathrm{I}}$, let us define a DL-vocabulary $\#(\Gamma, \Sigma) = {N_{\mathrm{C}}}^{\#(\Gamma,\Sigma)} \uplus {N_{\mathrm{R}}}^{\#(\Gamma,\Sigma)} \uplus {N_{\mathrm{I}}}^{\#(\Gamma,\Sigma)}$ such that:*

1. ${N_{\mathrm{C}}}^{\#(\Gamma,\Sigma)} = \{A_{\mathbf{d}}^{\mathbf{e}} \mid A \in N_{\mathrm{C}} \wedge \mathbf{d}, \mathbf{e} \in \mathfrak{D}_\Gamma\}$;
2. ${N_{\mathrm{R}}}^{\#(\Gamma,\Sigma)} = \{R_{\mathbf{d}}^{\mathbf{e}} \mid R \in N_{\mathrm{R}} \wedge \mathbf{d}, \mathbf{e} \in \mathfrak{D}_\Gamma\}$;
3. ${N_{\mathrm{I}}}^{\#(\Gamma,\Sigma)} = \{a^{\mathbf{e}} \mid a \in N_{\mathrm{I}} \wedge \mathbf{e} \in \mathfrak{D}_\Gamma\}$.

For each full dimensional vector $\mathbf{d} \in \Gamma$, we now define an operator $(\cdot)\#d$ which will be based on an embedding $g_{\mathbf{d}}$ of Σ into $\#(\Gamma, \Sigma)$.

Definition 19 ($\#\mathbf{d}$ operator). *For every full dimensional vector $\mathbf{d}$, $(\cdot)\#\mathbf{d}$ is defined as $g_{\mathbf{d}}^*(\cdot)$, where $g_{\mathbf{d}}$ is an embedding from Σ to $\#(\Gamma, \Sigma)$ defined as follows:*

- $g_{\mathbf{d}}(a) = a^{\mathbf{d}}$ *for every individual* a;
- $g_{\mathbf{d}}(X_{\mathbf{d}'_{\mathrm{B}}}) = X_{\mathbf{d}'_{\mathrm{B}}+\mathbf{d}}^{\mathbf{d}}$ *for every concept/role* $X_{\mathbf{d}'_{\mathrm{B}}}$;
- $\Sigma_c = \Sigma$; $\Sigma_e = \emptyset$.

Observe that in this case the split of Σ into the internal part Σ_c and the external part Σ_e is different: $\Sigma_c = \Sigma$ and $\Sigma_e = \emptyset$. This is in line with the fact that the single DL knowledge base which is the result of the transformation has complete information about all symbols in every context. That is, in terms of CKR we could see the transformed knowledge base as if placed on top of all contexts with respect to $\prec$. Using the $(\cdot)\#\mathbf{d}$ operator we now transform a CKR knowledge base $\mathfrak{K}$ into a DL theory $\#(\mathfrak{K})$ over $\#(\Gamma, \Sigma)$.

Definition 20 (Transformed CKR $\#(\mathfrak{K})$). *For every CKR $\mathfrak{K}$ over (Γ, Σ), let $\#(\mathfrak{K})$ be a DL knowledge base over $\#(\Gamma, \Sigma)$ such that for every individual a, concept A, role R, concept/role X (all atomic), and for every two contexts $\mathbf{d}, \mathbf{e}, \mathbf{f}$ it contains the following axioms:*[6]

1. $\top_{\mathbf{d}}^{\mathbf{f}} \sqsubseteq \top_{\mathbf{e}}^{\mathbf{f}}$ *for* $\mathbf{d} \prec \mathbf{e}$;
2. $A_{\mathbf{e}}^{\mathbf{d}} \sqsubseteq \top_{\mathbf{e}}^{\mathbf{d}}$;
3. $\exists R_{\mathbf{e}}^{\mathbf{d}}.\top \sqsubseteq \top_{\mathbf{e}}^{\mathbf{d}}$ *and* $\top \sqsubseteq \forall R_{\mathbf{e}}^{\mathbf{d}}.\top_{\mathbf{e}}^{\mathbf{d}}$;
4. $a^{\mathbf{d}} = a^{\mathbf{e}}$, *if* $\mathbf{d} \prec \mathbf{e}$;
6. $X_{\mathbf{d}}^{\mathbf{d}} \equiv X_{\mathbf{d}}^{\mathbf{e}}$ *if* $\mathbf{d} \prec \mathbf{e}$;
7. $A_{\mathbf{f}}^{\mathbf{d}} \equiv A_{\mathbf{f}}^{\mathbf{e}} \sqcap \top_{\mathbf{d}}^{\mathbf{d}}$ *if* $\mathbf{d} \prec \mathbf{e}$;
8. $I_{\mathbf{d}}^{\mathbf{d}} \circ R_{\mathbf{f}}^{\mathbf{e}} \circ I_{\mathbf{d}}^{\mathbf{d}} \sqsubseteq R_{\mathbf{f}}^{\mathbf{d}}$ *and* $R_{\mathbf{f}}^{\mathbf{d}} \sqsubseteq R_{\mathbf{f}}^{\mathbf{e}}$, *if* $\mathbf{d} \prec \mathbf{e}$;

[6] The numbering in the list below has a gap in order to maintain correspondence with Definition 9.

9. $\phi\#\mathbf{d}$ *for all* $\phi \in \mathrm{K}(\mathcal{C})$ *and* $\mathbf{d} = \dim(\mathcal{C})$.

Thanks to the transformation we are now able to check $\mathbf{d}$-satisfiability of a CKR knowledge base $\mathfrak{K}$ by checking for satisfiability/entailment in $\#(\mathfrak{K})$. This is formally established by the following two lemmata.

Lemma 2. *If* $\mathfrak{K}$ *is* $\mathbf{d}$*-satisfiable then* $\#(\mathfrak{K})$ *is satisfiable.*

Lemma 3. *If there is a* $\mathbf{d}$ *such that* $\#(\mathfrak{K}) \not\models \top_{\mathbf{d}}^{\mathbf{d}} \sqsubseteq \bot$*, then* $\mathfrak{K}$ *is* $\mathbf{d}$*-satisfiable.*

For any given CKR $\mathfrak{K}$, $\#(\mathfrak{K})$ is a regular $\mathcal{SROIQ}$ knowledge base. According [13], if a $\mathcal{SROIQ}$ KB is $\precsim$-stratified then subsumption is decidable. Hence we can prove decidability only for CKRs that are transformed into $\precsim$-stratified KBs. We say that a CKR is $\precsim$-stratified if the set of RIA $\bigcup_{\mathbf{d}\in\mathfrak{D}_\Gamma}\{(R \sqsubseteq S)\#\mathbf{d} | R \sqsubseteq S \in \mathrm{K}(\mathcal{C}_{\mathbf{d}})\}$ is $\precsim$-stratified. Furthermore, the RIAs introduced in step 8 are not $\precsim$-stratified, but it suffices to add $I_{\mathbf{d}}^{\mathbf{d}} \circ R_{\mathbf{f}}^{\mathbf{e}} \sqsubseteq S_1$, $S_1 \circ I_{\mathbf{d}}^{\mathbf{d}} \sqsubseteq R_{\mathbf{f}}^{\mathbf{d}}$, $R_{\mathbf{f}}^{\mathbf{e}} \circ I_{\mathbf{d}}^{\mathbf{d}} \sqsubseteq S_2$, $I_{\mathbf{d}}^{\mathbf{d}} \circ S_2 \sqsubseteq R_{\mathbf{f}}^{\mathbf{d}}$ where S_1 and S_2 are new roles w.r.t. each pair $R_{\mathbf{f}}^{\mathbf{d}}$ and $R_{\mathbf{f}}^{\mathbf{e}}$. Hence if a $\mathfrak{K}$ is $\precsim$-stratified, than there is a $\precsim$-stratified $\mathcal{SROIQ}$ KB equivalent to $\#(\mathfrak{K})$. This allows us to prove the following theorem.

Theorem 2. *If* $\mathfrak{K}$ *is* $\precsim$*-stratified, then checking if* $\mathfrak{K} \models \mathbf{d} : \phi$ *is decidable with the complexity upper-bound of* 2NEXPTIME.

6 Related Work

Both aRDF [23] and Context Description Framework [14] extend RDF triples by an n-tuple of qualification attributes with partially ordered domains. Apart from CKR being on top of OWL 2, it differs from these approaches by qualifying whole theories and not each formula separately. This approach is more compact as usually the context is shared a by group of formulae.

An extension of RDFS to cope with context was proposed by [10] and further developed in [2]. A new predicate $\mathsf{isin}(c, \phi)$ is used to assert that the triple ϕ occurs in the context c. A set of operators to combine contexts ($c_1 \wedge c_1$, $c_1 \vee c_2$, $\neg c$) and to relate contexts ($c \Rightarrow c_2$, $c \rightarrow c_2$) is defined, making the approach particularly suited for manipulating contexts. Unfortunately, no sound and complete axiomatization or decision procedure was provided so far.

The contextual DL $\mathcal{ALC}_{\mathcal{ALC}}$ [15] is a multi-modal extension of the $\mathcal{ALC}$ DL with the contextual modal operator $[C]_r A$ representing "all objects of type A in all contexts of type C reachable from the current context via relation r." In both $\mathcal{ALC}_{\mathcal{ALC}}$ and CKR contextual structure is formalized in a meta-language separated from the domain language used to describe the domain. The main difference between CKR and $\mathcal{ALC}_{\mathcal{ALC}}$ is that CKR is more expressive in the object language ($\mathcal{SROIQ}$ vs. $\mathcal{ALC}$) but less expressive in the contextual assertions, allowing qualification of knowledge only w.r.t. individual contexts rather than context classes as in $\mathcal{ALC}_{\mathcal{ALC}}$. The effect of this choice is that in CKR the complexity of reasoning is the same as in the object language (i.e., 2NEXPTIME) while in $\mathcal{ALC}_{\mathcal{ALC}}$ the complexity jumps to 2EXPTIME compared to EXPTIME for $\mathcal{ALC}$.

On the semantic level, CKR is also related to approaches such as multi-context systems [9], distributed description logics [5], $\mathcal{E}$-connections [16], but especially approaches concerned with importing such as package-based description logics (P-DL) [3] and semantic imports [19]. In P-DL imports of symbols are implemented by relating the elements of interpretation domains with one-to-one mappings. The work of Pan et al. goes even closer to our approach by assuming that the interpretation domains of distinct ontologies may overlap. In both cases additional semantic constraints are introduced to support various desired properties of the importing paradigm.

In our current work, we use similar techniques however we use them to meet different goals. Borrowing the viewpoint of the semantic imports paradigm, we may observe that imports are implemented between the contexts of CKR, however, to various extents depending on the relation of the two contexts in question. If the contexts are directly related by the coverage, all information from the narrower context is accessible in the broader context using the technique similar to importing. On the other hand, the narrower of the two contexts may only access part of the other context's information. If two contexts are related indirectly, then the importing is even more limited. Thus we can see that similar techniques are being used in order to characterize a complex scenario of information reuse in accordance with the underlaying ideas of the AI theories of context which is carefully crafted in the semantic conditions asserted in CKR models.

7 Conclusion

CKR is a novel framework for representing contextual knowledge in the SW. We have provided a sound and complete axiomatization and we have shown that reasoning in CKR is decidable at no additional complexity costs. We plan to investigate on the formal properties of CKR with more tractable fragments of OWL 2, e.g., OWL-Horst. For this fragment, we have developed a prototype on top of Sesame 2 RDF triple store, where contexts have been naturally implemented with named graphs [6]. We also want to study a distributed tableaux based reasoning technique for CKR.

References

1. F. Baader, D. Calvanese, D. McGuinness, D. Nardi, and P. Patel-Schneider, editors. *The Description Logic Handbook*. Cambridge University Press, 2003.
2. J. Bao, J. Tao, and D.L. McGuinness. Context representation for the semantic web. In *Procs. of WebSci10*, 2010.
3. Jie Bao, George Voutsadakis, Giora Slutzki, and Vasant Honavar. Package-based description logics. In *Modular Ontologies: Concepts, Theories and Techniques for Knowledge Modularization*, volume 5445 of *LNCS*, pages 349–371. Springer, 2009.
4. M. Benerecetti, P. Bouquet, and C. Ghidini. On the Dimensions of Context Dependence. In *Perspectives on Contexts*. CSLI, 2007.
5. Alexander Borgida and Luciano Serafini. Distributed description logics: Assimilating information from peer sources. *J. Data Semantics*, 1:153–184, 2003.
6. J. Carroll, C. Bizer, P. Hayes, and P. Stickler. Named graphs, provenance and trust. In *WWW'05*. ACM, 2005.
7. C. C. Chang and H. Jerome Keisler. *Model Theory, Third ed.* North Holland, 1990.
8. L. Ding, T. Finin, Y. Peng, P. Pinheiro da Silva, and D. McGuinness. Tracking RDF graph provenance using RDF molecules. In *ISWC*, 2005.
9. F. Giunchiglia and L. Serafini. Multilanguage hierarchical logics, or: how we can do without modal logics. *Artif. Intell.*, 65(1):29–70, 1994.
10. R. Guha, R. McCool, and R. Fikes. Contexts for the semantic web. In *ISWC*, 2004.
11. Ian Horrocks, Oliver Kutz, and Ulrike Sattler. The even more irresistible $\mathcal{SROIQ}$. In *Procs. of the 10th International Conference on Principles of Knowledge Representation and Reasoning (KR 2006)*, pages 57–67. AAAI Press, 2006.
12. Y. Kazakov. $\mathcal{RIQ}$ and $\mathcal{SROIQ}$ are harder than $\mathcal{SHOIQ}$. In *KR*, 2008.
13. Y. Kazakov. An extension of complex role inclusion axioms in the description logic $\mathcal{SROIQ}$. In *IJCAR*, 2010.

14. O. Khriyenko and V. Terziyan. A framework for context sensitive metadata description. *IJSMO*, 1(2):154–164, 2006.
15. S. Klarman and V. Gutiérrez-Basulto. $\mathcal{ALC}_{\mathcal{ALC}}$: a context description logic. In *JELIA*, 2010.
16. Oliver Kutz, Carsten Lutz, Frank Wolter, and Michael Zakharyaschev. $\mathcal{E}$-connections of abstract description systems. *Artificial Intelligence*, 156(1):1–73, 2004.
17. H. C. Liao and C. C. Tu. A RDF and OWL-based temporal context reasoning model for smart home. *Inform. Tech. J.*, 6:1130–1138, 2007.
18. J. McCarthy. Notes on formalizing context. In *IJCAI*, 1993.
19. Jeff Z. Pan, Luciano Serafini, and Yuting Zhao. Semantic import: An approach for partial ontology reuse. In *Procs. of the 1st International Workshop on Modular Ontologies (WoMo-06)*, volume 232 of *CEUR WS*, Athens, Georgia, USA, 2006.
20. D. Prawitz. *Natural Deduction: A Proof-Theoretical Study*. Almquist and Wiksell, 1965.
21. H. Stoermer. Introducing context into semantic web knowledge bases. In *CAiSE DC*, 2006.
22. Stephan Tobies. *Complexity results and practical algorithms for logics in knowledge representation*. PhD thesis, RWTH Aachen, Germany, 2001.
23. O. Udrea, D. Recupero, and V. S. Subrahmanian. Annotated RDF. *ACM Trans. Comput. Log.*, 11(2):1–41, 2010.
24. W3C. *OWL 2 Web Ontology Language Document Overview*. W3C Recommendation, 2009.

A Proofs

A.1 Proof of Lemma 1

Lemma 1. *If $\mathcal{I}$ and $\mathcal{I}'$ comply with $f : \Sigma \to \Sigma'$, then, for every concept C, $C^{\mathcal{I}} = f^*(C)^{\mathcal{I}'}$, for every role R, $R^{\mathcal{I}} = f^*(R)^{\mathcal{I}'}$, and for every axiom ϕ, $\mathcal{I} \models \phi$ iff $\mathcal{I}' \models f^*(\phi)$.*

Let us have two DL-alphabets Σ and Σ', a DL-embedding $f : \Sigma \to \Sigma'$ and two respective DL-interpretations $\mathcal{I}$ and $\mathcal{I}'$ complying with f. From Definition 15 this implies the following four facts which we denote by (†):

1. $a^{\mathcal{I}} = f(a)^{\mathcal{I}'}$, for all individuals a of Σ;
2. $X^{\mathcal{I}} = f(X)^{\mathcal{I}'}$, for all symbols $X \in \Sigma_c$;
3. $A^{\mathcal{I}} = f(A)^{\mathcal{I}'} \cap f(\top)^{\mathcal{I}'}$, for all concepts $A \in \Sigma_e$;
4. $R^{\mathcal{I}} = f(R)^{\mathcal{I}'} \cap f(\top)^{\mathcal{I}'} \times f(\top)^{\mathcal{I}'}$, for all roles $R \in \Sigma_e$.

Let us first realize how the domain $\Delta^{\mathcal{I}}$ of $\mathcal{I}$ is embedded into the domain $\Delta^{\mathcal{I}'}$ of Σ'. Later in the proof we will denote this observation by (‡):

$$\Delta^{\mathcal{I}} = \top^{\mathcal{I}} = f(\top)^{\mathcal{I}'} \subseteq \Delta^{\mathcal{I}'}$$

The second equation is due to $\top \in \Sigma_c$ and from the fact that $\mathcal{I}$ and $\mathcal{I}'$ comply with the embedding f. The other two equations trivially follow from $\mathcal{I}$ and $\mathcal{I}'$ being DL-interpretations.

We will now prove that for every concept or role X it holds that $X^{\mathcal{I}} = f^*(X)^{\mathcal{I}'}$. The proof is by structural induction on X.

1. $X = A \in \Sigma_c$:

$$\begin{aligned} f^*(A)^{\mathcal{I}'} &= f(A)^{\mathcal{I}'} \text{ by definition of } f^* \\ &= A^{\mathcal{I}} \text{ from (†,2)} \end{aligned}$$

2. $X = A \in \Sigma_e$:

$$\begin{aligned} f^*(A)^{\mathcal{I}'} &= f(A) \sqcap f(\top)^{\mathcal{I}'} \text{ by the definition of } f^* \\ &= f(A)^{\mathcal{I}'} \cap f(\top)^{\mathcal{I}'} \text{ by the interpretation of } \sqcap \\ &= (A)^{\mathcal{I}} \text{ from (†,3)} \end{aligned}$$

3. $X = \neg C$:

$$\begin{aligned} f^*(\neg C)^{\mathcal{I}'} &= f(\top) \sqcap \neg f^*(C)^{\mathcal{I}'} \text{ by definition of } f^* \\ &= f(\top)^{\mathcal{I}'} \cap \neg f^*(C)^{\mathcal{I}'} \text{ by interpretation of } \sqcap \\ &= f(\top)^{\mathcal{I}'} \cap (\Delta^{\mathcal{I}'} \setminus f^*(C)^{\mathcal{I}'}) \text{ by interpretation of } \neg \\ &= f(\top)^{\mathcal{I}'} \setminus f^*(C)^{\mathcal{I}'} \text{ due to } f(\top)^{\mathcal{I}'} \subseteq \Delta^{\mathcal{I}'} \\ &= \Delta^{\mathcal{I}} \setminus f^*(C)^{\mathcal{I}'} \text{ from (‡)} \\ &= \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \text{ by induction} \\ &= \neg C^{\mathcal{I}} \text{ by interpretation of } \neg \end{aligned}$$

4. $X = C \sqcap D$:

$$\begin{aligned} f^*(C \sqcap D)^{\mathcal{I}'} &= f^*(C) \sqcap f^*(D)^{\mathcal{I}'} \text{ by definition of } f^* \\ &= f^*(C)^{\mathcal{I}'} \cap f^*(D)^{\mathcal{I}'} \text{ by interpretation of } \sqcap \\ &= C^{\mathcal{I}} \cap D^{\mathcal{I}} \text{ by induction} \\ &= C \sqcap D^{\mathcal{I}} \text{ by interpretation of } \sqcap \end{aligned}$$

5. $X = C \sqcup D$:

$$\begin{aligned} f^*(C \sqcup D)^{\mathcal{I}'} &= f^*(C) \sqcup f^*(D)^{\mathcal{I}'} \text{ by definition of } f^* \\ &= f^*(C)^{\mathcal{I}'} \cup f^*(D)^{\mathcal{I}'} \text{ by interpretation of } \sqcup \\ &= C^{\mathcal{I}} \cup D^{\mathcal{I}} \text{ by induction} \\ &= C \sqcup D^{\mathcal{I}} \text{ by interpretation of } \sqcup \end{aligned}$$

6. $X = \exists R.C$ and $R \in \Sigma_c$:

$$\begin{aligned} f^*(\exists R.C)^{\mathcal{I}'} &= \exists f(R).f^*(C)^{\mathcal{I}'} \text{ by definition of } f^* \text{ and } R \in \Sigma_c \\ &= \{x \in \Delta^{\mathcal{I}'} \mid \exists y\ (x,y) \in f(R)^{\mathcal{I}'} \wedge y \in f^*(C)^{\mathcal{I}'}\} \text{ by interpretation of } \exists \\ &= \{x \in \Delta^{\mathcal{I}'} \mid \exists y\ (x,y) \in f(R)^{\mathcal{I}'} \wedge y \in C^{\mathcal{I}}\} \text{ by induction} \\ &= \{x \in \Delta^{\mathcal{I}'} \mid \exists y\ (x,y) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\} \text{ from } (\dagger,2) \\ &= \{x \in \Delta^{\mathcal{I}} \mid \exists y\ (x,y) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\} \text{ by } R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \text{ and } \Delta^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}'} \\ &= \exists R.C^{\mathcal{I}} \text{ by definition of } \exists \end{aligned}$$

7. $X = \exists R.C$ and $R \in \Sigma_e$:

$$\begin{aligned} f^*(\exists R.C)^{\mathcal{I}'} &= f(\top) \sqcap \exists f(R).f^*(C)^{\mathcal{I}'} \text{ by definition of } f^* \text{ and } R \in \Sigma_e \\ &= f(\top)^{\mathcal{I}'} \cap \exists f(R).f^*(C)^{\mathcal{I}'} \text{ by interpretation of } \sqcap \\ &= \Delta^{\mathcal{I}} \cap \exists f(R).f^*(C)^{\mathcal{I}'} \text{ from } (\ddagger) \\ &= \Delta^{\mathcal{I}} \cap \{x \in \Delta^{\mathcal{I}'} \mid \exists y\ (x,y) \in f(R)^{\mathcal{I}'} \wedge y \in f^*(C)^{\mathcal{I}'}\} \text{ by interpretation of } \exists \\ &= \Delta^{\mathcal{I}} \cap \{x \in \Delta^{\mathcal{I}'} \mid \exists y\ (x,y) \in f(R)^{\mathcal{I}'} \wedge y \in C^{\mathcal{I}}\} \text{ by induction} \\ &= \{x \in \Delta^{\mathcal{I}} \mid \exists y\ (x,y) \in f(R)^{\mathcal{I}'} \cap \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\} \text{ since } \mathcal{C}^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \\ &= \{x \in \Delta^{\mathcal{I}} \mid \exists y\ (x,y) \in f(R)^{\mathcal{I}'} \cap f(\top)^{\mathcal{I}'} \times f(\top)^{\mathcal{I}'} \wedge y \in C^{\mathcal{I}}\} \text{ from } (\ddagger) \\ &= \{x \in \Delta^{\mathcal{I}} \mid \exists y\ (x,y) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\} \text{ from } (\dagger,4) \\ &= \exists R.C^{\mathcal{I}} \text{ by interpretation of } \exists \end{aligned}$$

8. $X = \forall R.C$ and $R \in \Sigma_c$:

$$f^*(\forall R.C)^{\mathcal{I}'} = f(\top) \sqcap \forall f(R).f^*(C)^{\mathcal{I}'} \text{ by definition of } f^* \text{ and } R \in \Sigma_c$$
$$= f(\top)^{\mathcal{I}'} \cap \{x \in \Delta^{\mathcal{I}'} \mid \forall y\ (x,y) \in f(R)^{\mathcal{I}'} \to y \in f^*(C)^{\mathcal{I}'}\} \text{ by } \sqcap \text{ and } \forall$$
$$= \Delta^{\mathcal{I}} \cap \{x \in \Delta^{\mathcal{I}'} \mid \forall y\ (x,y) \in f(R)^{\mathcal{I}'} \to y \in f^*(C)^{\mathcal{I}'}\} \text{ by } (\ddagger)$$
$$= \{x \in \Delta^{\mathcal{I}} \mid \forall y\ (x,y) \in f(R)^{\mathcal{I}'} \to y \in f^*(C)^{\mathcal{I}'}\} \text{ as } \Delta^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}'} \text{ by } (\ddagger)$$
$$= \{x \in \Delta^{\mathcal{I}} \mid \forall y\ (x,y) \in R^{\mathcal{I}} \to y \in f^*(C)^{\mathcal{I}'}\} \text{ by } (\dagger,2)$$
$$= \{x \in \Delta^{\mathcal{I}} \mid \forall y\ (x,y) \in R^{\mathcal{I}} \to y \in C^{\mathcal{I}}\} \text{ by induction}$$
$$= \forall R.C^{\mathcal{I}} \text{ by interpretation of } \forall$$

9. $X = \forall R.C$ and $R \in \Sigma_e$:

$$f^*(\forall R.C)^{\mathcal{I}'} = f(\top) \sqcap \forall f(R).(\neg f(\top) \sqcup f^*(C))^{\mathcal{I}'} \text{ by definition of } f^* \text{ and } R \in \Sigma_e$$
$$= f(\top)^{\mathcal{I}'} \cap \{x \in \Delta^{\mathcal{I}'} \mid \forall y\ (x,y) \in f(R)^{\mathcal{I}'} \to y \notin f(\top)^{\mathcal{I}'} \vee y \in f^*(C)^{\mathcal{I}'}\} \text{ by } \forall, \sqcap, \sqcup, \neg$$
$$= \Delta^{\mathcal{I}} \cap \{x \in \Delta^{\mathcal{I}'} \mid \forall y\ (x,y) \in f(R)^{\mathcal{I}'} \to y \notin \Delta^{\mathcal{I}} \vee y \in f^*(C)^{\mathcal{I}'}\} \text{ by } (\ddagger)$$
$$= \{x \in \Delta^{\mathcal{I}} \mid \forall y\ (x,y) \in f(R)^{\mathcal{I}'} \to y \notin \Delta^{\mathcal{I}} \vee y \in f^*(C)^{\mathcal{I}'}\} \text{ as } \Delta^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}'} \text{ by } (\ddagger)$$
$$= \{x \in \Delta^{\mathcal{I}} \mid \forall y\ (x,y) \in f(R)^{\mathcal{I}'} \to y \notin \Delta^{\mathcal{I}} \vee y \in C^{\mathcal{I}}\} \text{ by induction}$$
$$= \{x \in \Delta^{\mathcal{I}} \mid \forall y\ (x,y) \in f(R)^{\mathcal{I}} \cap \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to y \in C^{\mathcal{I}}\} \text{ due to } C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$$
$$= \{x \in \Delta^{\mathcal{I}} \mid \forall y\ (x,y) \in f(R)^{\mathcal{I}'} \cap f(\top)^{\mathcal{I}'} \times f(\top)^{\mathcal{I}'} \to y \in C^{\mathcal{I}}\} \text{ from } (\ddagger)$$
$$= \{x \in \Delta^{\mathcal{I}} \mid \forall y\ (x,y) \in R^{\mathcal{I}} \to y \in C^{\mathcal{I}}\} \text{ from } (\dagger,4)$$
$$= (\forall R.C)^{\mathcal{I}} \text{ by definition of } \forall$$

10. $X = \geqslant nR.C$ and $R \in \Sigma_c$:

$$f^*(\geqslant nR.C)^{\mathcal{I}'} = \geqslant nf(R).f^*(C)^{\mathcal{I}'} \text{ by definition of } f^* \text{ and } R \in \Sigma_c$$
$$= \{x \in \Delta^{\mathcal{I}'} \mid \exists_{1 \le i \le n}\, y_i\ (x,y_i) \in f(R)^{\mathcal{I}'} \wedge y_i \in f^*(C)^{\mathcal{I}'}\} \text{ interpretation of } \geqslant n$$
$$= \{x \in \Delta^{\mathcal{I}'} \mid \exists_{1 \le i \le n}\, y_i\ (x,y_i) \in f(R)^{\mathcal{I}'} \wedge y_i \in C^{\mathcal{I}}\} \text{ by induction}$$
$$= \{x \in \Delta^{\mathcal{I}'} \mid \exists_{1 \le i \le n}\, y_i\ (x,y_i) \in R^{\mathcal{I}} \wedge y_i \in C^{\mathcal{I}}\} \text{ by } (\dagger,2)$$
$$= \{x \in \Delta^{\mathcal{I}} \mid \exists_{1 \le i \le n}\, y_i\ (x,y_i) \in R^{\mathcal{I}} \wedge y_i \in C^{\mathcal{I}}\} \text{ as } R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}, \Delta^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}'}$$
$$= \geqslant nR.C^{\mathcal{I}} \text{ by interpretation of } \geqslant n$$

11. $X = \geqslant nR.C$ and $R \in \Sigma_e$:

$$f^*(\geqslant nR.C)^{\mathcal{I}'} = f(\top) \sqcap \geqslant nf(R).f^*(C)^{\mathcal{I}'} \text{ by definition of } f^* \text{ and } R \in \Sigma_e$$
$$= f(\top)^{\mathcal{I}'} \cap \{x \in \Delta^{\mathcal{I}'} \mid \exists_{1 \le i \le n}\, y_i\ (x,y_i) \in f(R)^{\mathcal{I}'} \wedge y_i \in f^*(C)^{\mathcal{I}'}\} \text{ by } \sqcap, \geqslant n$$
$$= \Delta^{\mathcal{I}} \cap \{x \in \Delta^{\mathcal{I}'} \mid \exists_{1 \le i \le n}\, y_i\ (x,y_i) \in f(R)^{\mathcal{I}'} \wedge y_i \in f^*(C)^{\mathcal{I}'}\} \text{ by } (\ddagger)$$
$$= \{x \in \Delta^{\mathcal{I}} \mid \exists_{1 \le i \le n}\, y_i\ (x,y_i) \in f(R)^{\mathcal{I}'} \wedge y_i \in f^*(C)^{\mathcal{I}'}\} \text{ as } \Delta^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}'} \text{ by } (\ddagger)$$
$$= \{x \in \Delta^{\mathcal{I}'} \mid \exists_{1 \le i \le n}\, y_i\ (x,y_i) \in f(R)^{\mathcal{I}'} \wedge y_i \in C^{\mathcal{I}}\} \text{ by induction}$$
$$= \{x \in \Delta^{\mathcal{I}} \mid \exists_{1 \le i \le n}\, y_i\ (x,y_i) \in f(R)^{\mathcal{I}'} \cap \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \wedge y_i \in C^{\mathcal{I}}\} \text{ as } C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$$
$$= \{x \in \Delta^{\mathcal{I}} \mid \exists_{1 \le i \le n}\, y_i\ (x,y_i) \in f(R)^{\mathcal{I}'} \cap f(\top)^{\mathcal{I}'} \times f(\top)^{\mathcal{I}'} \wedge y_i \in C^{\mathcal{I}}\} \text{ by } (\ddagger)$$
$$= \{x \in \Delta^{\mathcal{I}} \mid \exists_{1 \le i \le n}\, y_i\ (x,y_i) \in R^{\mathcal{I}} \wedge y_i \in C^{\mathcal{I}}\} \text{ by } (\dagger,4)$$
$$= \geqslant nR.C^{\mathcal{I}} \text{ by definition of } \geqslant n$$

12. $X = \exists R.\text{Self}$ and $R \in \Sigma_c$:

$$\begin{aligned} f^*(\exists R.\text{Self})^{\mathcal{I}'} &= \exists f(R).\text{Self}^{\mathcal{I}'} \text{ by definition of } f^* \text{ and } R \in \Sigma_c \\ &= \{x \in \Delta^{\mathcal{I}'} \mid (x,x) \in f(R)^{\mathcal{I}'}\} \text{ by interpretation of } \exists R.\text{Self} \\ &= \{x \in \Delta^{\mathcal{I}'} \mid (x,x) \in R^{\mathcal{I}}\} \text{ by } (\dagger,2) \\ &= \{x \in \Delta^{\mathcal{I}} \mid (x,x) \in R^{\mathcal{I}}\} \text{ due to } R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \text{ and } \Delta^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}'} \\ &= \exists R.\text{Self}^{\mathcal{I}} \text{ by definition of } \exists R.\text{Self} \end{aligned}$$

13. $X = \exists R.\text{Self}$ and $R \in \Sigma_e$:

$$\begin{aligned} f^*(\exists R.\text{Self})^{\mathcal{I}'} &= f(\top) \sqcap \exists f(R).\text{Self}^{\mathcal{I}'} \text{ by the definition of } f^* \text{ with } R \in \Sigma_e \\ &= f(\top)^{\mathcal{I}'} \cap \{x \in \Delta^{\mathcal{I}'} \mid (x,x) \in f(R)^{\mathcal{I}'}\} \text{ by interpretation of } \sqcap \text{ and } \exists R.\text{Self} \\ &= \Delta^{\mathcal{I}} \cap \{x \in \Delta^{\mathcal{I}'} \mid (x,x) \in f(R)^{\mathcal{I}'}\} \text{ from } (\ddagger) \\ &= \{x \in \Delta^{\mathcal{I}} \mid (x,x) \in f(R)^{\mathcal{I}'}\} \text{ as } \Delta^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}'} \text{ by } (\ddagger) \\ &= \{x \in \Delta^{\mathcal{I}} \mid (x,x) \in f(R)^{\mathcal{I}'} \cap \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}\} \\ &= \{x \in \Delta^{\mathcal{I}} \mid (x,x) \in f(R)^{\mathcal{I}'} \cap f(\top)^{\mathcal{I}'} \times f(\top)^{\mathcal{I}'}\} \text{ from } (\ddagger) \\ &= \{x \in \Delta^{\mathcal{I}} \mid (x,x) \in R^{\mathcal{I}}\} \text{ from } (\dagger,4) \\ &= \exists R.\text{Self}^{\mathcal{I}} \text{ by definition of } \exists R.\text{Self} \end{aligned}$$

14. $X = \{a\}$:

$$\begin{aligned} f^*(\{a\})^{\mathcal{I}'} &= \{f(a)\}^{\mathcal{I}'} \text{ by definition of } f^* \\ &= \{f(a)^{\mathcal{I}'}\} \text{ by interpretation of nominals} \\ &= \{a^{\mathcal{I}}\} \text{ from } (\dagger,1) \\ &= \{a\}^{\mathcal{I}} \text{ by interpretation of nominals} \end{aligned}$$

15. $X = \leqslant nR.C$:

$$\begin{aligned} f^*(\leqslant nR.C)^{\mathcal{I}'} &= f(\top) \sqcap \leqslant n f(R).f^*(C)^{\mathcal{I}'} \text{ by definition of } f^* \\ &= f(\top)^{\mathcal{I}'} \cap \{x \in \Delta^{\mathcal{I}'} \mid \sharp\{y|(x,y) \in f(R)^{\mathcal{I}'} \wedge y \in f^*(C)^{\mathcal{I}'}\} \leq n\} \text{ by } \sqcap, \leqslant n \\ &= \Delta^{\mathcal{I}} \cap \{x \in \Delta^{\mathcal{I}'} \mid \sharp\{y|(x,y) \in f(R)^{\mathcal{I}'} \wedge y \in f^*(C)^{\mathcal{I}'}\} \leq n\} \text{ by } (\ddagger) \\ &= \{x \in \Delta^{\mathcal{I}} \mid \sharp\{y|(x,y) \in f(R)^{\mathcal{I}'} \wedge y \in f^*(C)^{\mathcal{I}'}\} \leq n\} \text{ as } \Delta^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}'} \text{ by } (\ddagger) \\ &= \{x \in \Delta^{\mathcal{I}} \mid \sharp\{y|(x,y) \in f(R)^{\mathcal{I}'} \wedge y \in C^{\mathcal{I}}\} \leq n\} \text{ by induction} \\ &= \{x \in \Delta^{\mathcal{I}} \mid \sharp\{y|(x,y) \in f(R)^{\mathcal{I}'} \cap \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\} \leq n\} \text{ as } C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \\ &= \{x \in \Delta^{\mathcal{I}} \mid \sharp\{y|(x,y) \in f(R)^{\mathcal{I}'} \cap f(\top)^{\mathcal{I}'} \times f(\top)^{\mathcal{I}'} \wedge y \in C^{\mathcal{I}}\} \leq n\} \text{ by } (\ddagger) \\ &= \{x \in \Delta^{\mathcal{I}} \mid \sharp\{y|(x,y) \in R)^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\} \leq n\} \text{ from } (\dagger,4) \\ &= \leqslant nR.C^{\mathcal{I}} \text{ by interpretation of } \leqslant n \end{aligned}$$

16. $X = R \in \Sigma_c$:

$$\begin{aligned} f^*(R)^{\mathcal{I}'} &= f(R)^{\mathcal{I}'} \text{ by definition of } f^* \\ &= R^{\mathcal{I}} \text{ by } (\dagger,2) \end{aligned}$$

17. $X = R \in \Sigma_e$:

$$\begin{aligned}
f^*(R)^{\mathcal{I}'} &= f(I) \circ f(R) \circ f(I)^{\mathcal{I}'} \text{ by definition of } f^* \\
&= \{(u,v) \mid \exists\, (x,y) \in f(R)^{\mathcal{I}'} \wedge (u,x),(v,y) \in f(I)^{\mathcal{I}'}\} \text{ by interpretation of } \circ \\
&= \{(x,y) \mid (x,y) \in f(R)^{\mathcal{I}'} \wedge (x,x),(y,y) \in f(I)^{\mathcal{I}'}\} \text{ as } I^{\mathcal{I}'} \text{ is identity on } f(\top)^{\mathcal{I}'} \\
&= \{(x,y) \mid (x,y) \in f(R)^{\mathcal{I}'} \cap f(\top)^{\mathcal{I}'} \times f(\top)^{\mathcal{I}'}\} \text{ as } I^{\mathcal{I}'} \text{ is identity on } f(\top)^{\mathcal{I}'} \\
&= \{(x,y) \mid (x,y) \in f(R)^{\mathcal{I}}\} \text{ from } (\dagger,4) \\
&= f(R)^{\mathcal{I}}
\end{aligned}$$

18. $X = R \circ S$:

$$\begin{aligned}
f^*(R \circ S)^{\mathcal{I}'} &= f^*(R) \circ f^*(S)^{\mathcal{I}'} \text{ by definition of } f^* \\
&= f^*(R)^{\mathcal{I}'} \circ f^*(S)^{\mathcal{I}'} \text{ by interpretation of } \circ \\
&= R^{\mathcal{I}} \circ S^{\mathcal{I}} \text{ by induction} \\
&= R \circ S^{\mathcal{I}} \text{ by interpretation of } \circ
\end{aligned}$$

Let us now continue with the second proposition of the lemma, i.e., that for every axiom ϕ, $\mathcal{I} \models \phi$ if and only if $\mathcal{I}' \models f^*(\phi)$. We must consider all the cases corresponding to the different forms of ϕ:

1. $\phi = C(a)$, $f^*(\phi) = f^*(C)(f(a))$: from (†,1) we have $a^{\mathcal{I}} = f(a)^{\mathcal{I}'}$ and we have proved above that $C^{\mathcal{I}} = f^*(C)^{\mathcal{I}'}$. Therefore $\mathcal{I} \models C(a)$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$ iff $f(a)^{\mathcal{I}'} \in f^*(C)^{\mathcal{I}'}$ iff $\mathcal{I}' \models f^*(C)(f(a))$;
2. $\phi = R(a,b)$, $f^*(\phi) = f(R)(f(a), f(b))$: from (†,1) we have $a^{\mathcal{I}} = f(a)^{\mathcal{I}'}$, $b^{\mathcal{I}} = f(b)^{\mathcal{I}'}$ and from (†,4) we have $R^{\mathcal{I}} = f(R)^{\mathcal{I}'}$. Rest of proof is analogous to the previouis case;
3. $\phi = \neg R(a,b)$, $f^*(\phi) = \neg f(R)(f(a), f(b))$: since $a^{\mathcal{I}} = f(a)^{\mathcal{I}'}$, $b^{\mathcal{I}} = f(b)^{\mathcal{I}'}$ and $R^{\mathcal{I}} = f(R)^{\mathcal{I}'}$, we have $\mathcal{I} \models \neg R(a,b)$ iff $(a^{\mathcal{I}}, b^{\mathcal{I}}) \notin R^{\mathcal{I}}$ iff $(f(a)^{\mathcal{I}'}, f(b)^{\mathcal{I}'}) \notin f(R)^{\mathcal{I}'}$ iff $\mathcal{I}' \models \neg f(R)(f(a), f(b))$;
4. $\phi = C \sqsubseteq D$, $f^*(\phi) = f^*(C) \sqsubseteq f^*(D)$: as we have already proved $C^{\mathcal{I}} = f^*(C)^{\mathcal{I}'}$ and $D^{\mathcal{I}} = f^*(D)^{\mathcal{I}'}$. Therefore $\mathcal{I} \models C \sqsubseteq D$ iff $C^{\mathcal{I}} \subseteq B^{\mathcal{I}}$ iff $f^*(C)^{\mathcal{I}'} \subseteq f^*(D)^{\mathcal{I}'}$ iff $\mathcal{I}' \models f^*(C) \sqsubseteq f^*(D)$;
5. $\phi = R \sqsubseteq S$, $f^*(\phi) = f^*(R) \sqsubseteq f(S)$: we have proved that $R^{\mathcal{I}} = f^*(R)^{\mathcal{I}'}$. If $S \in \Sigma_c$, we have from (†,2) that $S^{\mathcal{I}} = f(S)^{\mathcal{I}'}$. The proof of this case is analogous to the previous case.
   If $S \in \Sigma_e$ then from (†,4) and from (‡) we have $S^{\mathcal{I}} = f(S)^{\mathcal{I}'} \cap \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. For this case, let us first prove the if part: suppose $\mathcal{I}' \models f^*(R) \sqsubseteq f(S)$ and thererfore $f^*(R)^{\mathcal{I}'} \subseteq f(S)^{\mathcal{I}'}$. Then $R^{\mathcal{I}} = R^{\mathcal{I}} \cap \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} = f^*(R)^{\mathcal{I}'} \cap \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \subseteq f(S)^{\mathcal{I}'} \cap \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} = S^{\mathcal{I}}$. Which amounts to $\mathcal{I} \models R \sqsubseteq S$. The only-if part: Suppose $\mathcal{I} \models R \sqsubseteq S$, that is, $R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$. It follows that $f^*(R)^{\mathcal{I}'} = R^{\mathcal{I}} \subseteq S^{\mathcal{I}} = f(S)^{\mathcal{I}'} \cap \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \subseteq f(S)^{\mathcal{I}'}$.
6. ϕ is $a = b$, i.e., $f^*(\phi)$ is $f(a) = f(b)$: we know from (†,1) that $a^{\mathcal{I}} = f(a)^{\mathcal{I}'}$ and $b^{\mathcal{I}} = f(b)^{\mathcal{I}'}$. Hence $\mathcal{I} \models a = b$ iff $a^{\mathcal{I}} = b^{\mathcal{I}}$ iff $f(a)^{\mathcal{I}'} = f(b)^{\mathcal{I}'}$ iff $\mathcal{I}' \models f(a) = f(b)$;
7. ϕ is $a \neq b$, i.e., $f^*(\phi)$ is $f(a) \neq f(b)$: as a consequence of the previous case we have $\mathcal{I} \models a \neq b$ iff $\mathcal{I} \not\models a = b$ iff $\mathcal{I}' \not\models f(a) = f(b)$ iff $\mathcal{I}' \models f(a) \neq f(b)$.

A.2 Proof of Theorem 1

Theorem 1 states that our axiomatization is sound and complete, i.e., $\mathfrak{K} \vdash \mathbf{d} : \phi$ if and only if $\mathfrak{K} \models \mathbf{d} : \phi$. Let us first prove the soundness.

Lemma 4 (Soundness). *If* $\mathfrak{K} \vdash \mathbf{d} : \phi$ *then* $\mathfrak{K} \models \mathbf{d} : \phi$

Proof (Outline). For each of rule r of the form

$$\frac{\mathbf{d}_1 : \phi_1 \; \dots \; \mathbf{d}_n : \phi_n \quad \mu}{\mathbf{d} : \phi} r$$

with object premises $\mathbf{d}_1 : \phi_1, \dots, \mathbf{d}_n : \phi_n$, meta premises μ, and conclusion $\mathbf{d} : \phi$ we have to prove that if μ holds in the $\mathfrak{M}$, then

$$\left.\begin{array}{c} \Phi_1 \models \mathbf{d}_1 : \phi_1 \\ \vdots \\ \Phi_n \models \mathbf{d}_n : \phi_n \end{array}\right\} \Longrightarrow \Phi_1, \dots, \Phi_n \models \mathbf{d} : \phi$$

1. $r =$ LReas Let $\mathfrak{I}$ be a model for $\Phi_1, \dots, \Phi_n$. This implies that it also satisfies $\mathbf{d} : \phi_1, \dots, \mathbf{d} : \phi_n$, and therefore that $\mathcal{I}_\mathbf{d} \models \phi_i$ for $1 \leq i \leq n$. Soundness of DL reasoning implies that, since $\phi_1, \dots, \phi_n \models \phi$, we also have that $\mathcal{I}_\mathbf{d} \models \phi$, which ultimately implies that $\mathfrak{I} \models \mathbf{d} : \phi$.
2. $r =$ Top: By definition of CKR model $(C_\mathbf{d})^{\mathcal{I}_\mathbf{f}} \subseteq (\top_\mathbf{d})^{\mathcal{I}_\mathbf{f}}$ and that if $\mathbf{d} \preceq \mathbf{e}$, $(\top_\mathbf{d})^{\mathcal{I}_\mathbf{f}} \subseteq (\top_\mathbf{e})^{\mathcal{I}_\mathbf{f}}$. This implies that $(C_\mathbf{d})^{\mathcal{I}_\mathbf{f}} \subseteq (\top_\mathbf{e})^{\mathcal{I}_\mathbf{f}}$, and therefore that for every CKR model $\mathfrak{I}$, $\mathfrak{I} \models \mathbf{f} : C_\mathbf{d} \sqsubseteq \top_\mathbf{e}$. Similarly, by definition $R_\mathbf{d}{}^{\mathcal{I}_\mathbf{f}} \subseteq (\top_\mathbf{d})^{\mathcal{I}_\mathbf{f}} \times (\top_\mathbf{d})^{\mathcal{I}_\mathbf{f}}$, which implies that every CKR model $\mathfrak{I}$, $\mathfrak{I} \models \mathbf{f} : \exists R_\mathbf{d}.\top \sqsubseteq \top_\mathbf{d}$ and $\mathfrak{I} \models \top \sqsubseteq \forall R_\mathbf{d}.\top_\mathbf{d}$.
3. $r =$ Push: Suppose that $\Phi_0 \models \mathbf{e} : \phi@\mathbf{d}$, and for $1 \leq i \leq n$ $\Phi_i \models \mathbf{e} : \top_\mathbf{d}(a_i)$, where $a_1, \dots, a_n$ are all the individuals occurring in ϕ. We have to prove that $\Phi_0, \Phi_1, \dots, \Phi_n \models \mathbf{d} : \phi$. Let $\mathfrak{I}$ be a CKR model satisfying $\Phi_0, \Phi_1, \dots, \Phi_n$, then $\mathcal{I}_\mathbf{e} \models \phi@\mathbf{d}$ and $\mathcal{I}_\mathbf{e} \models \top_i(a_i)$ for $1 \leq i \leq n$. This implies that $\mathcal{I}_\mathbf{d}$ is defined for all a_i in ϕ, and therefore by Lemma 1 we have that $\mathcal{I}_\mathbf{d} \models \phi$, and therefore that $\mathfrak{I} \models \mathbf{d} : \phi$.
4. $r =$ Pop: Suppose that $\mathfrak{I} \models \Phi$, then by hypothesis $\mathfrak{I} \models \mathbf{d} : \phi$ which implies that $\mathcal{I}_\mathbf{d} \models \phi$. By Lemma 1 we have that $\mathcal{I}_\mathbf{e} \models \phi@\mathbf{d}$.

□

We now prove the completeness of the axiomatization.

Lemma 5 (Completeness). *If* $\mathfrak{K} \not\vdash \mathbf{d} : \top \sqsubseteq \bot$*, then* $\mathfrak{K}$ *is* $\mathbf{d}$*-satisfiable.*

Proof. The proof is a variation of the Henkin construction of a model based on constants (see e.g. [7]). Let $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the set of all top contexts of $\mathfrak{K}$. We first extend Σ with an infinite set of new constants $\Xi^\mathbf{e} = \{x_i^\mathbf{e} | i \geq 0\}$ one for each $\mathbf{e} \in E$.

Let us now construct a sequence $\mathfrak{K}^\emptyset, \mathfrak{K}^{\{\mathbf{e}_1\}}, \mathfrak{K}^{\{\mathbf{e}_1, \mathbf{e}_2\}}, \dots, \mathfrak{K}^E$ of CKRs such that $\mathfrak{K}^\emptyset = \mathfrak{K}$, and for $\mathbf{e} \notin F$ we have $\mathfrak{K}^{F \cup \{\mathbf{e}\}} := \mathfrak{K}^F$ if $\mathfrak{K}^F \vdash \mathbf{e} : \top \sqsubseteq \bot$, otherwise construct $\mathfrak{K}^{F \cup \{e\}}$ in two steps as follows.

First step. We will add witnesses for existential statements. Let $\phi_1, \phi_2 \dots$ be an enumeration of all the assertions in the extended language of the form $\exists R.C(a)$ or $\geqslant nR.C(a)$. Let $\mathfrak{K}^{F,0} = \mathfrak{K}^F$ and for every $m \geq 0$, $\mathfrak{K}^{F,m+1}$ is inductively defined as:

– if ϕ_m is of the form $\exists R.A(a)$:
  $\mathfrak{K}^{F,m+1} := \mathfrak{K}^{F,m} \cup \{\mathbf{e} : \neg\exists R.C \sqcup \exists R.(\{x^{\mathbf{e}}_k\} \sqcap C)(a)\}$
– if ϕ_m is of the form $\geqslant nR.C(a)$:
  $\mathfrak{K}^{F,m+1} := \mathfrak{K}^{F,m} \cup \{\mathbf{e} : \neg{\geqslant} nR.C \sqcup {\geqslant} nR.(\{x^{\mathbf{e}}_k, \dots, x^{\mathbf{e}}_{k+n-1}\} \sqcap C)(a)\}$

  where $x^{\mathbf{e}}_k, \dots, x^{\mathbf{e}}_{k+m-1}$ are constants of $\Xi^{\mathbf{e}}$ not appearing in $\mathfrak{K}^{F,m}$. Let $\mathfrak{K}^{F,*} = \bigcup_m \mathfrak{K}^{F,m}$.

Lemma 6. *For every assertion ϕ in Σ that does not contain any occurrence of $x^{\mathbf{e}}_i$, and every* $\mathbf{f}$, $\mathfrak{K} \vdash \mathbf{f} : \phi$ *iff* $\mathfrak{K}^{F,*} \vdash \mathbf{f} : \phi$.

Proof. The "only if" direction trivially follows due to monotonicity of the language. As $\mathfrak{K} \subseteq \mathfrak{K}^{F,*}$, everything that is proved from $\mathfrak{K}$ is also proved from its superset $\mathfrak{K}^{F,*}$.

The "if" direction. Suppose $\mathfrak{K}^{F,*} \vdash \mathbf{f} : \phi$. Observe that in this case there exists a finite subset of S of $\mathfrak{K}^{F,*}$ such that $S \vdash \mathbf{f} : \phi$. This is due to $\mathfrak{K}$ is finite, the length of the proof of $\mathbf{f} : \phi$ from $\mathfrak{K}^{F,*}$ is finite and in each step we use exactly one inference rule with derives its conclusion from a finite number of premisses. Let us denote the set of all premisses used by all the inference rules in the proof by P. Obviously the set P is finite and $P \vdash \mathbf{f} : \phi$. The formulae in P are either from $\mathfrak{K}^{F,*}$ or were derived as the proof goes on. Let $P' = \{\phi \in P | \phi \in \mathfrak{K}^{F,*}\}$. Since all the formulae which we discarded are consecutively derived as the proof goes on, then also $P' \vdash \mathbf{f} : \phi$, and by construction $P' \subseteq \mathfrak{K}^{F,*}$.

Since for every finite subset of $\mathfrak{K}^{F,*}$ there is a $\mathfrak{K}^{F,m}$ that contains such a subset, we have that $\mathfrak{K}^{F,m} \vdash \mathbf{f} : \phi$, for some m. If $m = 0$ then $\mathfrak{K}^{F,m} = \mathfrak{K}$ and we are done. If $m > 0$ we will show that in this case also $\mathfrak{K}^{F,m-1} \vdash \mathbf{f} : \phi$. We distinguish two cases:

1. If $\mathfrak{K}^{F,m} = \mathfrak{K}^{F,m-1} \cup \{\mathbf{e} : \neg\exists R.C \sqcup \exists R.(\{x^{\mathbf{e}}_k\} \sqcap C)(a)\}$, then starting from Π we can build the following deduction of $\mathbf{f} : \phi$ from $\mathfrak{K}^{F,m-1}$.

(1)	$\mathbf{e} : \neg\exists R.C \sqcup \exists R.C(a)$	$\mathfrak{K}^{F,m-1}$	from $\mathfrak{K}^{F,m-1}$ by LReas
(2)	$\mathbf{e} : \neg\exists R.C(a)$	(2)	By assumption
(3)	$\mathbf{e} : \neg\exists R.C \sqcup \exists R.(\{x^{\mathbf{e}}_k\} \sqcap C)(a)$	(2)	From (2) by LReas
(4)	$\mathbf{f} : \phi$	$(2), \mathfrak{K}^{F,m-1}$	From (3) by Π
(5)	$\mathbf{e} : \exists R.C(a)$	(5)	By assumption
(6)	$\mathbf{e} : R(a, x^{\mathbf{e}}_k)$	(6)	By assumption
(7)	$\mathbf{e} : C(x^{\mathbf{e}}_k)$	(7)	By assumption
(8)	$\mathbf{e} : \neg\exists R.C \sqcup \exists R.(\{x^{\mathbf{e}}_k\} \sqcap C)(a)$	$(6), (7)$	From (6) and (7) by LReas
(9)	$\mathbf{f} : \phi$	$(6), (7), \mathfrak{K}^{F,m-1}$	From (8) by Π
(10)	$\mathbf{f} : \phi$	$(5), \mathfrak{K}^{F,m-1}$	From (5) and (9) by $\exists$E, disc. (6) and (7)
(11)	$\mathbf{f} : \phi$	$\mathfrak{K}^{F,m-1}$	From (1), (4), and (10) by $\sqcup$E, disc. (2) and (5)

2. If $\mathfrak{K}^{F,m} = \mathfrak{K}^{F,m-1} \cup \{\mathbf{e} : \neg{\geqslant}nR.C \sqcup {\geqslant}nR.(\{x^{\mathbf{e}}_k, \ldots, x^{\mathbf{e}}_{k+n-1}\} \sqcap C)(a)\}$, as before, starting from Π, we can construct a deduction of $\mathbf{f} : \phi$ from $\mathfrak{K}^{F,m-1}$.

(1)	$\mathbf{e} : \neg{\geqslant}nR.C \sqcup {\geqslant}nR.C(a)$	$\mathfrak{K}^{F,m-1}$	from $\mathfrak{K}^{F,m-1}$ by LReas
(2)	$\mathbf{e} : \neg{\geqslant}nR.C(a)$	(2)	By assumption
(3)	$\mathbf{e} : \neg{\geqslant}nR.C \sqcup {\geqslant}nR.(\{x^{\mathbf{e}}_k, \ldots, x^{\mathbf{e}}_{k+n-1}\} \sqcap C)(a)$	(2)	From (2) by LReas
(4)	$\mathbf{f} : \phi$	$(2), \mathfrak{K}^{F,m-1}$	From (3) by Π
(5)	$\mathbf{e} : {\geqslant}nR.C(a)$	(5)	By assumption
(6_i)	$\mathbf{e} : R(a, x^{\mathbf{e}}_{k+i})$	(6_i)	By assumption (for $0 \leq i < n$)
(7_i)	$\mathbf{e} : C(x^{\mathbf{e}}_{k+i})$	(7_i)	By assumption (for $0 \leq i < n$)
(8_{ij})	$\mathbf{e} : x^{\mathbf{e}}_i \neq x^{\mathbf{e}}_j$	(8_{ij})	By assumption (for $0 \leq i \neq j < n$)
(9)	$\mathbf{e} : \neg{\geqslant}nR.C \sqcup {\geqslant}nR.(\{x^{\mathbf{e}}_k, \ldots, x^{\mathbf{e}}_{k+n-1}\} \sqcap C)(a)$	$(6_i), (7_i), (8_i)_{0 \leq i \neq j < n}$	From (6) and (7) by LReas
(10)	$\mathbf{f} : \phi$	$(6_i), (7_i), (8_i)_{0 \leq i \neq j < n}, \mathfrak{K}^{F,m-1}$	From (9) by Π
(11)	$\mathbf{f} : \phi$	$(5), \mathfrak{K}^{F,m-1}$	From (5) and (10) by $({\geqslant}n)$E, disc. (6_i), (7_i) and (8_{ij})
(12)	$\mathbf{f} : \phi$	$\mathfrak{K}^{F,m-1}$	From (1), (4), and (11) by $\sqcup$E, disc. (2) and (5)

In both cases $\mathfrak{K}^{F,m-1} \vdash \mathbf{f} : \phi$. Since this holds for any $m > 0$ it follows by induction that $\mathfrak{K} \vdash \mathbf{f} : \phi$. ☐

Second step. We now saturate $\mathfrak{K}^{F \cup \{e\}}$ with respect to the assertions on the constants of $\Xi^{\mathbf{e}}$. Let $\phi_1, \phi_2, \ldots$ be a complete enumeration of assertions of the form $A(a)$, $R(a, b)$ and $a = b$ in the extended language (A and R are are atomic), such that for each constant x appearing in ϕ we have $\mathfrak{K}^{F,*} \vdash \mathbf{e} : \top(x)$. Let us define by induction $\mathfrak{K}^{F,*}_m$ for $m \geq 0$, as follows:

$$\mathfrak{K}^{F,*}_0 = \mathfrak{K}^{F,*}$$

$$\mathfrak{K}^{F,*}_{m+1} = \begin{cases} \mathfrak{K}^{F,*}_m \cup \{\mathbf{e} : \phi\} & \text{if } \mathfrak{K}^{F,*}_m \cup \{\mathbf{e} : \phi\} \text{ is } \mathbf{d}\text{-consistent} \\ \mathfrak{K}^{F,*}_m \cup \{\mathbf{e} : \neg\phi\} & \text{otherwise} \end{cases}$$

We define $\mathfrak{K}^{F \cup \{\mathbf{e}\}} = \bigcup_m \mathfrak{K}^{F,*}_m$.

Lemma 7. *For every $F \subseteq E$ and for every $\mathbf{e} \in E \setminus F$, $\mathfrak{K}^{F \cup \{\mathbf{e}\}}$ is $\mathbf{d}$-consistent.*

Proof. We prove by induction on m that $\mathfrak{K}^{F,*}_m$ is $\mathbf{d}$-consistent. $\mathfrak{K}^{F,*}_0$ is $\mathbf{d}$-consistent because $\mathfrak{K}$ is $\mathbf{d}$-consistent and by Lemma 6 we have that $\mathfrak{K}^{F,*}_0$ is $\mathbf{d}$-consistent. Suppose that $\mathfrak{K}^{F,*}_m$ is $\mathbf{d}$-consistent, and let us show that $\mathfrak{K}^{F,*}_{m+1}$ is $\mathbf{d}$-consistent. Suppose by contradiction that $\mathfrak{K}^{F,*}_{m+1}$ is not $\mathbf{d}$-consistent. By definition this means that both $\mathfrak{K}^{F,*}_m \cup \{\mathbf{e} : \phi\}$ and $\mathfrak{K}^{F,*}_m \cup \{\mathbf{e} : \neg\phi\}$ are not $\mathbf{d}$-consistent. Let Π_1 and Π_2 be two deductions of $\mathbf{d} : \top \sqsubseteq \bot$ from $\mathfrak{K}^{F,*}_m \cup \{\mathbf{e} : \phi\}$ and $\mathfrak{K}^{F,*}_m \cup \{\mathbf{e} : \neg\phi\}$ respectively. Starting from these proofs we can construct the following deduction of $\mathbf{d} : \top \sqsubseteq \bot$ from $\mathfrak{K}^{F,*}_m$:

1. if ϕ is $A(a)$

(1)	$\mathbf{e} : \top(a)$	$\mathfrak{K}^{F,*}_m$	by construction, for evey constant x occurring in ϕ $\mathfrak{K}^{F,*}_m \vdash \mathbf{e} : \top(x)$
(2)	$\mathbf{e} : A \sqcup \neg A(a)$	$\mathfrak{K}^{F,*}_m$	from (1) by LReas
(3)	$\mathbf{e} : A(a)$	(3)	assumption
(4)	$\mathbf{d} : \top \sqsubseteq \bot$	$(3), \mathfrak{K}^{F,*}_m$	From (3) by Π_1
(5)	$\mathbf{e} : \neg A(a)$	(5)	assumption
(6)	$\mathbf{d} : \top \sqsubseteq \bot$	$(5), \mathfrak{K}^{F,*}_m$	From (5) by Π_2
(7)	$\mathbf{d} : \top \sqsubseteq \bot$	$\mathfrak{K}^{F,*}_m$	From (2), (4), and (6) by $\sqcup$E, disc. (3) and (5)

2. if ϕ is $R(a, b)$

(1)	$\mathbf{e} : \top(a)$	$\mathfrak{K}^{F,*}_m$	by construction, for evey constant x occurring in ϕ $\mathfrak{K}^{F,*}_m \vdash \mathbf{e} : \top(x)$
(2)	$\mathbf{e} : \top(b)$	$\mathfrak{K}^{F,*}_m$	as for (1)
(3)	$\mathbf{e} : (\exists R.\{b\} \sqcup \neg\exists R.\{b\})(a)$	$\mathfrak{K}^{F,*}_m$	from (1) and (2) by LReas
(4)	$\mathbf{e} : \exists R.\{b\}(a)$	(4)	assumption
(5)	$\mathbf{e} : R(a, b)$	(4)	from (4) by LReas
(6)	$\mathbf{d} : \top \sqsubseteq \bot$	$(4), \mathfrak{K}^{F,*}_m$	From (5) by Π_1
(7)	$\mathbf{e} : \neg\exists R.\{b\}(a)$	(7)	assumption
(8)	$\mathbf{e} : \neg R(a, b)$	(7)	from (7) by LReas
(9)	$\mathbf{d} : \top \sqsubseteq \bot$	$(7), \mathfrak{K}^{F,*}_m$	From (7) by Π_2
(10)	$\mathbf{d} : \top \sqsubseteq \bot$	$\mathfrak{K}^{F,*}_m$	From (3), (6), and (9) by $\sqcup$E, disc. (4) and (7)

3. if ϕ is $a = b$

(1)	$\mathbf{e} : \top(a)$	$\mathfrak{K}^{F,*}_m$	by construction, for evey constant x occurring in ϕ $\mathfrak{K}^{F,*}_m \vdash \mathbf{e} : \top(x)$
(2)	$\mathbf{e} : \top(b)$	$\mathfrak{K}^{F,*}_m$	as for (1)
(3)	$\mathbf{e} : \{b\} \sqcup \neg\{b\}(a)$	$\mathfrak{K}^{F,*}_m$	from (1) and (2) by LReas
(4)	$\mathbf{e} : \{b\}(a)$	(4)	assumption
(5)	$\mathbf{e} : a = b$	(4)	from (4) by LReas
(6)	$\mathbf{d} : \top \sqsubseteq \bot$	$(4), \mathfrak{K}^{F,*}_m$	From (5) by Π_1
(7)	$\mathbf{e} : \neg\{b\}(a)$	(7)	assumption
(8)	$\mathbf{e} : \neg a = b$	(7)	from (7) by LReas
(9)	$\mathbf{d} : \top \sqsubseteq \bot$	$(7), \mathfrak{K}^{F,*}_m$	From (7) by Π_2
(10)	$\mathbf{d} : \top \sqsubseteq \bot$	$\mathfrak{K}^{F,*}_m$	From (3), (6), and (9) by $\sqcup$E, disc. (4) and (7)

We therefore conclude that if $\mathfrak{K}^{F,*}_{m+1} \vdash \mathbf{d} : \top \sqsubseteq \bot$, then $\mathfrak{K}^{F,*}_m \vdash \mathbf{d} : \top \sqsubseteq \bot$. This implies that $\mathfrak{K}^{F \cup \{e\}}$ is $\mathbf{d}$-consistent $\Box$.

Lemma 8. *If all the individuals x occurring in a complex concept C are such that $\mathfrak{K}^E \vdash \mathbf{e} : \top(x)$, and a is an individual such that $\mathfrak{K}^E \vdash \mathbf{e} : \top(a)$, then $\mathfrak{K}^E \vdash \mathbf{e} : C(a)$ or $\mathfrak{K}^E \vdash \mathbf{e} : \neg C(a)$.*

Proof. We proceed by induction on the complexity of C.

1. If C is atomic: then the lemma follows directly from the construction of $\mathfrak{K}^E$;
2. C is $A \sqcap B$: suppose $\mathfrak{K}^E \not\vdash \mathbf{e} : A \sqcap B(a)$. Then $\mathfrak{K}^E \not\vdash \mathbf{e} : A(a)$ or $\mathfrak{K}^E \not\vdash \mathbf{e} : B(a)$ due to LReas. By induction hypothesis $\mathfrak{K}^E \vdash \mathbf{e} : \neg A(a)$ or $\mathfrak{K}^E \vdash \mathbf{e} : \neg B(a)$. Finally $\mathfrak{K}^E \vdash \mathbf{e} : \neg(A \sqcap B)(a)$ by LReas;
3. C is $A \sqcup B$: suppose $\mathfrak{K}^E \not\vdash \mathbf{e} : A \sqcup B(a)$. This implies that $\mathfrak{K}^E \not\vdash \mathbf{e} : A(a)$ and $\mathfrak{K}^E \not\vdash \mathbf{e} : B(a)$ by LReas. By induction hypothesis we have $\mathfrak{K}^E \vdash \mathbf{e} : \neg A(a)$ and $\mathfrak{K}^E \vdash \mathbf{e} : \neg B(a)$ which implies $\mathfrak{K}^E \vdash \mathbf{e} : \neg(A \sqcup B)(a)$ by LReas;
4. C is $\neg A$: By induction we have that either $\mathfrak{K}^E \vdash \mathbf{e} : A(a)$ or $\mathfrak{K}^E \vdash \mathbf{e} : \neg A(a)$, which implies that either $\mathfrak{K}^E \vdash \mathbf{e} : \neg\neg A(a)$ or $\mathfrak{K}^E \vdash \mathbf{e} : \neg A(a)$;
5. C is $\exists R.A$: From the construction of $\mathfrak{K}^E$ we have $\mathfrak{K}^E \vdash \mathbf{e} : \neg \exists R.A \sqcup \exists R.(\{x^{\mathbf{e}}\} \sqcap A)(a)$ for some x^e.
   From the construction of $\mathfrak{K}^E$ we know that either $\mathfrak{K}^E \vdash \mathbf{e} : \phi$ or $\mathfrak{K}^E \vdash \mathbf{e} : \neg\phi$ for any assertion ϕ. Since $R(a, x^{\mathbf{e}})$ and $A(x^{\mathbf{e}})$ are assertions, one of the three cases must occur:
   - $\mathfrak{K}^E \vdash \mathbf{e} : R(a, x^{\mathbf{e}})$ and $\mathfrak{K}^E \vdash \mathbf{e} : A(x^{\mathbf{e}})$: in this case $\mathfrak{K}^E \vdash \mathbf{e} : \exists R.A(a)$ directly by LReas;
   - $\mathfrak{K}^E \vdash \mathbf{e} : \neg R(a, x^{\mathbf{e}})$: in this case $\mathfrak{K}^E \vdash \neg \exists R.(\{x^{\mathbf{e}}\} \sqcap A)(a)$ and since we have $\mathfrak{K}^E \vdash \mathbf{e} : \neg \exists R.A \sqcup \exists R.(\{x^{\mathbf{e}}\} \sqcap A)(a)$ then $\mathfrak{K}^E \vdash \mathbf{e} : \neg \exists R.A$, both steps by LReas;
   - $\mathfrak{K}^E \vdash \mathbf{e} : \neg A(x^{\mathbf{e}})$: in this case again $\mathfrak{K}^E \vdash \neg \exists R.(\{x^{\mathbf{e}}\} \sqcap A)(a)$ and hence $\mathfrak{K}^E \vdash \mathbf{e} : \neg \exists R.A$ by LReas;
6. C is $\geqslant nR.A$: analogously to the previous case;
7. C is $\forall R.A$: can be rewritten as $\neg \exists R.\neg A$;
8. C is $\leqslant nR.A$: can be rewritten as $\neg {\geqslant} n{+}1 R.A$;
9. C is $\exists R.\mathrm{Self}$: by construction we have that $\mathfrak{K}^E \vdash \mathbf{e} : R(a, a)$ or that $\mathfrak{K}^E \vdash \mathbf{e} : \neg R(a, a)$. By LReas this implies that $\mathfrak{K}^E \vdash \mathbf{e} : \exists R.\mathrm{Self}(a)$ or $\mathfrak{K}^E \vdash \mathbf{e} : \neg \exists R.\mathrm{Self}(a)$;
10. C is $\{b\}$: by construction we have $\mathfrak{K}^E \vdash \mathbf{e} : a = b$ or $\mathfrak{K}^E \vdash \neg \mathbf{e} : a = b$. By LReas this gives us $\mathfrak{K}^E \vdash \mathbf{e} : \{b\}(a)$ or $\mathfrak{K}^E \vdash \mathbf{e} : \neg\{b\}(a)$.

□

The last step of the proof is dedicated to the construction of the a CKR interpretation starting from $\mathfrak{K}^E$. The elements of the domain of each interpretations are the equivalence classes of the constants of the extended language with respect the following equivalence relations

$$a \sim_{\mathbf{e}} b \text{ iff } \mathfrak{K}^E \vdash \mathbf{e} : a = b$$

We define $[x]_{\mathbf{e}}$ as the set $\{y | \mathfrak{K}^E \vdash x = y\}$

Lemma 9. *$\sim_{\mathbf{e}}$ is an equivalence relation on the set* $\{a | \mathfrak{K}^E \vdash \mathbf{e} : \top(a)\}$

Proof.
- $\sim_{\mathbf{e}}$ is reflexive: $\mathfrak{K}^E \vdash \mathbf{e} : \top(a)$ implies $\mathfrak{K}^E \vdash \mathbf{e} : a = a$ which implies that $a \sim_{\mathbf{e}} a$.
- $\sim_{\mathbf{e}}$ is symmetric: $a \sim_{\mathbf{e}} b$ implies that $\mathfrak{K}^E \vdash \mathbf{e} : a = b$ which implies that $\mathfrak{K}^E \vdash \mathbf{e} : b = a$, and therefore that $b \sim_{\mathbf{e}} a$.
- $\sim_{\mathbf{e}}$ is transitive: $a \sim_{\mathbf{e}} b$ and $b \sim_{\mathbf{e}} c$ imply that $\mathfrak{K}^E \vdash \mathbf{e} : a = b$ and $\mathfrak{K}^E \vdash \mathbf{e} : b = c$. By LReas we have that $\mathfrak{K}^E \vdash \mathbf{e} : a = c$ and therefore $a \sim_{\mathbf{e}} c$.

□

Lemma 10. $[a]_{\mathbf{d}} \neq \emptyset$ *iff* $\mathfrak{K}^E \vdash \mathbf{e} : \top(a)$

Proof. $[a]_{\mathbf{d}} \neq \emptyset$ iff there is a b such $\mathfrak{K}^E \vdash \mathbf{e} : a = b$, which implies that $\mathfrak{K}^E \vdash \mathbf{e} : \top(a)$. Vice-versa if $\mathfrak{K}^E \vdash \mathbf{e} : \top(a)$, then by LReas, $\mathfrak{K}^E \vdash \mathbf{e} : a = a$ and therefore $a \in [a]_{\mathbf{e}}$, i.e., $[a]_{\mathbf{e}} \neq \emptyset$ □

Lemma 11. *If* $[a]_{\mathbf{d}} \neq \emptyset$ *and* $\mathbf{d} \prec \mathbf{e}$*, then* $[a]_{\mathbf{d}} = [a]_{\mathbf{e}}$*.*

Proof. Let's prove that $[a]_{\mathbf{d}} \subseteq [a]_{\mathbf{e}}$: $b \in [a]_{\mathbf{d}}$ iff $\mathfrak{K}^E \vdash \mathbf{d} : a = b$ only if $\mathfrak{K}^E \vdash \mathbf{e} : a = b$ iff $b \in [a]_{\mathbf{e}}$. Vice-versa let us prove that $[a]_{\mathbf{e}} \subseteq [a]_{\mathbf{d}}$: $b \in [a]_{\mathbf{e}}$ iff $\mathfrak{K}^E \vdash \mathbf{e} : a = b$. By the following proof we have that $\mathfrak{K}^E \vdash \mathbf{d} : a = b$, and therefore that $b \in [a]_{\mathbf{d}}$

(1)	$\mathbf{d} : \top(a)$	$\mathfrak{K}^E$	By hypothesis
(2)	$\mathbf{e} : \top_{\mathbf{d}}(a)$	$\mathfrak{K}^E$	from (1) by Pop
(3)	$\mathbf{e} : a = b$	$\mathfrak{K}^E$	by hypothesis
(4)	$\mathbf{e} : \top_{\mathbf{d}}(b)$	$\mathfrak{K}^E$	from (3) by LReas
(5)	$\mathbf{d} : a = b$	$\mathfrak{K}^E$	from (3), (2), and (4) by Push

□

For every $\mathbf{e}$ the local interpretation $\mathcal{I}_{\mathbf{e}}$ is defined as follows:

- $\Delta_{\mathbf{e}} = \{[x]_{\mathbf{e}} | \mathfrak{K}^E \vdash \mathbf{e} : \top(x)\}$
- $a^{\mathcal{I}_{\mathbf{e}}} = [a]_{\mathbf{e}}$ if $\mathfrak{K}^E \vdash \mathbf{e} : \top(a)$ otherwise $a^{\mathcal{I}_{\mathbf{e}}}$ is undefined
- $A^{\mathcal{I}_{\mathbf{d}}} = \{[x]_{\mathbf{d}} \mid \mathfrak{K}^E \vdash \mathbf{d} : A(x)\}$
- $R^{\mathcal{I}_{\mathbf{d}}} = \{\langle [x]_{\mathbf{d}}, [y]_{\mathbf{d}} \rangle \mid \mathfrak{K}^E \vdash \mathbf{d} : R(x, y)\}$

We first prove that complex concepts are well defined:

Lemma 12. $[a]_{\mathbf{d}} \in C^{\mathcal{I}_{\mathbf{d}}}$ *iff* $\mathfrak{K}^E \vdash \mathbf{d} : C(a)$.

Proof. By structural induction:

1. $C = A$ is atomic: this case follows directly from the construction;
2. $C = \neg D$: $[a]_{\mathbf{d}} \in (\neg D)^{\mathcal{I}_{\mathbf{d}}}$ iff $[a]_{\mathbf{d}} \in \Delta_{\mathbf{d}} \setminus D^{\mathcal{I}_{\mathbf{d}}}$ iff $\mathfrak{K}^E \vdash \mathbf{d} : \top(a)$ and $\mathfrak{K}^E \not\vdash \mathbf{d} : D(a)$ (first from the construction, second from the induction hypothesis) iff $\mathfrak{K}^E \vdash \mathbf{d} : \top(a)$ and $\mathfrak{K}^E \vdash \mathbf{d} : \neg D(a)$ (from Lemma 8) iff $\mathfrak{K}^E \vdash \mathbf{d} : \neg D(a)$ (by LReas);
3. $C = F \sqcap G$: $[a]_{\mathbf{d}} \in (F \sqcap G)^{\mathcal{I}_{\mathbf{d}}}$ iff $[a]_{\mathbf{d}} \in F^{\mathcal{I}_{\mathbf{d}}}$ and $[a]_{\mathbf{d}} \in G^{\mathcal{I}_{\mathbf{d}}}$ iff $\mathfrak{K}^E \vdash \mathbf{d} : F(a)$ and $\mathfrak{K}^E \vdash \mathbf{d} : G(a)$ (by induction hypothesis) if $\mathfrak{K}^E \vdash \mathbf{d} : F \sqcap G(a)$ (by LReas);
4. $C = F \sqcup G$: $[a]_{\mathbf{d}} \in (F \sqcup G)^{\mathcal{I}_{\mathbf{d}}}$ iff $[a]_{\mathbf{d}} \in F^{\mathcal{I}_{\mathbf{d}}}$ or $[a]_{\mathbf{d}} \in G^{\mathcal{I}_{\mathbf{d}}}$ iff $\mathfrak{K}^E \vdash \mathbf{d} : F(a)$ or $\mathfrak{K}^E \vdash \mathbf{d} : G(a)$ (by induction hypothesis) iff $\mathfrak{K}^E \vdash \mathbf{d} : F \sqcup G(a)$ (by LReas);
5. $C = \exists R.D$: $[a]_{\mathbf{d}} \in (\exists R.D)^{\mathcal{I}_{\mathbf{d}}}$ iff for some $[b]_{\mathbf{d}}$ $\langle [a]_{\mathbf{d}}, [b]_{\mathbf{d}} \rangle \in R^{\mathcal{I}_{\mathbf{d}}}$ and $[b]_{\mathbf{d}} \in D^{\mathcal{I}_{\mathbf{d}}}$ iff for some b we have $\mathfrak{K}^E \vdash \mathbf{d} : R(a, b)$ and $\mathfrak{K}^E \vdash \mathbf{d} : D(b)$ (by the construction and induction hypothesis) iff $\mathfrak{K}^E \vdash \mathbf{d} : \exists R.D(a)$ (by LReas);
6. $C = \geqslant nR.D$: analogously to the previous case;
7. $C = \forall R.D$: can be rewritten as $\neg \exists R.\neg D$;
8. $C = \leqslant nR.D$: can be rewritten as $\neg \geqslant n{+}1R.D$;

9. $C = \exists R.\mathrm{Self}$: $[a]_{\mathbf{d}} \in (\exists R.\mathrm{Self})^{\mathcal{I}_{\mathbf{d}}}$ iff $\langle [a]_{\mathbf{d}}, [a]_{\mathbf{d}} \rangle \in R^{\mathcal{I}_{\mathbf{d}}}$ iff $\mathfrak{K}^E \vdash \mathbf{d} : R(a, a)$ (from the construction) iff $\mathfrak{K}^E \vdash \mathbf{d} : \exists R.\mathrm{Self}(a)$ (by LReas);
10. $C = \{b\}$: $[a]_{\mathbf{d}} \in \{b\}^{\mathcal{I}_{\mathbf{d}}}$ iff $[a]_{\mathbf{d}} = [b]_{\mathbf{d}}$ iff $a \sim_{\mathbf{d}} b$ iff $\mathfrak{K}^E \vdash \mathbf{d} : a = b$ (by definition of $\sim_{\mathbf{d}}$) iff $\mathfrak{K}^E \vdash \mathbf{d} : \{b\}(a)$.

□

Let us show that $\mathfrak{I} = \{\mathcal{I}_{\mathbf{d}}\}_{\mathbf{d} \in D_{\mathbf{A}}}$ is a CKR model for $\mathfrak{K}^E$. We show that all the conditions of Definition 9 are satisfied:

1. $(\top_{\mathbf{d}})^{\mathcal{I}_{\mathbf{f}}} \subseteq (\top_{\mathbf{e}})^{\mathcal{I}_{\mathbf{f}}}$ if $\mathbf{d} \prec \mathbf{e}$: $[x]_{\mathbf{f}} \in (\top_{\mathbf{d}})^{\mathcal{I}_{\mathbf{f}}}$ iff $\mathfrak{K}^E \vdash \mathbf{f} : \top_{\mathbf{d}}(x)$. Since $\mathfrak{K}^E \vdash \mathbf{f} : \top_{\mathbf{d}} \sqsubseteq \top_{\mathbf{e}}$ by LReas, we have that $\mathfrak{K}^E \vdash \mathbf{f} : \top_{\mathbf{e}}(x)$, which implies that $[x]_{\mathbf{f}} \in (\top_{\mathbf{e}})^{\mathcal{I}_{\mathbf{f}}}$
2. $(C_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} \subseteq (\top_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}}$: $[x]_{\mathbf{d}} \in (\mathcal{C}_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}}$ iff $\mathfrak{K}^E \vdash \mathbf{d} : \mathcal{C}_{\mathbf{f}}(x)$. Since $\mathfrak{K}^E \vdash \mathbf{d} : C_{\mathbf{f}} \sqsubseteq \top_{\mathbf{f}}$ by LReas, we have that $\mathfrak{K}^E \vdash \mathbf{d} : \top_{\mathbf{f}}(x)$, which implies that $[x]_{\mathbf{d}} \in (\top_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}}$.
3. $(R_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} \subseteq (\top_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} \times (\top_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}}$: If $\langle [x]_{\mathbf{d}}, [y]_{\mathbf{d}} \rangle \in (R_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}}$, then $\mathfrak{K}^E \vdash \mathbf{d} : R_{\mathbf{f}}(x, y)$. Furthermore we have that $\mathfrak{K}^E \vdash \mathbf{d} : \top \sqsubseteq \forall R_{\mathbf{f}}.\top_{\mathbf{f}}$, and $\mathfrak{K}^E \vdash \mathbf{d} : \exists R_{\mathbf{f}}.\top \sqsubseteq \top_{\mathbf{f}}$, which implies that $\mathfrak{K}^E \vdash \mathbf{d} : \top_{\mathbf{f}}(x)$ and $\mathfrak{K}^E \vdash \mathbf{d} : \top_{\mathbf{f}}(y)$. This implies that $\langle [x]_{\mathbf{d}}, [y]_{\mathbf{d}} \rangle \in (\top_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}}$.
4. if $\mathbf{d} \prec \mathbf{e}$, and $a^{\mathcal{I}_{\mathbf{e}}} \in \Delta_{\mathbf{d}}$ then $a^{\mathcal{I}_{\mathbf{e}}} = a^{\mathcal{I}_{\mathbf{d}}}$. $[a]_{\mathbf{e}} \in \Delta_{\mathbf{d}}$ iff there is a b, such that $[b]_{\mathbf{d}} = [a]_{\mathbf{e}}$, and $[b]_{\mathbf{d}} \in \Delta_{\mathbf{d}}$. By Lemma 11 we have that $[b]_{\mathbf{d}} = [b]_{\mathbf{e}}$ which implies that $[b]_{\mathbf{e}} = [a]_{\mathbf{e}}$. This implies that $\mathfrak{K}^E \vdash \mathbf{e} : a = b$, and therefore that $\mathfrak{K}^E \vdash \mathbf{d} : a = b$, which implies $[b]_{\mathbf{d}} = [a]_{\mathbf{d}}$. Summing up, $a^{\mathcal{I}_{\mathbf{e}}} = [a]_{\mathbf{e}} = [b]_{\mathbf{d}} = [a]_{\mathbf{d}} = a^{\mathcal{I}_{\mathbf{d}}}$.
5. $(X_{\mathbf{d_B}})^{\mathcal{I}_{\mathbf{e}}} = (X_{\mathbf{d_B+e}})^{\mathcal{I}_{\mathbf{e}}}$: Let X be a concept C. $[x]_{\mathbf{e}} \in C_{\mathbf{d_B}}$, iff $\mathfrak{K}^E \vdash \mathbf{e} : C_{\mathbf{d_B}}(x)$. Since $\mathfrak{K}^E \vdash \mathbf{e} : C_{\mathbf{d_B}} \equiv C_{\mathbf{d_B+e}}$, we have that $\mathfrak{K}^E \vdash \mathbf{e} : C_{\mathbf{d_B}}(x)$ iff $\mathfrak{K}^E \vdash \mathbf{e} : C_{\mathbf{d_B+e}}(x)$, which holds iff $[x]_{\mathbf{e}} \in C_{\mathbf{d_B+e}}$. An analogous argument can be done if X is a role symbol.
6. $(X_{\mathbf{d}})^{\mathcal{I}_{\mathbf{e}}} = (X_{\mathbf{d}})^{\mathcal{I}_{\mathbf{d}}}$ if $\mathbf{d} \prec \mathbf{e}$. Let X be a concept symbol C. $[x]_{\mathbf{e}} \in (C_{\mathbf{d}})^{\mathcal{I}_{\mathbf{e}}}$ iff $\mathfrak{K}^E \vdash \mathbf{e} : C_{\mathbf{d}}(x)$ iff $\mathfrak{K}^E \vdash \mathbf{d} : C_{\mathbf{d}}(x)$ iff $[x]_{\mathbf{e}} \in (C_{\mathbf{d}})^{\mathcal{I}_{\mathbf{d}}}$. An analogous argument can be done if X is a role symbol.
7. $(C_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} = (C_{\mathbf{f}})^{\mathcal{I}_{\mathbf{e}}} \cap \Delta_{\mathbf{d}}$, if $\mathbf{d} \prec \mathbf{e}$: $[x]_{\mathbf{d}} \in (C_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}}$ iff $\mathfrak{K}^E \vdash \mathbf{d} : C_{\mathbf{f}}(x)$ iff $\mathfrak{K}^E \vdash \mathbf{e} : \top_{\mathbf{d}} \sqcap C_{\mathbf{f}}(x)$ iff $\mathfrak{K}^E \vdash \mathbf{e} : \top_{\mathbf{d}}$ and $\mathfrak{K}^E \vdash \mathbf{e} : C_{\mathbf{f}}(x)$ iff $[x]_{\mathbf{e}} \in (C_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}}$ and $[x]_{\mathbf{e}} \in (\top_{\mathbf{d}})^{\mathcal{I}_{\mathbf{d}}}$ iff $[x]_{\mathbf{e}} \in (C_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} \cap \Delta_{\mathbf{d}}$ iff $[x]_{\mathbf{d}} \in (C_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} \cap \Delta_{\mathbf{d}}$.
8. $(R_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} = (R_{\mathbf{f}})^{\mathcal{I}_{\mathbf{e}}} \cap (\Delta_{\mathbf{d}} \times \Delta_{\mathbf{d}})$, if $\mathbf{d} \prec \mathbf{e}$ The same argument as the previous point.
9. $\mathcal{I}_{\mathbf{d}} \models \phi$ for all $\mathbf{d} : \phi \in \mathfrak{K}^E$. Consider the four different axioms:
   (a) ϕ is $C(a)$: in this case $\mathfrak{K}^E \vdash \mathbf{d} : C(a)$, therefore $[a]_{\mathbf{d}} \in C^{\mathcal{I}_{\mathbf{d}}}$ follows from Lemma 12;
   (b) ϕ is $R(a, b)$ and ϕ is $\neg R(a, b)$: by construction $\langle [a]_{\mathbf{d}}, [b]_{\mathbf{d}} \rangle \in R^{\mathcal{I}_{\mathbf{d}}}$ or $\langle [a]_{\mathbf{d}}, [b]_{\mathbf{d}} \rangle \notin R^{\mathcal{I}_{\mathbf{d}}}$;
   (c) ϕ is $C \sqsubseteq D$: if $[x]_{\mathbf{d}} \in C^{\mathcal{I}_{\mathbf{d}}}$, then $\mathfrak{K}^E \vdash \mathbf{d} : C(x)$ by Lemma 12. Since in this case $\mathfrak{K}^E \vdash \mathbf{d} : C \sqsubseteq D$ by LReas we have that $\mathfrak{K}^E \vdash \mathbf{d} : D(x)$ and therefore that $[x]_{\mathbf{d}} \in D^{\mathcal{I}_{\mathbf{d}}}$ again by Lemma 12;
   (d) ϕ is $R_1 \circ \cdots \circ R_n \sqsubseteq R$: if $\langle [x]_{\mathbf{d}}, [y]_{\mathbf{d}} \rangle \in (R_1 \circ \cdots \circ R_n)^{\mathcal{I}_{\mathbf{d}}}$ then there must be $[z_1]_{\mathbf{d}}, \ldots, [z_{n-1}]_{\mathbf{d}}$ such that $\langle [x]_{\mathbf{d}}, [z_1]_{\mathbf{d}} \rangle \in R_1^{\mathcal{I}_{\mathbf{d}}}$, $\langle [z_1]_{\mathbf{d}}, [z_2]_{\mathbf{d}} \rangle \in R_2^{\mathcal{I}_{\mathbf{d}}}$, $\ldots$, $\langle [z_{n-1}]_{\mathbf{d}}, [y]_{\mathbf{d}} \rangle \in R_n^{\mathcal{I}_{\mathbf{d}}}$. From the construction we have $\mathfrak{K}^E \vdash \mathbf{d} : R_1(x, z_1)$, $\mathfrak{K}^E \vdash \mathbf{d} : R_2(z_1, z_2), \ldots, \mathfrak{K}^E \vdash \mathbf{d} : R_n(z_{n-1}, y)$. Since in this case also $\mathfrak{K}^E \vdash \mathbf{d} : R_1 \circ \cdots \circ R_n \sqsubseteq R$ than by LReas we have $\mathfrak{K}^E \vdash R(x, y)$ and therefore $\langle [x]_{\mathbf{d}}, [y]_{\mathbf{d}} \rangle \in R$. Please note that this holds also in case that any of $R, R_1, \ldots, R_n$ is an inverse role.

□□

A.3 Proof of Lemma 2

Lemma 2 states: if $\mathfrak{K}$ is $\mathbf{d}$-satisfiable then $\sharp(\mathfrak{K})$ is satisfiable.

Let $\mathfrak{I}$ be a models of $\mathfrak{K}$. We define the DL interpretation $\sharp(\mathfrak{I})$ over $\sharp\langle\Sigma,\Gamma\rangle$ as follows:

1. $\Delta_{\sharp(\mathfrak{I})} = \bigcup_{\mathbf{d}\in\mathfrak{M}} \Delta_{\mathbf{d}}$;
2. $a^{\sharp(\mathfrak{I})} = a^{\mathcal{I}_{\mathbf{d}}}$ for some $\mathbf{d} \in D_{\mathbf{A}}$
3. $(C^{\mathbf{d}}_{\mathbf{e}})^{\sharp(\mathfrak{I})} = (C_{\mathbf{e}})^{\mathcal{I}_{\mathbf{d}}}$ for every $C \in \Sigma$ and for every $\mathbf{d}, \mathbf{e} \in D_{\mathbf{A}}$;
4. $(R^{\mathbf{d}}_{\mathbf{e}})^{\sharp(\mathfrak{I})} = (R_{\mathbf{e}})^{\mathcal{I}_{\mathbf{d}}}$ for every $R \in \Sigma$ and for every $\mathbf{d}, \mathbf{e} \in D_{\mathbf{A}}$;

It remains to prove that $\sharp(\mathfrak{I})$ satisfies $\sharp(\mathfrak{K})$. The satisfaction of each axiom schema in items 1..8 of Definition 20 is entailed by the corresponding condition of $\sharp(\mathfrak{I})$ being a CKR-model (Definition 9). The fact that $\sharp(\mathfrak{I})$ satisfies also the axioms schemata $\phi\sharp\mathbf{d}$ (item 9) is a consequence of the fact that $(\cdot)\sharp$ is defined on the basis of an embedding of Σ into $\sharp(\Gamma,\Sigma)$, and that each pair of interpretations $\mathcal{I}_{\mathbf{d}}$ of Σ and $\sharp(\mathfrak{I})$ of $\sharp(\Gamma,\Sigma)$ satisfy the conditions of Lemma 1.

A.4 Proof of Lemma 3

Lemma 3 states: if there is a $\mathbf{d}$ such that $\sharp(\mathfrak{K}) \not\models \top^{\mathbf{d}}_{\mathbf{d}} \sqsubseteq \bot$, then $\mathfrak{K}$ is $\mathbf{d}$-satisfiable.

Given a CKR $\mathfrak{K}$ let $\mathcal{I}$ be a model of $\sharp(\mathfrak{K})$ such that $\top^{\mathbf{d}}_{\mathbf{d}}$ is not empty. This model exists since by hypothesis $\sharp(\mathfrak{K}) \not\models \top^{\mathbf{d}}_{\mathbf{d}} \sqsubseteq \bot$. Let us define the CKR model $\mathfrak{I} = \{\mathcal{I}_{\mathbf{d}}\}_{\mathbf{d}\in\mathfrak{M}}$, $\mathcal{I}_{\mathbf{d}} = \langle\Delta_{\mathbf{d}}, \mathcal{I}_{\mathbf{d}}\rangle$, as follows:

1. $\Delta_{\mathbf{d}} = (\top^{\mathbf{d}}_{\mathbf{d}})^{\mathcal{I}}$
2. $a^{\mathcal{I}_{\mathbf{d}}} = a^{\mathcal{I}}$, if $\mathcal{I} \models \top^{\mathbf{d}}_{\mathbf{d}}(a)$
3. $(X_{\mathbf{d}'_{\mathbf{B}}})^{\mathcal{I}_{\mathbf{d}}} = (X^{\mathbf{d}}_{\mathbf{d}'_{\mathbf{B}}+\mathbf{d}})^{\mathcal{I}}$

We show that $\mathfrak{I}$ is a model of $\mathfrak{K}$. By construction we have that there is a $\mathbf{d}$ such that $\Delta_{\mathbf{d}}$ is not empty. Let us show that all the conditions of Definition 9 are satisfied by $\mathfrak{I}$.

1. If $\mathbf{d} \prec \mathbf{e}$, then $\mathcal{I} \models \top^{\mathbf{f}}_{\mathbf{d}} \sqsubseteq \top^{\mathbf{f}}_{\mathbf{e}}$. This implies that $(\top^{\mathbf{f}}_{\mathbf{d}})^{\mathcal{I}} \subseteq (\top^{\mathbf{f}}_{\mathbf{e}})^{\mathcal{I}}$, which implies $(\top_{\mathbf{d}})^{\mathcal{I}_{\mathbf{f}}} \subseteq (\top_{\mathbf{e}})^{\mathcal{I}_{\mathbf{f}}}$.
2. $\mathcal{I} \models C^{\mathbf{d}}_{\mathbf{e}} \sqsubseteq \top^{\mathbf{d}}_{\mathbf{e}}$ implies that $(C^{\mathbf{d}}_{\mathbf{e}})^{\mathcal{I}} \subseteq (\top^{\mathbf{d}}_{\mathbf{e}})^{\mathcal{I}}$, which implies $(C_{\mathbf{e}})^{\mathcal{I}_{\mathbf{d}}} \subseteq (\top_{\mathbf{e}})^{\mathcal{I}_{\mathbf{e}}}$.
3. $\mathcal{I} \models \exists R^{\mathbf{d}}_{\mathbf{e}}.\top \sqsubseteq \top^{\mathbf{d}}_{\mathbf{e}}$ and $\mathcal{I} \models \top \sqsubseteq \forall R^{\mathbf{d}}_{\mathbf{e}}.\top^{\mathbf{d}}_{\mathbf{e}}$ implies that $(R^{\mathbf{d}}_{\mathbf{e}})^{\mathcal{I}} \subseteq (\top^{\mathbf{d}}_{\mathbf{e}})^{\mathcal{I}} \times (\top^{\mathbf{d}}_{\mathbf{e}})^{\mathcal{I}}$, which implies $(R_{\mathbf{e}})^{\mathcal{I}_{\mathbf{d}}} \subseteq (\top_{\mathbf{e}})^{\mathcal{I}_{\mathbf{e}}} \times (\top_{\mathbf{e}})^{\mathcal{I}_{\mathbf{e}}}$.
4. Suppose that $a^{\mathcal{I}_{\mathbf{d}}} \in \Delta_{\mathbf{e}}$, then this implies that $a^{\mathcal{I}_{\mathbf{d}}}$ is defined, and therefore $\mathcal{I} \models \top^{\mathbf{d}}_{\mathbf{d}}(a)$, the fact that belongs to $\Delta_{\mathbf{e}}$ implies that $\mathcal{I} \models \top^{\mathbf{e}}_{\mathbf{e}}(a)$. Since $\mathcal{I} \models (\neg\top^{\mathbf{d}}_{\mathbf{d}} \sqcup \neg\top^{\mathbf{e}}_{\mathbf{e}} \sqcup \top^{\mathbf{e}}_{\mathbf{d}})(a)$, this implies that $\mathcal{I} \models \top^{\mathbf{e}}_{\mathbf{d}}(a)$, which implies that $a^{\mathcal{I}_{\mathbf{d}}} = a^{\mathcal{I}_{\mathbf{e}}} \in (\top_{\mathbf{d}})^{\mathcal{I}_{\mathbf{e}}}$.
5. By construction of $\mathfrak{I}$, we have that $(X_{\mathbf{d}_{\mathbf{B}}})^{\mathcal{I}_{\mathbf{e}}} = (X^{\mathbf{e}}_{\mathbf{d}_{\mathbf{B}}+\mathbf{e}})^{\mathcal{I}} = (X_{\mathbf{d}_{\mathbf{B}}+\mathbf{e}})^{\mathcal{I}_{\mathbf{e}}}$
6. We have that $\mathcal{I} \models X^{\mathbf{d}}_{\mathbf{d}} \equiv X^{\mathbf{e}}_{\mathbf{d}}$. This implies that $(X_{\mathbf{d}})^{\mathcal{I}_{\mathbf{e}}} = (X^{\mathbf{e}}_{\mathbf{d}})^{\mathcal{I}} = (X^{\mathbf{d}}_{\mathbf{d}})^{\mathcal{I}} = (X_{\mathbf{d}})^{\mathcal{I}_{\mathbf{d}}}$;
7. If $\mathbf{d} \prec \mathbf{e}$, we have that $\mathcal{I}\mathcal{C}^{\mathbf{d}}_{\mathbf{f}} \equiv C^{\mathbf{e}}_{\mathbf{f}} \sqcap \top^{\mathbf{d}}_{\mathbf{d}}$. This implies that $(\mathcal{C}^{\mathbf{d}}_{\mathbf{f}})^{\mathcal{I}} = (C^{\mathbf{e}}_{\mathbf{f}})^{\mathcal{I}} \cap (\top^{\mathbf{d}}_{\mathbf{d}})^{\mathcal{I}}$, which implies $(C_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} = (C_{\mathbf{f}})^{\mathcal{I}_{\mathbf{e}}} \cap \Delta_{\mathbf{d}}$;
8. $\mathcal{I} \models I^{\mathbf{d}}_{\mathbf{d}} R^{\mathbf{e}}_{\mathbf{f}} I^{\mathbf{d}}_{\mathbf{d}} \sqsubseteq R^{\mathbf{d}}_{\mathbf{f}}$ implies that $(R_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} \supseteq (R_{\mathbf{f}})^{\mathcal{I}_{\mathbf{e}}} \cap {\Delta_{\mathbf{d}}}^2$. The fact that $\mathcal{I} \models R^{\mathbf{d}}_{\mathbf{f}} \sqsubseteq R^{\mathbf{e}}_{\mathbf{f}}$, implies that $(R_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} \subseteq (R_{\mathbf{f}})^{\mathcal{I}_{\mathbf{e}}} \cap {\Delta_{\mathbf{d}}}^2$.
9. Let $\mathbf{d} = \dim(\mathcal{C})$, if $\phi \in \mathrm{K}(\mathcal{C})$, then we have that $\mathcal{I} \models \phi\sharp\mathbf{d}$. It can be proven by induction that $\mathcal{I}_{\mathbf{d}} \models \phi$.

A.5 Proof of Theorem 2

Theorem 2. *If $\mathfrak{K}$ is $\precsim$-stratified, then checking if $\mathfrak{K} \models \mathbf{d} : \phi$ is decidable with the complexity upper-bound of* 2NEXPTIME.

The decidability follows from Lemmata 2 and 3, as the problem of checking $\mathfrak{K} \models \mathbf{e} : \phi$ can be rewritten into the problem of checking if $\#(\mathfrak{K}) \models \phi\sharp\mathbf{d}$. We will show that the transformation $\#(\cdot)$ is polynomial. Since $\#(\mathfrak{K})$ is $\mathcal{SROIQ}$ knowledge base and deciding entailment id 2NEXPTIME-hard for $\mathcal{SROIQ}$ [12] it follows that checking if $\mathfrak{K} \models \mathbf{d} : \phi$ is possible within the upper bound of 2NEXPTIME worst case complexity.

Without loss of generality, we will consider the size of the input to be the total number of occurrences of all symbols from Σ and Γ in both $\mathfrak{K}$ and ϕ summed together with the total number of all DL constructors in $\mathfrak{K}$ and ϕ and with the number of formulae in $\mathfrak{K}$ and ϕ. We will denote this number by m. The real size of input to be processed depends on the encoding of symbols. As the number of symbols used in any particular knowledge base is always finite, suitable encoding can always be found such that the real size of input is $c \times m$ for some constant c [22].

As explained before, the number of contextual dimensions is assumed to be a fixed constant k. While in theory the number of possible dimensional values may be large, in practise the number of contexts n is always smaller than m. This is because whenever a new context $\mathcal{C}$ is initialized, also k new formulae are added in the meta-knowledge, by which the dimensional values are associated with $\mathcal{C}$.

Let us now determine the size of $\#(\mathfrak{K})$. We will go through the construction in Definition 20 point by point:

1. one axiom $\top^{\mathbf{f}}_{\mathbf{d}} \sqsubseteq \top^{\mathbf{f}}_{\mathbf{e}}$ for any three initialized contexts $\mathcal{C}_{\mathbf{d}}$, $\mathcal{C}_{\mathbf{e}}$ and $\mathcal{C}_{\mathbf{f}}$, with $\mathbf{d} \prec \mathbf{e}$. These are maximum n^3 of axioms of size 3, i.e., with total size bounded with $3 \times n^3$;
2. one axiom $A^{\mathbf{d}}_{\mathbf{e}} \sqsubseteq \top^{\mathbf{d}}_{\mathbf{e}}$ for any two initialized contexts $\mathcal{C}_{\mathbf{d}}$, $\mathcal{C}_{\mathbf{e}}$ and for any $A_{\mathbf{e}}$ occurring in $\mathfrak{K}$. Note that $A_{\mathbf{e}}$ occurs in $\mathfrak{K}$ whenever $A_{\mathbf{e_B}}$ occurs in $\mathcal{C}_{\mathbf{g}}$ with $\mathbf{e} = \mathbf{e_B}+\mathbf{g}$ (below this sense will be also used w.r.t. roles). This means that for each such occurrence of $A_{\mathbf{e}}$ in $\mathfrak{K}$ there is at least one actual occurrence of some $A_{\mathbf{e_B}}$ with possibly incomplete dimensional vector. Therefore at most m atomic symbols (concepts, roles and individuals) in total occur in $\mathfrak{K}$ in this sense. This implies that most $m \times n^2$ axioms of size 3 are added in this step, with total size bounded with $3 \times m \times n^2$;
3. a pair of axioms $\exists R^{\mathbf{d}}_{\mathbf{e}}.\top \sqsubseteq \top^{\mathbf{d}}_{\mathbf{e}}$ and $\top \sqsubseteq \forall R^{\mathbf{d}}_{\mathbf{e}}.\top^{\mathbf{d}}_{\mathbf{e}}$; for any two initialized contexts $\mathcal{C}_{\mathbf{d}}$, $\mathcal{C}_{\mathbf{e}}$ and for any $R_{\mathbf{e}}$ occurring in $\mathfrak{K}$. Similarly to the previous step this yields at most $2 \times m \times n^2$ axioms of size 3, i.e., with total size bounded with $6 \times m \times n^2$;
4. one axiom $a^{\mathbf{d}} = a^{\mathbf{e}}$ for for any two initialized contexts $\mathcal{C}_{\mathbf{d}}$, $\mathcal{C}_{\mathbf{e}}$ and for any individual a occurring in $\mathfrak{K}$. This is at most $m \times n^2$ axioms of size 3 with total size bounded with $3 \times m \times n^2$;
6. one axiom $X^{\mathbf{d}}_{\mathbf{d}} \equiv X^{\mathbf{e}}_{\mathbf{d}}$ for any two initialized contexts $\mathcal{C}_{\mathbf{d}}$, $\mathcal{C}_{\mathbf{e}}$ with $\mathbf{d} \preceq \mathbf{e}$ and for any atomic concept or role $X_{\mathbf{d}}$ occurring in $\mathfrak{K}$. This again leads to the maximum of $m \times n^2$ axioms of size 3 with total size bounded with $3 \times m \times n^2$;
7. $A^{\mathbf{d}}_{\mathbf{f}} \equiv A^{\mathbf{e}}_{\mathbf{f}} \sqcap \top^{\mathbf{d}}_{\mathbf{d}}$ for any two initialized contexts $\mathcal{C}_{\mathbf{d}}$, $\mathcal{C}_{\mathbf{e}}$ with $\mathbf{d} \preceq \mathbf{e}$ and for any atomic concept $A_{\mathbf{f}}$ occurring in $\mathfrak{K}$. This leads to the maximum of $m \times n^2$ axioms of size 5, i.e., with total size bounded with $5 \times m \times n^2$;

8. a pair of axioms $I^{\mathbf{d}}_{\mathbf{d}} \circ R^{\mathbf{e}}_{\mathbf{f}} \circ I^{\mathbf{d}}_{\mathbf{d}} \sqsubseteq R^{\mathbf{d}}_{\mathbf{f}}$ and $R^{\mathbf{d}}_{\mathbf{f}} \sqsubseteq R^{\mathbf{e}}_{\mathbf{f}}$ for any two initialized contexts $\mathcal{C}_{\mathbf{d}}$, $\mathcal{C}_{\mathbf{e}}$ with $\mathbf{d} \prec \mathbf{e}$ and for any $R_{\mathbf{f}}$ occurring in $\mathfrak{K}$. This leads to the maximum of $m \times n^2$ axioms of size 7 together with maximum of $m \times n^2$ axioms of size 3. Total size of both these sets together is therefore bounded with $10 \times m \times n^2$;
9. one axiom $\phi\#\mathbf{d}$ for every axiom ϕ occurring in any context $\mathrm{K}(\mathcal{C})$ of $\mathfrak{K}$. In this step less than m axioms are added. All of these axioms are transformed by the $\#(\cdot)$ operator which yields to a blow up in the axiom size because each symbol may be replaced by up to 5 new symbols (i.e., the transformation is linear). Therefore the total size of the axioms added in this step is bounded with $5 \times m$.

Summing up, the transformed knowledge base $\#(\mathfrak{K})$ is bounded in size with $33 \times m \times n^2 + 5 \times m$ which is under $O(m^3)$ since $n \leq m$.