Conflict Resolution in Assumption-Based Frameworks*

Martin Baláž[1], Jozef Frtús[1], Giorgos Flouris[2], Martin Homola[1], and Ján Šefránek[1]

[1] Comenius University in Bratislava, Slovakia
[2] FORTH-ICS, Greece

Abstract. A number of non-monotonic logics have been considered for formal representation of agents' knowledge and their ability to draw conclusions. Assumption-Based Framework (ABF) is a well-established and general formalism which captures multiple existing non-monotonic logics based on default reasoning. In this paper, we show how also defeasible reasoning can be embedded into ABF. Differently from other similar proposals, we do not encode the conflict resolution mechanism for defeasible rules into the ABF's deductive systems. Instead, we formalize the notions of conflict and conflict resolution and make them part of the extended ABF framework. The user thus gains increased control over the entire conflict resolution process. Such an approach also allows to devise different domain-dependent conflict resolution strategies and to compare them. We also show, that no matter which conflict resolution strategy is used, our framework is able to guarantee certain desired properties.

1 Introduction

Logic-based Multi-Agent Systems (LMAS) concentrate on agents' ability to represent knowledge abstracted from their environment, reason about it, and act accordingly. A number of logics from Non-Monotonic Reasoning (NMR) were investigated and tried for agent reasoning capabilities, especially due to their ability to deal with incompleteness, uncertainty, possible inconsistency, and other aspects of knowledge posed by practical environments. This has lead to an ever increasing number of formalisms to be considered, and selectively applied under suitable circumstances.

Assumption-based framework (ABF) [5,10] constitutes a powerful abstraction, known to generalize a number of non-monotonic reasoning formalisms. It assumes a language of sentences and relies on a general notion of a deductive system with inference rules. Some of the sentences are thought of as *assumptions*, in the non-monotonic sense. They are hypothetical statements, assumed to be true, unless contrary evidence is derivable. If a conflict between an assumption and a derived contrary statement arises, it is called *undermining* (cf. Fig. 1 (a)).

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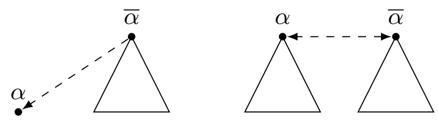
ABF is particularly suited to capture default reasoning, which is a widely accepted approach that enables agents to deal with incomplete knowledge: if hard evidence is missing, agents may act upon hypothetical assumptions. In ABF, undermining conflicts are resolved by dropping the assumption – a strategy used by most default reasoning formalisms which ABF is known to generalize [5].



(a) Undermining (b) Rebutting

Fig. 1. Different types of conflicts in ABF

Another form of reasoning often associated with ABF is defeasible reasoning. This approach enables agents to deal with inconsistent knowledge. Such a conflicts, dubbed *rebutting*, arise between two conflicting statements that are supported by two different derivations (cf. Fig. 1 (b)). Rebutting conflicts may be resolved by tracing the respective derivations and finding a derivation rule or an assumption that is to be dropped. However, there is often more than one possibility to do this, therefore multiple conflict resolution strategies may be applied, sometimes depending on the application domain. Two such strategies are known as last link principle and weakest link principle [20].

Defeasible reasoning can also be embedded into ABF, however, since rebutting conflicts are not resolved directly by the ABF semantics, workarounds are often used. One way to deal with them is to use names of rules as new assumptions and to encode the conflict resolution strategy using the ABF's deductive system [16,3]. One of the advantages of ABF is its employment of argumentation semantics which allows not only to compute the solution, but also to provide intuitive explanations why necessary assumptions had to be dropped. As we show in Sect. 2, by relying upon workarounds as cited above such explanations are no longer easily found.

We propose the eXtended Assumption-Based Framework (XABF) in which *conflict resolution strategy* (CRS) need not to be encoded into the deductive system; it is elevated to a first-class citizen of the formalism. Our main results are summarized as follows:

- CRS is a parameter of the framework, a number of different CRSs may be defined and used according to the specific needs of the application domain.
- Treatment of the CRS in XBF enables to identify the reasons why conflicts were resolved in any particular way much more clearly, in comparison to encoding a CRS into the deductive system.
- We show that consistency and closure, two widely accepted desiderata for defeasible reasoning proposed by Caminada and Amgoud [6], are satisfied in our approach in general, for any given CRS. These properties are important;

they guarantee that conflicts are not resolved just cosmetically, and then consequently derived again by the deductive system.
- Apart from extending ABF, we show that CRS can be incorporated also into Dung's Abstract Argumentation Framework (AAF) [9] and that both approaches yield equivalent results.
- Unlike the ASPIC$^+$ framework [20,17] we avoid using transposed rules which cause problems when embedding formalisms with strictly directional rules (such as Defeasible Logic Programs (DeLP) [12]) into the framework, whose meaning is affected by transposition (see Sect. 7 for details).

The rest of this paper is structured as follows: we start by motivating our design choices by outlining some desiderata for our formalism (Section 2); in Section 3 we give some preliminary definitions, whereas in Section 4 we define conflict resolution strategies and related notions. In Section 5 defined the semantics of XABF but also the alternative approach derived by extending AAF. Section 6 proves the properties of the proposed formalism. Finally, Section 7 we discuss the related works and the we conclude in Section 8. Proofs are included in the Appendix A

2 Motivating Example

To motivate our approach, we will use a running example featuring a DeLP borrowed from [6]:

Example 1 (marriage example). Consider the following set of rules:

$\rightarrow \mathit{wears\_ring}$	$\rightarrow \mathit{goes\_out}$
$r_1:\ \mathit{wears\_ring} \Rightarrow \mathit{married}$	$r_2:\ \mathit{goes\_out} \Rightarrow \mathit{bachelor}$
$\mathit{married} \rightarrow \mathit{has\_wife}$	$\mathit{bachelor} \rightarrow \sim \mathit{has\_wife}$

The rules in the above example lead us to conclude that a man wearing a ring is married, and therefore has a wife, whereas a man that goes out is a bachelor and therefore does not have a wife. Thus, for a given man who wears a ring and goes out, we identify a conflict, caused by the fact that our rules would allow us to conclude that the man has and does not have a wife. Note that some of the rules are strict ($\rightarrow$) whereas others are defeasible ($\Rightarrow$), and that defeasible rules are associated with a name of the form r_i.

To obtain an argumentative semantics for the program we first try a direct embedding into Dung's AAF. In a direct embedding arguments constructed as proof trees. The following arguments are respective to the program above:

$$A_1 = [\rightarrow \mathit{wears\_ring}] \qquad A_2 = [\rightarrow \mathit{goes\_out}]$$
$$A_3 = [A_1 \Rightarrow \mathit{married}] \qquad A_4 = [A_2 \Rightarrow \mathit{bachelor}]$$
$$A_5 = [A_3 \rightarrow \mathit{has\_wife}] \qquad A_6 = [A_4 \rightarrow \sim \mathit{has\_wife}]$$

However, we see that this embedding does not work. Arguments A_5 and A_6 are the only conflicting deductive arguments with opposite conclusions. For the remaining arguments A_1, A_2, A_3 and A_4, there does not exist a conflicting deductive argument with opposite conclusion. Therefore we should believe each literal in $\{wears\_ring, goes\_out, married, bachelor\}$. Since strict rules have to be always satisfied, the set of believed literals has to be closed under strict rules, and we should also believe in *has_wife* and $\sim has\_wife$. Although we defeat one of the arguments A_5 or A_6, the conflict between *has_wife* and $\sim has\_wife$ reappears. The problem is that the conflict has to be resolved by defeating one of the arguments A_3 or A_5 which are not directly involved in the conflict.

The direct embedding is not correct, therefore any working embedding must be indirect, in the sense that it introduces additional arguments (indeed our embedding in Sect. 5.2 falls into this category). This type of embeddings are known for ABF: all rules are treated as strict, and for each defeasible rule r_i an additional literal of the form r_i is added. The meaning of r_i is "r_i is defeated", and the meaning of $\sim r_i$ (the assumption whose contrary is r_i) is "r_i is undefeated". The encoding of the program is as follows [16,3]:

$$\begin{array}{rl|rl} & \rightarrow wears\_ring & & \rightarrow goes\_out \\ wears\_ring, \sim r_1 & \rightarrow married & goes\_out, \sim r_2 & \rightarrow bachelor \\ married & \rightarrow has\_wife & bachelor & \rightarrow \sim has\_wife \\ \sim r_1 & \rightarrow r_2 & \sim r_2 & \rightarrow r_1 \end{array}$$

Now, application of each formerly defeasible rule r_i is guarded by the new assumption $\sim r_i$. the rules $\sim r_1 \rightarrow r_2$ and $\sim r_2 \rightarrow r_1$ serve as implementation of the conflict resolution strategy: in order to resolve the conflict between possible derivation of *has_wife* and its contrary $\sim has\_wife$ only one of $\sim r_1$ may hold while the other must be defeated.

However, the link between *has_wife* and $\sim r_1$ is hard to see from the two rules $\sim r_1 \rightarrow r_2$ and $\sim r_2 \rightarrow r_1$. Therefore it is not straightforward to obtain an explanation.

3 Preliminaries

An *abstract argumentation framework* is a tuple $\mathcal{F} = (Arg, Att)$ where Arg is a set of *arguments* and $Att \subseteq Arg \times Arg$ is an *attack* relation. An *extension* is a set $\mathcal{E}$ of arguments. An extension $\mathcal{E}$ *attacks* an argument A iff $\mathcal{E}$ contains an argument attacking A; *defends* an argument A iff each extension attacking A contains an argument attacked by $\mathcal{E}$. An extension $\mathcal{E}$ is *attack-free* iff $\mathcal{E}$ does not attack an argument in $\mathcal{E}$. An attack-free extension $\mathcal{E}$ is *admissible* iff $\mathcal{E}$ defends each argument in $\mathcal{E}$. An extension $\mathcal{E}$ is *complete* iff $\mathcal{E}$ is admissible and contains all arguments defended by $\mathcal{E}$; *grounded* iff $\mathcal{E}$ is a subset-maximal admissible extension contained in all complete extensions; *preferred* iff $\mathcal{E}$ is a subset-maximal admissible extension; *ideal* iff $\mathcal{E}$ is a subset-maximal admissible extension contained in all preferred extensions; *stable* iff $\mathcal{E}$ is an attack-free extension attacking each argument which does not belong to $\mathcal{E}$.

A *language* is a set $\mathcal{L}$ of well-formed sentences. An *inference rule* over a language $\mathcal{L}$ is an expression r of the form $\frac{\varphi_1,\ldots,\varphi_n}{\varphi_0}$ where $0 \leq n$ and each φ_i,

$0 \leq i \leq n$, is a sentence in $\mathcal{L}$. The sentences $prem(r) = \{\varphi_1, \ldots, \varphi_n\}$ are called the *premises* of r and the sentence $cons(r) = \varphi_0$ is called the *consequence* of r. A *deductive system* is a pair $(\mathcal{L}, \mathcal{R})$ where $\mathcal{L}$ is a language and $\mathcal{R}$ is a set of inference rules over $\mathcal{L}$.

A *default derivation for* a sentence $\varphi \in \mathcal{L}$ is an expression of the form $D = [\varphi]$. We denote:

$$\begin{aligned} cons(D) &= \varphi \\ prem(D) &= \{\varphi\} \\ subderiv(D) &= \{D\} \end{aligned}$$

A *deductive derivation for* a sentence φ is defined recursively as an expression $D = [D_1, \ldots, D_n \to \varphi]$ where D_i is a default or deductive derivation for φ_i $(0 < i \leq n)$, and $\varphi_1, \ldots, \varphi_n \to \varphi$ is an inference rule in $\mathcal{R}$. We denote:

$$\begin{aligned} cons(D) &= \varphi \\ prem(D) &= prem(D_1) \cup \cdots \cup prem(D_n) \\ subderiv(D) &= \{D\} \cup subderiv(D_1) \cup \cdots \cup subderiv(D_n) \end{aligned}$$

A *derivation*[3] *for* a sentence φ is a default or deductive derivation for φ. We will say that derivation D' is a *subderivation* of D (denoted by $D' \sqsubseteq D$) iff $D' \in subderiv(D)$; similarly, we will say that derivation D' is a *proper subderivation* of D (denoted by $D' \sqsubset D$) iff $D' \sqsubseteq D$ and $D' \neq D$. A *theory* is a set S of sentences. A sentence φ is a *consequence* of a theory S iff there exists a derivation D for φ such that $prem(D) \subseteq S$. By $Cn_{\mathcal{R}}(S)$ we will denote the set of all consequences of S.

An *assumption-based framework* is a tuple $\mathcal{F} = (\mathcal{L}, \mathcal{R}, \mathcal{A}, \overline{})$ where $(\mathcal{L}, \mathcal{R})$ is a deductive system, $\mathcal{A} \subseteq \mathcal{L}$ is a set of *assumptions*, and $\overline{}\colon \mathcal{A} \mapsto \mathcal{L}$ is a mapping called *contrariness function*. We say that the sentence $\overline{\alpha}$ is the *contrary* of an assumption α. A *context* is a set Δ of assumptions. We say that Δ is *conflict-free* iff $\{\alpha, \overline{\alpha}\} \nsubseteq Cn_{\mathcal{R}}(\Delta)$ for each assumption α; and that Δ is *closed* iff $Cn_{\mathcal{R}}(\Delta) \cap \mathcal{A} \subseteq \Delta$. An assumption-based framework is *flat* iff each context is closed. A context Δ *attacks* an assumption α iff $\overline{\alpha} \in Cn_{\mathcal{R}}(\Delta)$. A context Δ *defends* an assumption α iff each closed context attacking α contains an assumption attacked by Δ. A closed context Δ is *attack-free* iff Δ does not attack an assumption in Δ. An attack-free context Δ is *admissible* iff Δ defends each assumption in Δ. A closed context Δ is *complete* iff Δ is admissible and contains all assumptions defended by Δ; *grounded*[4] iff Δ is a subset-maximal admissible context contained in all complete contexts; *preferred* iff Δ is a subset-maximal admissible context; *ideal* iff Δ is a subset-maximal admissible context contained in all preferred contexts; *stable* iff Δ is an attack-free context attacking each assumption which does not belong to Δ.

[3] Derivations are usually called arguments. We call them derivations to avoid confusion with arguments in the abstract argumentation framework (they do not correspond).

[4] Grounded context is called well-founded in [5]; we call it grounded to be consistent with [9].

Example 2 (marriage example continued). The language of Example 1 is a language of literals. It is build from the set of atoms $\mathcal{L}_0 = \{wears\_ring, goes\_out, bachelor, married, has\_wife, r_1, r_2\}$ and the set of assumptions (default literals) $\mathcal{A} = \{\sim a \mid a \in \mathcal{L}_0\}$ in a standard manner as $\mathcal{L} = \mathcal{L}_0 \cup \mathcal{A}$. The considered set of rules $\mathcal{R}$ are as given in Section 2 (second listing). The contrariness function is defined in the obvious way, i.e., $\overline{\sim a} = a$ for each atom $a \in \mathcal{L}_0$. Based on $\mathcal{R}$, we can conclude that $D_1 = [\sim r_1]$ is a default derivation, whereas $D_2 = [[\rightarrow wears\_ring], [\sim r_1] \rightarrow married]$ is a deductive derivation. Obviously, $D_1 \sqsubset D_2$; thus, $married \in Cn_{\mathcal{R}}(\{\sim r_1\})$.

4 Conflict Resolution Strategies

Conflict resolution in standard ABF is performed using sets of assumptions (i.e., contexts). Other formalisms, like ASPIC$^+$ [20,17], or the framework proposed in [25] use structures similar to derivations. Here, we propose to use derivations to define conflicts and conflict resolution in ABF. Thus, each different pair of derivations (that allow us to conclude an assumption and its contrary) leads to a different conflict, even if these different conflicts are all generated by the same pair of contexts. Consequently, a conflict resolution deals with one possible "cause" of conflict (where "cause" here means a pair of derivations). This approach allows a very fine-grained treatment of conflicts and resolutions, as motivated in Section 2, as well as the distinction between different kinds of conflicts (rebutting, undermining). Formally:

Definition 1 (Conflict). *We say that derivation D_1* is in conflict with *derivation D_2 iff there is some $\alpha \in \mathcal{A}$ such that $cons(D_1) = \alpha$ and $cons(D_2) = \overline{a}$. A* conflict *is a pair (D_1, D_2) such that D_1 is in conflict with D_2.*

Now let's consider two conflicts (D_1, D_2) and (D_1', D_2'). It is clear that if $D_1 \sqsubseteq D_1'$ and $D_2 \sqsubseteq D_2'$, then by resolving the first conflict, the second is automatically resolved as well. Therefore, it makes sense to resolve "smaller" conflicts first. This leads us to introduce the notion of *subconflict*, as follows:

Definition 2 (Subconflict). *Let (D_1, D_2) and (D_1', D_2') be conflicts. We say that (D_1', D_2') is a* subconflict of *(D_1, D_2) iff $D_1 \sqsubseteq D_1'$ and $D_2 \sqsubseteq D_2'$. Abusing notation, we will write $(D_1, D_2) \sqsubseteq (D_1', D_2')$ to denote that (D_1, D_2) is a subconflict of (D_1', D_2'). We say that (D_1, D_2) is a proper subconflict of (D_1', D_2') (denoted by $(D_1, D_2) \sqsubset (D_1', D_2')$) iff $(D_1, D_2) \sqsubseteq (D_1', D_2')$ and either $D_1 \sqsubset D_1'$ or $D_2 \sqsubset D_2'$.*

Conflict resolution can be simply defined as a triple, where the first two elements indicate a conflict, whereas the third indicates the assumption chosen to abandon in order to eliminate that conflict. Formally:

Definition 3 (Conflict Resolution). *A* conflict resolution *is a triple $\rho = (D_1, D_2, \alpha)$ such that D_1 is in conflict with D_2 and $\alpha \in prem(D_1) \cup prem(D_2)$.*

The assumption α is called the resolution *of ρ, and denoted by $res(\rho)$. The* context *of a conflict resolution, denoted by $ctx(\rho)$, is the set:*

$$ctx(\rho) = \begin{cases} prem(D_1) \cup prem(D_2) & \textit{whenever } \alpha \in prem(D_1) \cap prem(D_2) \\ (prem(D_1) \cup prem(D_2)) \setminus \{\alpha\} & \textit{otherwise} \end{cases}$$

In Definition 3, $res(\rho)$ intuitively refers to the assumption that the conflict resolution chose to drop from our set of assumptions, whereas the set $ctx(\rho)$ refers to the assumptions that essentially "cause" $res(\rho)$ to be dropped; in argumentation terminology, $ctx(\rho)$ attacks $res(\rho)$ (see also Section 5). Note that this is true only as far as the specific ρ is concerned; the interplay between different conflict resolutions and the choices they encode need also to be considered, as explained in detail in Section 5.

Definition 4. *A* conflict resolution strategy *σ is a mapping which assigns to an ABF a set of conflict resolutions of this ABF.*

Intuitively, a conflict resolution strategy takes an ABF and returns a set of conflict resolutions; note that a conflict resolution strategy does not necessarily resolve all conflicts that appear in an ABF, i.e., it may opt to leave some of the conflicts unresolved. In the following example, we explain the notions of conflict, conflict resolution and conflict resolution strategy:

Example 3 (Marriage example revisited). Continuing Example 1, we note that the following derivations can be created:
$D_1 = [[[\to goes\_out], [\sim r_2] \to bachelor] \to \sim has\_wife]$
$D_2 = [[[\to wears\_ring], [\sim r_1] \to married] \to has\_wife]$
which show that there is a conflict (D_1, D_2). Using a direct embedding into ABF, the only conflict resolution which we get is that $\sim has\_wife$ is defeated while *has_wife* is upheld. However, we can see that there are also other alternatives for resolving this conflict, e.g., by defeating one of the two defeasible rules (i.e., $\sim r_1, \sim r_2$). Thus, we can define the following conflict resolutions: $\rho_1 = (D_1, D_2, \sim r_2)$, $\rho_2 = (D_1, D_2, \sim r_1)$. It follows that $ctx(\rho_1) = \{\sim r_1\}$, $res(\rho_1) = r_2$, $ctx(\rho_2) = \{\sim r_2\}$, $res(\rho_2) = r_1$. We define a conflict resolution strategy that includes both resolutions, i.e., $\sigma(\mathcal{F}) = \{\rho_1, \rho_2\}$.

The example below shows how our approach deals with the thorny issue of self-defeating assumption, and justifies the chosen definition of ctx, res (Definition 3).

Example 4 (Self-defeating). Let us now consider a similar example, with rules $\sim a \to b$ and $\sim a \to \sim b$, which lead to a *self-defeating* assumption $(\sim a)$. The derivations $D_1 = [[\sim a] \to \sim b]$, $D_2 = [[\sim a] \to b]$ lead to the conflict (D_1, D_2), whose corresponding conflict resolution would be $\rho = (D_1, D_2, \sim a)$. Following Definition 3, we get $res(\rho) = \sim a$, $ctx(\rho) = \{\sim a\}$. The fact that $res(\rho) \in ctx(\rho)$ is a consequence of our definitions for self-defeating assumptions, and intuitively means that $\sim a$ forces itself to be dropped, so $\sim a$ can never be accepted. Note that for non-self-defeating assumptions, $res(\rho) \notin ctx(\rho)$ (e.g., Example 3)

Proposing specific conflict resolution strategies, or techniques for defining them, is out of the scope of this paper, and will be addressed in future work. In this respect, some popular options like specificity [19,23,4], or the weakest link principle [1,11] could be considered, as well as strategies based on the "utility" of each assumption (see, e.g., [7]). Note that many of these strategies are based on the notion of *preference*, i.e., some kind of priority (ordering) on assumptions or contexts [15,14,13]. Such strategies (and preferences) can be combined using various operators to produce even more complicated ones, as described in [22,8,15,14,13]. All these intuitions can be easily supported in our framework by just using the appropriate conflict resolution strategy that captures the corresponding policy; due to space limitations, we don't evaluate this claim in the present paper, but leave it for future work.

5 Argumentation Semantics

In this section, we will show how we can model conflict resolution and conflict resolution strategies in two different, but equivalent, ways. The first way uses a generalization of the Assumption-Based Framework (ABF) [5,10] that we call the *eXtended Assumption-Based Framework (XABF)*. The second method uses the information in XABF to instantiate Abstract Argumentation Frameworks (AAF) [9]. As we will see, both models are suitable for describing very well the interplay and interrelationship between different conflict resolutions. Further, we will show that their semantics are equivalent (see Proposition 1).

In the rest of this section, we assume an arbitrary, but fixed, ABF $\mathcal{F}$ and the set of conflict resolutions $\mathcal{P} = \sigma(\mathcal{F})$, for an arbitrary, but fixed, conflict resolution strategy σ.

5.1 Extended Assumption-Based Framework

Each conflict resolution ρ in $\mathcal{P}$ represents a choice as to how a conflict should be resolved; this choice actually determines the assumption $res(\rho)$ to be dropped in order for the conflict to disappear. This essentially implies that certain assumptions invalidate other assumptions (more precisely, the assumptions in $ctx(\rho)$ invalidate $res(\rho)$). This idea can be captured nicely using the notion of "attack" appearing in ABFs [5,10], where a context Δ "attacking" an assumption α intuitively means that $\Delta \subseteq \mathcal{A}$ would imply that $\alpha \notin \mathcal{A}$. This is of course generalized to all super-contexts of Δ, which attack contexts including α.

However, the above viewpoint considers only the effects of a single conflict resolution, not taking into account the interplay between conflict resolutions in the conflict resolution strategy. In effect, the choices made by the different conflict resolutions in the strategy are not independent, because the resolution proposed by a conflict resolution may implicitly resolve other conflicts as well, thereby making another conflict resolution void. As a result, a certain context (that is causing a certain conflict, which is being resolved in a manner prescribed by the chosen strategy), may prevent another conflict from appearing, therefore it may *defend* some assumption (resulting from the corresponding conflict resolution) from attack. This notion of defence can also be described using ABFs, via their inherent notion of defence.

In Definitions 5, 6 we formally define the notions of attack and defence, which depend on the actual conflict resolution strategy considered and are thus an extension of the corresponding notions described in Section 3 and in [5,10].

Definition 5 (Attack-Freeness). *A context* Δ attacks *an assumption* α *iff there exists some* $\rho \in \mathcal{P}$ *with* $ctx(\rho) \subseteq \Delta$ *and* $res(\rho) = \alpha$. *We denote:*

$$Attack_{\mathcal{P}}(\Delta) = \{\alpha \in \mathcal{A} \mid \exists \rho \in \mathcal{P} \colon \alpha = res(\rho) \wedge ctx(\rho) \subseteq \Delta\}$$

A context Δ *is* attack-free *iff* Δ *does not attack an assumption in* Δ, *i.e. iff* $Attack_{\mathcal{P}}(\Delta) \cap \Delta = \emptyset$.

Definition 6 (Admissibility). *A context* Δ defends *an assumption* α *iff* Δ *attacks an assumption in each context attacking* α. *We will denote*

$$Defence_{\mathcal{P}}(\Delta) = \{\alpha \in \mathcal{A} \mid \forall \rho \in \mathcal{P} \colon \alpha = res(\rho) \Rightarrow ctx(\rho) \cap Attack_{\mathcal{P}}(\Delta) \neq \emptyset\}$$

An attack-free context Δ *is* admissible *iff* Δ *defends each assumption in* Δ, *i.e. iff* $\Delta \subseteq Defence_{\mathcal{P}}(\Delta)$.

Definition 7 (Extension). *A context* Δ *is*

- complete *iff* Δ *is admissible and* $Defence_{\mathcal{P}}(\Delta) \subseteq \Delta$
- grounded *iff* Δ *is a subset-maximal admissible context contained in all complete contexts*
- preferred *iff* Δ *is a subset-maximal admissible context*
- ideal *iff* Δ *is a subset-maximal admissible context contained in all preferred contexts*
- stable *iff* $\Delta = \mathcal{A} \setminus Attack_{\mathcal{P}}(\Delta)$

If Δ *is a* $\Sigma_{\mathcal{P}}$*-context then* $\mathcal{E} = Cn_{\mathcal{R}}(\Delta)$ *is a* $\Sigma_{\mathcal{P}}$*-extension of* $\mathcal{F}$ *for each* $\Sigma \in \{complete, grounded, preferred, ideal, stable\}$.

In the following, we will use the term *standard semantics* to refer to any of the semantics of extensions that appear in Definition 7. An extension corresponds to a set of assumptions (aka, context) that is acceptable, under the given conflict resolution strategy. This extension is essentially the result of the conflict resolution process, where all different conflict resolutions in the strategy, as well as their interplay, have been considered in selecting what to drop and what to keep. Recall that this process does not guarantee a conflict-free set of assumptions $\mathcal{A}$, as, by design, we allow some conflicts to remain unresolved. An important difference of the above viewpoint compared to standard approaches is that the actual reason for defeating an assumption is not an argument, but the conflicts themselves (and their resolutions), which force us to drop some of the assumptions.

Example 5 (marriage example continued). If we take the ABF $\mathcal{F}$ with the set $\mathcal{P}$ of conflict resolutions from Example 3, we have three complete contexts, namely $\Delta_1 = \{\sim r_1, \sim bachelor\}$, $\Delta_2 = \{\sim r_2, \sim married, \sim has\_wife\}$, and $\Delta_3 = \{\}$. They correspond to three extensions $\mathcal{E}_1 = Cn_{\mathcal{R}}(\Delta_1) = \Delta_1 \cup \{married, has\_wife\}$, $\mathcal{E}_2 = Cn_{\mathcal{R}}(\Delta_2) = \Delta_2 \cup \{bachelor\}$, and $\mathcal{E}_3 = Cn_{\mathcal{R}}(\Delta_3) = \{\}$. In the context Δ_1, the conflict resolution ρ_1 explains why the assumption $\sim r_2$ is defeated. Since $ctx(\rho_1) \subseteq \Delta_1$ and $res(\rho_1) = r_2$, the assumption $\sim r_2$ is defeated in order to resolve conflicts between derivations D_1 and D_2. Similarly, the conflict resolution $\rho_2 \in \mathcal{P}$ is an explanation for defeating $\sim r_1$ in the context Δ_2.

5.2 Instantiation of Abstract Argumentation Framework

The constructed XABF can be used to construct an instantiation of the Abstract Argumentation Framework (AAF) [9] using similar ideas. In this case, conflict resolutions are seen as arguments in the AAF. The notion of attack captures the fact that some conflict resolutions prevent other conflicts from emerging, thereby voiding other conflict resolutions; in other words, an argument (i.e., a conflict resolution) ρ_1 attacks ρ_2 iff the resolution of ρ_1 prevents the conflict of ρ_2 from appearing. Formally:

Definition 8 (AAF instantiation). *Given an ABF $\mathcal{F}$ and the set of conflict resolutions $\mathcal{P} = \sigma(\mathcal{F})$, we construct an instantiation of an AAF $\mathcal{F}_{AAF} = (Arg, Att)$ such that $Arg = \mathcal{P}$ and $Att = \{(\rho_1, \rho_2) \in \mathcal{P} \times \mathcal{P} \mid res(\rho_1) \in ctx(\rho_2)\}$.*

The intuition explained above implies that extensions of the AAF instantiated in Definition 8 correspond to the set of conflict resolutions that should be applied. As a result, the accepted assumptions of XABF are determined by the conflict resolutions (arguments) in some extension $\mathcal{E}$, by taking those assumptions that are defended by $\mathcal{E}$, in the sense of the following definition.

Definition 9 (Acceptable assumptions of $\mathcal{E}$). *Let $\mathcal{E}$ be an extension of the AAF of Definition 8 per the standard semantics (i.e., a set of conflict resolutions). Then, the set of acceptable assumptions of $\mathcal{E}$ is defined as $\Delta_{\mathcal{E}} = \{\alpha \in \mathcal{A} \mid \forall \rho \in \mathcal{E} \colon \alpha = res(\rho) \Rightarrow \mathcal{E} \text{ attacks } \rho\}$.*

Example 6. Continuing Example 5, we would get $Arg = \{\rho_1, \rho_2\}$ and $Att = \{(\rho_1, \rho_2), (\rho_2, \rho_1)\}$. The computation of complete extensions would result to the following: $\mathcal{E}_1 = \{\rho_1\}$, $\mathcal{E}_2 = \{\rho_2\}$, $\mathcal{E}_3 = \{\}$. Each extension corresponds to the following sets of accepted assumptions: $\Delta_{\mathcal{E}_1} = \{\sim r_1, \sim bachelor\}$, $\Delta_{\mathcal{E}_2} = \{\sim r_2, \sim married, \sim has\_wife\}$ and $\Delta_{\mathcal{E}_3} = \{\}$,

6 Properties

In this section, we show the properties of the constructions defined in Section 5. In particular, the semantics of the XABF (Definitions 5, 6) and the instantiated AAF (Definition 8) are shown to be equivalent (Proposition 1). Further, we show that XABF behaves in a reasonable manner, according to well-established properties present in [6] (Propositions 2, 3, 4, 5). Finally, we show the role of "minimal" subconflicts (see Definitions 11, 12 and Propositions 6, 7).

In the rest of this section, we assume an arbitrary, but fixed, ABF $\mathcal{F}$ and the set of conflict resolutions $\mathcal{P} = \sigma(\mathcal{F})$ for an arbitrary, but fixed, conflict resolution strategy σ, as well as any given standard semantics $\Sigma_{\mathcal{P}}$.

Our first result shows the equivalence of semantics defined in Sections 5.1 and 5.2 (cf. Examples 5, 6).

Proposition 1. *Let $\mathcal{F}_{ABF} = (\mathcal{L}, \mathcal{R}, \mathcal{A}, ^{-})$ be an ABF, $\mathcal{P}$ be a set of conflict resolutions of $\mathcal{F}_{ABF}$, and $\mathcal{F}_{AAF}$ be the instantiation of AAF defined in Definition 8. Then Δ is a $\Sigma_{\mathcal{P}}$-context of $\mathcal{F}_{ABF}$ iff there exists a Σ-extension $\mathcal{E}$ of $\mathcal{F}_{AAF}$ such that $\Delta = \Delta_{\mathcal{E}}$.*

Our next result shows that there exists an instantiation of conflict resolution strategy, such that conflict-free $\Sigma_{\mathcal{P}}$-extensions of XABF corresponds to Σ-extensions of standard ABF. This allows us to apply to our setting the rich set of results proven for AAF literature.

Proposition 2. *Let $\mathcal{F} = (\mathcal{L}, \mathcal{R}, \mathcal{A}, ^{-})$ be an assumption-based framework and $\mathcal{P} = \{(D_1, D_2, \alpha) \mid \alpha \in \mathcal{A}, cons(D_1) = \alpha, cons(D_2) = \overline{\alpha}\}$. If $\mathcal{F}$ is flat, then $\mathcal{E}$ is a conflict-free $\Sigma_{\mathcal{P}}$-extension of $\mathcal{F}$ iff $\mathcal{E}$ is an Σ-extension of $\mathcal{F}$ as in [5,10].*

As already mentioned, a conflict resolution strategy does not need to resolve all conflicts. However, strategies that do resolve all conflicts have some interesting properties and will be called *total*. Formally:

Definition 10. *A set of conflict resolutions $\mathcal{P}$ is* total *iff for each context Δ, which is not conflict-free, there is a resolution ρ with $ctx(\rho) \subseteq \Delta$ and $res(\rho) \in \Delta$.*

The following results show that our framework satisfies a generalized version of the rationality conditions proposed in [6]. Note that for Proposition 3, the hypothesis of totality is crucial.

Proposition 3. *If $\mathcal{P}$ is total then each $\Sigma_{\mathcal{P}}$-extension is conflict-free.*

Proposition 4. *If $\emptyset$ is not conflict-free then each $\Sigma_{\mathcal{P}}$-extension is not conflict-free.*

Proposition 5. *Each $\Sigma_{\mathcal{P}}$-extension is closed.*

Subconflicts (Definition 2) were introduced to capture the intuition that the resolution of "smaller" conflicts (in the sense of $\sqsubseteq$) also resolves "larger" ones. Thus, resolutions resolving the $\sqsubseteq$-minimal conflicts are of special interest.

Definition 11 (Bottom). *The bottom of $\mathcal{P}$ is a set $\lfloor \mathcal{P} \rfloor = \{(D_1, D_2, \alpha) \in \mathcal{P} \mid \forall (D'_1, D'_2) \sqsubset (D_1, D_2) \colon (D'_1, D'_2, \alpha) \notin \mathcal{P}\}$.*

Proposition 6. *A theory $\mathcal{E}$ is a $\Sigma_{\mathcal{P}}$-extension iff $\mathcal{E}$ is a $\Sigma_{\lfloor \mathcal{P} \rfloor}$-extension.*

Definition 12 (Downward Closure). *We say that* $\mathcal{P}$ *is* downward closed *iff for each conflict resolution* $(D_1, D_2, \alpha) \in \mathcal{P}$ *there exists a minimal subconflict* (D_1', D_2') *of* (D_1, D_2) *with* $(D_1', D_2', \alpha) \in \mathcal{P}$.

Proposition 7. *The bottom of a downward closed set* $\mathcal{P}$ *of conflict resolutions contains only* $\sqsubseteq$*-minimal conflicts.*

Proposition 6 implies that all conflict resolutions that are not in the bottom of the original set can be dropped without changing the semantics. If the set $\mathcal{P}$ of conflict resolution is in addition downward closed then $\sqsubseteq$-minimal conflicts take precedence during resolution. As a special case, undermining takes precedence over rebutting as already suggested by Prakken and Sartor [21]. In our case, the same precedence in conflict resolutions is given also in the case of two $\sqsubseteq$-related rebutting conflicts (i.e., the subconflict of the two should be removed).

An important consequence of Proposition 7 is that, for constructing a conflict resolution strategy, one only needs to be concerned with minimal conflicts. This is a very useful property from the practical viewpoint, as it allows not dealing with all conflicts during the construction of a strategy, only with minimal ones. Note that this intuition cannot be extended to minimal derivations, as they do not always lead to minimal conflicts: a pair of minimal derivations for an assumption could "hide" a subconflict on another assumption.

7 Related Work

The main difference between ABF [5,10] and XABF is the introduction of conflict resolution strategies into the framework as first-class citizens. Such an extension has many advantages as already discussed in this paper, e.g., any CRS can be used according to domain specific preferences the semantics of XABF always ensured required properties, explanations can be easily provided. We have also seen that XABF can be viewed as an instantiation of AAF [9] easily identified and provided to the user, where arguments are conflict resolutions instead of derivations.

The formal notion of CRS occurs in previous works [2,3], where the focus was however entirely on DeLP. Declarative semantics [2] and transformational semantics [3] for DeLP based on CRS were defined. In the current work, these ideas are largely pushed forward resulting into the extension of ABF as a generic framework with improved capabilities and properties. Any non-monotonic formalisms embedded into XABF, not only DeLP as in the previous works. The embedding into AAF is also entirely new.

Another well known argumentation formalism is ASPIC$^+$ [20,17], which also extends ABF with certain new features like preferences on arguments. ASPIC$^+$ is also known to satisfy the consistency and closure properties, though it relies on the addition of transposed rules. Transposition of strict rules may however introduce new undesired consequences. For instance given the logic programming (LP) rule $\neg b \to \neg a$ which does not allow to derive anything from the theory $\{a\}$, after the addition of the transposed rule $a \to b$ we derive b as a consequence of the theory $\{a\}$. This does not allow a direct embedding[5] of LP into ASPIC$^+$ as this would lead into derivation of undesired consequences. In addition, the conflict resolution in ASPIC$^+$ is rather fixed, as conflicts are always resolved by defeating one of the last used defeasible rules. XABF allows to defeat any rule, depending on a suitable conflict resolution strategy.

Toni [24] proposed so called generalized ABF with the motivation to obtain a more intuitive argumentation semantics for defeasible reasoning. The approach of Toni also assures the closure and consistency properties. On the other hand, the properties of generalized ABF are not investigated in detail on the abstract level. While sharing several goals with Toni, we proposed a principal extension of ABF witch offers well motivated and useful generalizations, such as flexible conflict resolution strategies (which also capture domain specific preferences), which are not investigated by Toni.

There are other approaches for defeasible reasoning with argumentation semantics such as Defeasible Logic Programming [12], Extended Logic Programming with Defeasible Priorities [21], and Defeasible Logic [18]. However these approaches are pertinent to logic programming and do not constitute generic argumentation frameworks such as ABF, ASPIC$^+$, or XABF. Moreover they rely on specific argumentation semantics in contrast to more general admissibility-based semantics family as established Dung [9], which is now a widely accepted standard. Neither of them satisfies both the closure and consistency properties [6].

8 Conclusions

In this paper we were concerned with ABF, a powerful abstract framework which captures a number of existing NMR formalisms by means of an argumentation semantics. Particularly focusing at defeasible reasoning, we observed that capturing it in ABF and similar frameworks requires various workarounds, such as encoding conflict resolution into the ABF's deductive system, or relying upon transposed rules as done in ASPIC$^+$, which has a number of disadvantages like obscuring explanations (the former) and affecting the semantics (the latter).

Observing further that the preferred way to resolve conflicts may be domain dependent, we formalize the notion of conflict resolution strategy (CRS) and combine it with ABF. In the extended framework, dubbed eXtended ABF (XABF), CRS is a separate first-class citizen: it can be specified by the users, as much as the language, assumptions, and deductive rules in the original ABF.

Such treatment of the CRS in XABF enables to identify the reasons why conflicts were resolved in any particular way and thus to provide for explanations. Moreover, the semantics we proposed for XABF takes care that the widely accepted properties of consistency and closure [6] are satisfied *for any given CRS*. These properties are important, as they guarantee that conflicts are not resolved just cosmetically and then consequently derived again by the deductive system.

[5] Direct embedding is such that LP rules will become the rules of the deductive system in ASPIC$^+$. Note that we do not claim that a more complex, indirect embedding cannot be done.

The fact that multiple CRSs may be formalized and used with XABF allows to easily switch between them according to the specific needs of the application domain. But, what is more, it also allows to compare different CRSs and to study their formal properties. To demonstrate this, we formally investigate the relation between subconflicts and superconflicts and the corresponding notion of minimality on conflicts. As a side-result in this paper, we characterize a class of CRSs which are minimal in the sense that resolution of superconflicts can be propagated to resolving their minimal subconflicts; that is, in these CRSs all conflicts can be resolved more effectively by considering a smaller number of cases.

Apart from our study of ABF and the resulting proposal of XABF, we also showed that CRS can also be incorporated also into Dung's Abstract Argumentation Framework [9] and that both approaches yield equivalent results.

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A Proofs

Definition 13 (Accepted Assumption). *An assumption α is* accepted *by a set of conflict resolutions $\mathcal{E}$ iff all conflict resolutions in $\mathcal{P}$ with $res(\rho) = \alpha$ are attacked by $\mathcal{E}$. We denote $\Delta_{\mathcal{E}} = \{\alpha \in \mathcal{A} \mid \forall \rho \in \mathcal{P} \colon \alpha = res(\rho) \Rightarrow \mathcal{E}$ attacks $\rho\}$.*

Definition 14 (Accepted Conflict Resolution). *A conflict resolution ρ is* accepted *by a context Δ iff $ctx(\rho) \subseteq \Delta$. We denote $\mathcal{E}_{\Delta} = \{\rho \in \mathcal{P} \mid ctx(\rho) \subseteq \Delta\}$.*

Lemma 1. *Let $\mathcal{F}_{ABF} = (\mathcal{L}, \mathcal{R}, \mathcal{A}, ^{-})$ be an ABF, $\mathcal{P}$ be a set of conflict resolutions of $\mathcal{F}_{ABF}$, and $\mathcal{F}_{AAF}$ be the instantiation of AAF defined in Definition 8. If Δ is a complete context of $\mathcal{F}_{ABF}$ with respect to $\mathcal{P}$ then $\mathcal{E}_{\Delta}$ is a complete extension of $\mathcal{F}_{AAF}$.*

Proof. Let Δ be a complete context of $\mathcal{F}_{ABF}$ with respect to $\mathcal{P}$. We show that $\mathcal{E}_{\Delta}$ is a complete extension of $\mathcal{F}_{AAF}$, i.e. that $\mathcal{E}_{\Delta}$ is attack-free, $\mathcal{E}_{\Delta}$ defends all conflict resolutions in $\mathcal{E}_{\Delta}$, and $\mathcal{E}_{\Delta}$ contains all conflict resolutions defended by $\mathcal{E}_{\Delta}$.

Let $\rho, \rho' \in \mathcal{E}_{\Delta}$ be conflict resolutions such that $(\rho, \rho') \in Att$. According to Definition 14, $ctx(\rho) \subseteq \Delta$, $ctx(\rho') \subseteq \Delta$, and according to Definition 8, $res(\rho) \in ctx(\rho') \subseteq \Delta$. According to Definition 5, Δ is not attack-free because $ctx(\rho) \subseteq \Delta$ and $res(\rho) \in \Delta$. Since Δ is complete, according to Definitions 6 and 7, Δ is attack-free and we have a contradiction. Therefore $\mathcal{E}_{\Delta}$ is attack-free.

Let $\rho \in \mathcal{E}_{\Delta}$ and $\rho' \in \mathcal{P}$ be conflict resolutions such that $(\rho', \rho) \in Att$. According to Definition 14, $ctx(\rho) \subseteq \Delta$, and according to Definition 5, $res(\rho') \in ctx(\rho) \subseteq \Delta$. Since Δ is complete, according to Definition 7, $Defence_{\mathcal{P}}(\Delta) \subseteq \Delta$. Because $res(\rho') \in \Delta$, according to Definitions 5 and 6, there exists a conflict resolution $\rho'' \in \mathcal{P}$ such that $ctx(\rho'') \subseteq \Delta$ and $res(\rho'') \in ctx(\rho')$. According to Definition 14, $\rho'' \in \mathcal{E}_{\Delta}$, and according to Definition 8, $(\rho'', \rho') \in Att$. Therefore $\mathcal{E}_{\Delta}$ defends all conflict resolutions in $\mathcal{E}_{\Delta}$.

Let $\rho \in \mathcal{P}$ be a conflict resolution defended by $\mathcal{E}$. According to Definitions 5 and 6, each conflict resolution $\rho' \in \mathcal{P}$ attacking ρ is attacked by a conflict resolution $\rho'' \in \mathcal{E}_{\Delta}$. Let ρ' attacks ρ and $\rho'' \in \mathcal{E}_{\Delta}$ attacks ρ'. According to Definition 14, $ctx(\rho'') \subseteq \Delta$ and according to Definition 8, $res(\rho'') \in ctx(\rho')$. Since Δ is complete, according to Definition 7, $Defence_{\mathcal{P}}(\Delta) \subseteq \Delta$. Then $ctx(\rho) \subseteq \Delta$ and according to Definition 14, $\rho \in \mathcal{E}_{\Delta}$. Therefore $\mathcal{E}_{\Delta}$ contains all conflict resolutions defended by $\mathcal{E}_{\Delta}$. □

Lemma 2. *Let $\mathcal{F}_{ABF} = (\mathcal{L}, \mathcal{R}, \mathcal{A}, ^{-})$ be an ABF, $\mathcal{P}$ be a set of conflict resolutions of $\mathcal{F}_{ABF}$, and $\mathcal{F}_{AAF}$ be the instantiation of AAF defined in Definition 8. If $\mathcal{E}$ is a complete extension of $\mathcal{F}_{AAF}$ then $\Delta_{\mathcal{E}}$ is a complete context of $\mathcal{F}_{ABF}$ with respect to $\mathcal{P}$.*

Proof. Let $\mathcal{E}$ be a complete extension of $\mathcal{F}_{AAF}$. We show that $\Delta_{\mathcal{E}}$ is a complete extension of $\mathcal{F}_{ABF}$ with respect to $\mathcal{P}$, i.e. that $\Delta_{\mathcal{E}}$ is attack-free, $\Delta_{\mathcal{E}}$ defends all assumptions in $\Delta_{\mathcal{E}}$, and $\Delta_{\mathcal{E}}$ contains all assumptions defended by $\Delta_{\mathcal{E}}$.

Let $\alpha \in \Delta_{\mathcal{E}}$ be attacked by $\Delta_{\mathcal{E}}$. According to Definition 5, there exists a conflict resolution $\rho \in \mathcal{P}$ such that $\alpha = res(\rho)$ and $ctx(\rho) \subseteq \Delta_{\mathcal{E}}$. According to Definition 13, $\mathcal{E}$ attacks ρ. According to Definition 8, there exists $\rho' \in \mathcal{E}$ such that $res(\rho') \in ctx(\rho) \subseteq \Delta_{\mathcal{E}}$. According to Definition 13, $\mathcal{E}$ attacks ρ' and consequently $\mathcal{E}$ is not attack-free. Since $\mathcal{E}$ is complete and thus attack-free, we have a contradiction. Therefore $\Delta_{\mathcal{E}}$ is attack-free.

Let $\alpha \in \Delta_{\mathcal{E}}$ and $\rho \in \mathcal{P}$ be a conflict resolution with $\alpha = res(\rho)$. According to Definition 13, $\mathcal{E}$ attacks ρ. According to Definition 8, there exists $\rho' \in \mathcal{E}$ such that $res(\rho') \in ctx(\rho)$. Since $\mathcal{E}$ is complete, $ctx(\rho') \subseteq \Delta_{\mathcal{E}}$. According to Definitions 5 and 6, $\Delta_{\mathcal{E}}$ defends α.

Let $\alpha \in \mathcal{A}$ such that for each $\rho \in \mathcal{P}$ with $res(\rho) = \alpha$ there exists $\rho' \in \mathcal{E}$ attacking ρ. According to Definitions 6 and 13, for each $\rho \in \mathcal{P}$, $ctx(\rho) \cap Attack_{\mathcal{P}}(\Delta_{\mathcal{E}}) \neq \emptyset$ and consequently $\alpha \in Defence_{\mathcal{P}}(\Delta_{\mathcal{E}}) \cap \Delta_{\mathcal{E}}$. Therefore $\Delta_{\mathcal{E}}$ contains all assumptions defended by $\Delta_{\mathcal{E}}$. □

Lemma 3. *Let $\mathcal{F}_{ABF} = (\mathcal{L}, \mathcal{R}, \mathcal{A}, ^{-})$ be an ABF, $\mathcal{P}$ be a set of conflict resolutions of $\mathcal{F}_{ABF}$, and $\mathcal{F}_{AAF}$ be the instantiation of AAF defined in Definition 8. Then $\Delta = \Delta_{\mathcal{E}}$ iff $\mathcal{E} = \mathcal{E}_{\Delta}$ for each complete context Δ of $\mathcal{F}_{ABF}$ and each complete extension $\mathcal{E}$ of $\mathcal{F}_{AAF}$.*

Proof. The lemma can be divided into two implications.

1. Let Δ be a complete context of $\mathcal{F}_{ABF}$ and $\mathcal{E}$ a complete extension of $\mathcal{F}_{AAF}$ such that $\Delta = \Delta_{\mathcal{E}}$.
   Let $\rho \in \mathcal{E}$. According to Definition 8, every resolution ρ' attacking ρ is attacked by some $\rho'' \in \mathcal{E}$. Let $\alpha \in ctx(\rho)$ and $\rho' \in \mathcal{P}$ be a resolution with $res(\rho') = \alpha$. From Definition 13 and assumption that $\mathcal{E}$ is a complete extension we have $\alpha \in \Delta_{\mathcal{E}}$. Consequently $\alpha \in \Delta$ and $ctx(\rho) \subseteq \Delta$. Therefore, according to Definition 14, $\rho \in \mathcal{E}_{\Delta}$.
   Let $\rho \in \mathcal{E}_{\Delta}$. From Definition 14 we have $ctx(\rho) \subseteq \Delta$. Let $\rho' \in \mathcal{P}$ be a resolution attacking ρ. From $ctx(\rho) \subseteq \Delta$, $\Delta = \Delta_{\mathcal{E}}$ and Definition 13 we have $\mathcal{E}$ attacks ρ'. Therefore $\mathcal{E}$ defends ρ. Since $\mathcal{E}$ is complete extension, $\rho \in \mathcal{E}$.
2. Let Δ be a complete context of $\mathcal{F}_{ABF}$ and $\mathcal{E}$ a complete extension of $\mathcal{F}_{AAF}$ such that $\mathcal{E} = \mathcal{E}_{\Delta}$.
   Let $\alpha \in \Delta$ and $\rho \in \mathcal{P}$ be a resolution with $res(\rho) = \alpha$. Since Δ is a complete context, according to Definitions 6 and 8, there is a resolution $\rho' \in \mathcal{P}$ attacking ρ such that $ctx(\rho') \subseteq \Delta$. From the previous Definition 14 we have $\rho' \in \mathcal{E}_{\Delta}$ and from assumption $\mathcal{E}_{\Delta} = \mathcal{E}$ also $\rho' \in \mathcal{E}$. Therefore $\mathcal{E}$ attacks ρ and consequently from Definition 13 $\alpha \in \Delta_{\mathcal{E}}$.
   Let $\alpha \in \Delta_{\mathcal{E}}$ and $\rho \in \mathcal{P}$ be a resolution with $res(\rho) = \alpha$. According to Definition 13, there is a resolution $\rho' \in \mathcal{E}$ attacking ρ. Assumption $\mathcal{E} = \mathcal{E}_{\Delta}$ implies $ctx(\rho') \subseteq \Delta$ and consequently from Definitions 5 and 6 and assumption that Δ is a complete context we have $\alpha \in \Delta$.

□

Lemma 4. *Let $\mathcal{F}_{ABF} = (\mathcal{L}, \mathcal{R}, \mathcal{A}, ^{-})$ be an ABF and $\mathcal{P}$ be a set of conflict resolutions of $\mathcal{F}_{ABF}$. Grounded context Δ of $\mathcal{F}_{ABF}$ is the least complete context of $\mathcal{F}_{ABF}$.*

Proof. Let Δ be the grounded context of $\mathcal{F}_{ABF}$. According to Definition 7, Δ is subset-maximal admissible context contained in all complete contexts. Assume by contradiction Δ is not complete. That is, there is an assumption $\alpha \in \mathit{Defence}_{\mathcal{P}}(\Delta) \setminus \Delta$. Since α is defended by Δ, Δ is a subset of every complete context and $\mathit{Defence}_{\mathcal{P}}$ is a monotonic function, α is defended by every complete context and consequently α is contained by every complete context. Since Δ is subset-maximal admissible context contained in all complete contexts, $\alpha \in \Delta$. Contradiction. Therefore Δ is complete. Fact that Δ is the least complete context directly follows from the assumption that Δ is a subset of all complete contexts.

Let Δ be the least complete context. Definition 7 and the assumption that Δ is the least complete context implies that Δ is admissible context contained by all complete contexts. Assume by contradiction Δ is not subset-maximal admissible context contained by all complete contexts. Therefore, there is an admissible context Δ' contained by all complete contexts such that $\Delta \subset \Delta'$. Since Δ is a complete context, $\Delta' \subseteq \Delta$. Contradiction. Therefore Δ is the grounded context of $\mathcal{F}_{ABF}$. □

Lemma 5. *Let $\mathcal{F}_{ABF} = (\mathcal{L}, \mathcal{R}, \mathcal{A}, \overline{\ \ })$ be an ABF and $\mathcal{P}$ be a set of conflict resolutions of $\mathcal{F}_{ABF}$. An ideal context Δ of $\mathcal{F}_{ABF}$ is a complete context of $\mathcal{F}_{ABF}$.*

Proof. Let Δ be an ideal context of $\mathcal{F}_{ABF}$. According to Definition 7, Δ is a subset-maximal admissible context contained in all preferred contexts. Let $\alpha \in \mathit{Defence}_{\mathcal{P}}(\Delta)$. Since Δ is a subset of all preferred contexts, preferred contexts are complete, and $\mathit{Defence}_{\mathcal{P}}$ is a monotonic function, α is contained by every preferred context as well. Hence $\alpha \in \Delta$ and Δ is a complete context of $\mathcal{F}_{ABF}$.

Lemma 6. *Let $\mathcal{F}_{ABF} = (\mathcal{L}, \mathcal{R}, \mathcal{A}, \overline{\ \ })$ be an ABF and $\mathcal{P}$ be a set of conflict resolutions of $\mathcal{F}_{ABF}$. A stable context Δ of $\mathcal{F}_{ABF}$ is a complete context of $\mathcal{F}_{ABF}$.*

Proof. Let Δ be a stable context of $\mathcal{F}_{ABF}$. According to Definition 7, $\Delta = \mathcal{A} \setminus \mathit{Attack}_{\mathcal{P}}(\Delta)$. We need to show that Δ is attack-free, admissible and contains all assumptions which Δ defends.

Attack-freeness follows directly from the assumption that Δ is stable context and Definition 7.

Let $\alpha \in \Delta$ and $\rho \in \mathcal{P}$ be a conflict resolution with $\mathit{res}(\rho) = \alpha$. Since Δ is attack-free and $\alpha \in \Delta$, it must be the case that $\mathit{ctx}(\rho) \nsubseteq \Delta$ and according to Definition 7, $\mathit{ctx}(\rho) \cap \mathit{Attack}_{\mathcal{P}}(\Delta) \neq \emptyset$. From Definition 6 we have, $\alpha \in \mathit{Defence}_{\mathcal{P}}(\Delta)$. Hence Δ is admissible.

Let $\alpha \in \mathit{Defence}_{\mathcal{P}}(\Delta)$. Assume by contradiction that $\alpha \notin \Delta$. Then, according to Definition 7, $\alpha \in \mathit{Attack}_{\mathcal{P}}(\Delta)$ and consequently there is a conflict resolution $\rho \in \mathcal{P}$ with $\mathit{res}(\rho) = \alpha$ and $\mathit{ctx}(\rho) \subseteq \Delta$. But since $\alpha \in \mathit{Defence}_{\mathcal{P}}(\Delta)$, $\mathit{Attack}_{\mathcal{P}}(\Delta) \cap \mathit{ctx}(\rho)$ is nonempty and consequently Δ attacks an assumption in Δ, what contradicts attack-freeness of Δ. Therefore Δ is a complete context of $\mathcal{F}_{ABF}$.

Lemma 7. *Let $\mathcal{F}_{ABF} = (\mathcal{L}, \mathcal{R}, \mathcal{A}, \overline{\ \ })$ be an ABF, $\mathcal{P}$ be a set of conflict resolutions of $\mathcal{F}_{ABF}$, and $\mathcal{F}_{AAF}$ be the instantiation of AAF defined in Definition 8.*

If $\mathcal{E}' \subset \mathcal{E}$ are complete extensions of $\mathcal{F}_{AAF}$, then $\Delta_{\mathcal{E}'} \subset \Delta_{\mathcal{E}}$. If $\Delta' \subset \Delta$ are complete contexts of $\mathcal{F}_{ABF}$, then $\mathcal{E}_{\Delta'} \subset \mathcal{E}_{\Delta}$.

Proof. Assume $\mathcal{E}'$, $\mathcal{E}$ are complete extensions of $\mathcal{F}_{AAF}$ such that $\mathcal{E}' \subset \mathcal{E}$. Let $\alpha \in \Delta_{\mathcal{E}'}$ and $\rho \in \mathcal{P}$ is a conflict resolution with $res(\rho) = \alpha$. Then $\mathcal{E}'$ attacks ρ. Then, according to Definition 13, there is a resolution $\rho' \in \mathcal{E}'$ attacking ρ. Since $\mathcal{E}' \subset \mathcal{E}$, $\mathcal{E}$ attacks ρ and consequently $\alpha \in \Delta_{\mathcal{E}}$. We have shown $\Delta_{\mathcal{E}'} \subseteq \Delta_{\mathcal{E}}$. Let $\rho^* \in \mathcal{E} \setminus \mathcal{E}'$. Then $\mathcal{E}'$ does not defend ρ^* and consequently $ctx(\rho^*) \nsubseteq \Delta_{\mathcal{E}'}$. On the other hand, $ctx(\rho^*) \subseteq \Delta_{\mathcal{E}}$. Hence $\Delta_{\mathcal{E}'} \subset \Delta_{\mathcal{E}}$.

Assume Δ', Δ are complete contexts of $\mathcal{F}_{ABF}$ such that $\Delta' \subset \Delta$. Let $\rho \in \mathcal{E}_{\Delta'}$. According to Definition 14, $ctx(\rho) \subseteq \Delta'$ and consequently from $\Delta' \subset \Delta$ also $ctx(\rho) \subseteq \Delta$. Therefore $\rho \in \mathcal{E}_{\Delta}$ and $\mathcal{E}_{\Delta'} \subseteq \mathcal{E}_{\Delta}$. Let $\alpha \in \Delta \setminus \Delta'$. Then there is a conflict resolution $\rho^* \in \mathcal{P}$ with $res(\rho^*) = \alpha$ such that Δ' does not attack an assumption in $ctx(\rho^*)$. Furthermore, there is a conflict resolution $\rho' \in \mathcal{P}$ attacking ρ^* such that $ctx(\rho') \subseteq \Delta$. From $\alpha \notin \Delta'$ we have, $ctx(\rho^*) \nsubseteq \Delta'$ and consequently $\rho^* \in \mathcal{E}_{\Delta} \setminus \mathcal{E}_{\Delta'}$. Hence $\mathcal{E}_{\Delta'} \subset \mathcal{E}_{\Delta}$. □

Proposition 1. *Let $\mathcal{F}_{ABF} = (\mathcal{L}, \mathcal{R}, \mathcal{A}, ^-)$ be an ABF, $\mathcal{P}$ be a set of conflict resolutions of $\mathcal{F}_{ABF}$, and $\mathcal{F}_{AAF}$ be the instantiation of AAF defined in Definition 8. Then Δ is a $\Sigma_{\mathcal{P}}$-context of $\mathcal{F}_{ABF}$ iff there exists a Σ-extension $\mathcal{E}$ of $\mathcal{F}_{AAF}$ such that $\Delta = \Delta_{\mathcal{E}}$.*

Proof. Five different semantics $\Sigma \in \{complete, grounded, preferred, ideal, stable\}$ have to be considered.

1. $\Sigma = complete$.
   Let Δ be a complete context of $\mathcal{F}_{ABF}$. Lemma 1 implies $\mathcal{E}_{\Delta}$ is a complete extension of $\mathcal{F}_{AAF}$. Fact that $\Delta = \Delta_{\mathcal{E}_{\Delta}}$ directly follows from Lemma 3.
   Let $\mathcal{E}$ be a complete extension of $\mathcal{F}_{AAF}$ and $\Delta \subseteq \mathcal{A}$ a context such that $\Delta = \Delta_{\mathcal{E}}$. Lemma 2 implies $\Delta = \Delta_{\mathcal{E}}$ is a complete context of $\mathcal{F}_{ABF}$.
2. $\Sigma = grounded$.
   Let Δ be the grounded context of $\mathcal{F}_{ABF}$. Lemma 4 implies Δ is a complete context of $\mathcal{F}_{ABF}$ and Lemma 1 implies $\mathcal{E}_{\Delta}$ is a complete extension of $\mathcal{F}_{AAF}$. Assume by contradiction $\mathcal{E}_{\Delta}$ is not the grounded extension of $\mathcal{F}_{AAF}$. That is, there is a complete extension $\mathcal{E}'$ of $\mathcal{F}_{AAF}$ such that $\mathcal{E}' \subset \mathcal{E}_{\Delta}$. But then Lemma 3 and 7 implies $\Delta_{\mathcal{E}'} \subset \Delta_{\mathcal{E}_{\Delta}} = \Delta$ what contradicts Lemma 4 that the grounded context Δ is the least complete context. Therefore $\mathcal{E}_{\Delta}$ is the grounded extension of $\mathcal{F}_{AAF}$. Fact that $\Delta = \Delta_{\mathcal{E}_{\Delta}}$ directly follows from Lemma 3.
   Let $\mathcal{E}$ be the grounded extension of $\mathcal{F}_{AAF}$ and $\Delta \subseteq \mathcal{A}$ a context such that $\Delta = \Delta_{\mathcal{E}}$. Since grounded extension is complete [9] and $\Delta_{\mathcal{E}} = \Delta$, Lemma 2 implies Δ is a complete context of $\mathcal{F}_{ABF}$. Assume by contradiction there is a complete context of $\mathcal{F}_{ABF}$ such that $\Delta' \subset \Delta$. But then Lemma 1 and 7 implies $\mathcal{E}_{\Delta'}$, $\mathcal{E}_{\Delta}$ are complete extensions of $\mathcal{F}_{AAF}$ such that $\mathcal{E}_{\Delta'} \subset \mathcal{E}_{\Delta}$. From the previous, the assumption that $\Delta = \Delta_{\mathcal{E}}$ and Lemma 3 we have, $\mathcal{E}_{\Delta'} \subset \mathcal{E} = \mathcal{E}_{\Delta}$, what contradicts assumption that $\mathcal{E}$ is the grounded extension. Therefore Δ is the grounded context of $\mathcal{F}_{ABF}$.

3. $\Sigma = \mathit{preferred}$.
   Let Δ be a preferred context of $\mathcal{F}_{ABF}$. Definition 7 implies Δ is a complete context of $\mathcal{F}_{ABF}$ and Lemma 1 implies $\mathcal{E}_\Delta$ is a complete extension of $\mathcal{F}_{AAF}$. Assume by contradiction $\mathcal{E}_\Delta$ is not a preferred extension of $\mathcal{F}_{AAF}$. That is, there is a complete extension $\mathcal{E}'$ of $\mathcal{F}_{AAF}$ such that $\mathcal{E}' \supset \mathcal{E}_\Delta$. But then Lemma 3 and 7 implies $\Delta_{\mathcal{E}'} \supset \Delta_{\mathcal{E}_\Delta} = \Delta$ what contradicts assumption that Δ is a subset-maximal admissible context. Therefore $\mathcal{E}_\Delta$ is a preferred extension of $\mathcal{F}_{AAF}$. Fact that $\Delta = \Delta_{\mathcal{E}_\Delta}$ directly follows from Lemma 3.
   Let $\mathcal{E}$ be a preferred extension of $\mathcal{F}_{AAF}$ and $\Delta \subseteq \mathcal{A}$ a context such that $\Delta = \Delta_\mathcal{E}$. Since preferred extension is complete [9] and $\Delta_\mathcal{E} = \Delta$, Lemma 2 implies Δ is a complete context of $\mathcal{F}_{ABF}$. Assume by contradiction there is a complete context of $\mathcal{F}_{ABF}$ such that $\Delta' \supset \Delta$. But then Lemma 1 and 7 implies $\mathcal{E}_{\Delta'}$, $\mathcal{E}_\Delta$ are complete extensions of $\mathcal{F}_{AAF}$ such that $\mathcal{E}_{\Delta'} \supset \mathcal{E}_\Delta$. From the previous, the assumption that $\Delta = \Delta_\mathcal{E}$ and Lemma 3 we have, $\mathcal{E}_{\Delta'} \supset \mathcal{E} = \mathcal{E}_\Delta$, what contradicts assumption that $\mathcal{E}$ is a preferred extension. Therefore Δ is a preferred context of $\mathcal{F}_{ABF}$.
4. $\Sigma = \mathit{ideal}$.
   Let Δ be an ideal context of $\mathcal{F}_{ABF}$. Lemma 5 implies Δ is a complete context of $\mathcal{F}_{ABF}$ and Lemma 1 implies $\mathcal{E}_\Delta$ is a complete extension of $\mathcal{F}_{AAF}$. We need to show that $\mathcal{E}_\Delta$ is a maximal complete extension contained by every preferred extension of $\mathcal{F}_{AAF}$. Let $\mathcal{E}$ be a preferred extension of $\mathcal{F}_{AAF}$ and $\rho \in \mathcal{E}_\Delta$. Since Δ is an ideal context of $\mathcal{F}_{ABF}$ and Δ is a subset of all preferred contexts of $\mathcal{F}_{ABF}$, according to Definition 14, ρ is contained by every preferred extension of $\mathcal{F}_{AAF}$. Therefore $\rho \in \mathcal{E}$. Assume by contradiction there is a complete extension $\mathcal{E}'$ contained by every preferred extension of $\mathcal{F}_{AAF}$ such that $\mathcal{E}' \supset \mathcal{E}_\Delta$. Lemma 7 and 3 and Proposition 1 for special case $\Sigma = \mathit{preferred}$, proved above, implies that $\Delta_{\mathcal{E}'}$ is a context contained by every preferred context of $\mathcal{F}_{ABF}$ and $\Delta_{\mathcal{E}'} \supset \Delta = \Delta_{\mathcal{E}_\Delta}$, what contradicts the assumption that Δ is an ideal context of $\mathcal{F}_{ABF}$. Therefore $\mathcal{E}_\Delta$ is an ideal extension of $\mathcal{F}_{AAF}$. Fact that $\Delta = \Delta_{\mathcal{E}_\Delta}$ directly follows from Lemma 3.
   Let $\mathcal{E}$ be an ideal extension of $\mathcal{F}_{AAF}$ and $\Delta \subseteq \mathcal{A}$ a context such that $\Delta = \Delta_\mathcal{E}$. Since ideal extensions are complete [10], Lemma 2 implies $\Delta_\mathcal{E}$ is a complete context of $\mathcal{F}_{ABF}$. We need to show that Δ is a maximal complete context contained by every preferred context of $\mathcal{F}_{ABF}$. Let Δ_p be a preferred context of $\mathcal{F}_{ABF}$ and $\alpha \in \Delta$. From $\Delta \subseteq \Delta_p$, monotonicity of $\mathit{Defence}_\mathcal{P}$ function and fact that preferred contexts are complete we have, $\alpha \in \Delta_p$. Assume by contradiction there is a complete context Δ' contained by every preferred context of $\mathcal{F}_{ABF}$ such that $\Delta' \supset \Delta$. Lemma 7 and 3 and Proposition 1 for special case $\Sigma = \mathit{preferred}$, proved above, implies that $\mathcal{E}_{\Delta'}$ is an extension contained by every preferred extension of $\mathcal{F}_{AAF}$ and $\mathcal{E}_{\Delta'} \supset \mathcal{E}_\Delta = \mathcal{E}_{\Delta_\mathcal{E}} = \mathcal{E}$, what contradicts the assumption that $\mathcal{E}$ is an ideal extension of $\mathcal{F}_{AAF}$. Therefore Δ is an ideal context of $\mathcal{F}_{ABF}$.
5. $\Sigma = \mathit{stable}$.
   Let Δ be a stable context of $\mathcal{F}_{ABF}$. Lemma 6 implies Δ is a complete context of $\mathcal{F}_{ABF}$ and Lemma 1 implies $\mathcal{E}_\Delta$ is a complete extension of $\mathcal{F}_{AAF}$. We need to show that $\mathcal{E}_\Delta$ is an attack-free extension attacking each conflict resolution

which does not belong to $\mathcal{E}_\Delta$. Assume by contradiction that $\mathcal{E}_\Delta$ is not attack-free. Then, there are conflict resolutions $\rho_1, \rho_2 \in \mathcal{P}$ such that $ctx(\rho_1) \subseteq \Delta$, $ctx(\rho_2) \subseteq \Delta$ and ρ_1 attacks ρ_2, what contradicts Lemma 6 that stable contexts are attack-free. Hence $\mathcal{E}_\Delta$ is attack-free in $\mathcal{F}_{AAF}$. Let $\rho \in \mathcal{P} \setminus \mathcal{E}_\Delta$. From Definition 14 we have, $ctx(\rho) \nsubseteq \Delta$ and the assumption that Δ is a stable context implies $ctx(\rho) \cap Attack_{\mathcal{P}}(\Delta)$ is nonempty. Therefore, according to Definitions 5 and 8, there is a conflict resolution $\rho^* \in \mathcal{P}$ attacking ρ such that $ctx(\rho^*) \subseteq \Delta$. Consequently $\rho^* \in \mathcal{E}_\Delta$ and $\mathcal{E}_\Delta$ attacks ρ. Hence $\mathcal{E}_\Delta$ is a stable extension of $\mathcal{F}_{AAF}$. Fact that $\Delta = \Delta_{\mathcal{E}_\Delta}$ directly follows from Lemma 3.

Let $\mathcal{E}$ be a stable extension of $\mathcal{F}_{AAF}$ and $\Delta \subseteq \mathcal{A}$ a context such that $\Delta = \Delta_{\mathcal{E}}$. Since stable extensions are complete [9], Lemma 2 implies $\Delta_{\mathcal{E}}$ is a complete context of $\mathcal{F}_{ABF}$. We need to show that Δ attacks every assumption which does not belong to Δ. Let $\alpha \in \mathcal{A} \setminus \Delta$. Assume by contradiction that Δ does not attack α. From $\alpha \notin \Delta = \Delta_{\mathcal{E}}$ we have, there is a conflict resolution $\rho \in \mathcal{P}$ such that $res(\rho) = \alpha$ and $\mathcal{E}$ does not attack ρ. From the previous and the assumption that $\mathcal{E}$ is a stable extension of $\mathcal{F}_{AAF}$ we have, $\rho \in \mathcal{E}$ and consequently, Definition 13 and fact that stable extensions are complete imply $ctx(\rho) \subseteq \Delta_{\mathcal{E}} = \Delta$, what contradicts the assumption that Δ does not attack α. Therefore Δ attacks α and consequently Δ is a stable context of $\mathcal{F}_{ABF}$.

□

Proposition 2. *Let $\mathcal{F} = (\mathcal{L}, \mathcal{R}, \mathcal{A}, ^-)$ be an assumption-based framework and $\mathcal{P} = \{(D_1, D_2, \alpha) \mid \alpha \in \mathcal{A}, cons(D_1) = \alpha, cons(D_2) = \overline{\alpha}\}$. If $\mathcal{F}$ is flat, then $\mathcal{E}$ is a conflict-free $\Sigma_{\mathcal{P}}$-extension of $\mathcal{F}$ iff $\mathcal{E}$ is an Σ-extension of $\mathcal{F}$ as in [5,10].*

Proof. Five different semantics $\Sigma \in \{complete, grounded, preferred, ideal, stable\}$ have to be considered. Assume $\mathcal{F}$ is flat. Observe, that this implies D_1 is a default derivation $[\alpha]$, for each conflict resolution $(D_1, D_2, \alpha) \in \mathcal{P}$.

1. $\Sigma = complete$.
   Let Δ be a complete context of $\mathcal{F}$ and $\mathcal{E} = Cn_{\mathcal{R}}(\Delta)$ a complete conflict-free extension of $\mathcal{F}$. Then Δ is closed. We need to show that Δ is admissible and contains every assumption which it defends with respect to [5,10].
   Let $\alpha \in \Delta$ and $\Delta'' \subseteq \mathcal{A}$ be a context attacking (with respect to [5,10]) α. Then $\overline{\alpha} \in Cn_{\mathcal{R}}(\Delta'')$. From the previous, construction of $\mathcal{P}$ and definition of derivations we have, there exists a conflict resolution $\rho \in \mathcal{P}$ such that $\rho = (\alpha, D_2, \alpha)$, $cons(D_2) = \overline{\alpha}$ and $prem(D_2) \subseteq \Delta''$.
   Since Δ is a complete context of $\mathcal{F}$ and $\alpha \in \Delta$, there is a conflict resolution ρ' such that $res(\rho') \in ctx(\rho)$ and $ctx(\rho') \subseteq \Delta$. Furthermore, it holds that $res(\rho') \notin \Delta$. From the previous and the construction of $\mathcal{P}$ it follows that $res(\rho') \in ctx(\rho)$ and $\overline{res(\rho')} \in Cn_{\mathcal{R}}(ctx(\rho'))$. Since $ctx(\rho') \subseteq \Delta$ and $res(\rho') \in \Delta'' \setminus \Delta$, Δ attacks (with respect to [5,10]) $\Delta'' \setminus \Delta$. So Δ is admissible with respect to [5,10].
   Let α' be an assumption defended (with respect to [5,10]) by Δ and ρ'' be a conflict resolution with $res(\rho'') = \alpha'$. Then there is a derivation D_2 such that

$\rho'' = (\alpha', D_2, \alpha')$, $prem(D_2) = ctx(\rho'')$, $\overline{\alpha'} \in Cn_{\mathcal{R}}(ctx(\rho''))$ and consequently Δ attacks (with respect to [5,10]) $ctx(\rho'') \setminus \Delta$. Then there is an assumption β such that $\overline{\beta} \in Cn_{\mathcal{R}}(\Delta)$, $\beta \in ctx(\rho'') \setminus \Delta$ and construction of $\mathcal{P}$ guarantees existence of a derivation D_2 such that $cons(D_2) = \overline{\beta}$, $prem(D_2) \subseteq \Delta$ and $(\beta, D_2, \beta) \in \mathcal{P}$. From the previous we have $\alpha' \in Defence_{\mathcal{P}}(\Delta)$ and since Δ is complete, $\alpha' \in \Delta$. Therefore Δ is also complete with respect to [5,10] and $Cn_{\mathcal{R}}(\Delta)$ is a complete extension of $\mathcal{F}$ as in [5,10].

Let Δ be a complete context and $\mathcal{E} = Cn_{\mathcal{R}}(\Delta)$ a complete extension of $\mathcal{F}$ as in [5,10]. We need to show that $\mathcal{E}$ is conflict-free and Δ is a complete context of $\mathcal{F}$ with respect to Definition 7.

Assume by contradiction that $\mathcal{E} = Cn_{\mathcal{R}}(\Delta)$ is not conflict-free. Then there is an assumption $\alpha \in \Delta$ such that $\overline{\alpha} \in Cn_{\mathcal{R}}(\Delta)$. Then Δ attacks Δ. Since Δ is admissible (with respect to [5,10]), Δ attacks an assumption in $\Delta \setminus \Delta$. Contradiction. Hence $\mathcal{E}$ is conflict-free and consequently, from the construction of $\mathcal{P}$, Δ is attack-free with respect to Definition 6.

Let $\alpha \in \Delta$ and $\rho \in \mathcal{P}$ be a conflict resolution such that $res(\rho) = \alpha$. We show that Δ defends α with respect to Definition 6. The construction of $\mathcal{P}$ implies there is a derivation D such that $\rho = (\alpha, D, \alpha)$, and $cons(D) = \overline{\alpha}$. Therefore $ctx(\rho)$ attacks (with respect to [5,10]) Δ and consequently Δ attacks (with respect to [5,10]) $ctx(\rho) \setminus \Delta$. That means, there is an assumption $\beta \in ctx(\rho)$ such that $\overline{\beta} \in Cn_{\mathcal{R}}(\Delta)$ and consequently there is a derivation D' such that $cons(D') = \overline{\beta}$, $\rho' = (\beta, D', \beta) \in \mathcal{P}$ and $prem(D') = ctx(\rho') \subseteq \Delta$. Acorrding to Definitions 5 and 6, $\alpha \in Defence_{\mathcal{P}}(\Delta)$. Therefore Δ is admissible with respect to Definition 6.

Let $\alpha \in Defence_{\mathcal{P}}(\Delta)$ and $\rho = (\alpha, D, \alpha) \in \mathcal{P}$ be a conflict resolution such that $cons(D) = \overline{\alpha}$. Then there is a conflict resolution $\rho' = (\beta, D', \beta) \in \mathcal{P}$ such that $cons(D') = \overline{\beta}$, $\beta \in ctx(\rho)$ and $prem(D') \subseteq \Delta$. Furthermore $\beta \notin \Delta$ is guaranteed by attack-freeness of Δ. From the previous we have Δ defends α with respect to [5,10]. Since Δ is complete with respect to [5,10], $\alpha \in \Delta$. Therefore Δ is also a complete context and $\mathcal{E} = Cn_{\mathcal{R}}(\Delta)$ a complete extension of $\mathcal{F}$ with respect to Definition 7.

2. $\Sigma = grounded$.

Let Δ be the grounded context of $\mathcal{F}$ and $\mathcal{E} = Cn_{\mathcal{R}}(\Delta)$ the grounded conflict-free extension of $\mathcal{F}$. We need to show that Δ is the subset-least complete context with respect to [5,10]. Since the grounded context is also a complete context (Lemma 4), the first case of Proposition 2 implies, Δ is a complete context with respect to [5,10]. We show it is the least one. Assume by contradiction there is a context $\Delta' \subset \Delta$ such that Δ' is complete with respect to [5,10]. We show Δ' is also a complete context with respect to Definition 7.

Attack-freeness of Δ' follows from the assumption that Δ is grounded and $\Delta' \subset \Delta$. Let $\alpha \in \Delta'$ and $\rho = (\alpha, D, \alpha) \in \mathcal{P}$ such that $cons(D) = \overline{\alpha}$. Then $ctx(\rho)$ attacks Δ' and Δ' attacks $ctx(\rho) \setminus \Delta'$ with respect to [5,10]. Therefore there is an assumption $\beta \in ctx(\rho)$ and a derivation D' such that $(\beta, D', \beta) \in \mathcal{P}$, where $cons(D') = \overline{\beta}$ and $prem(D') \subseteq \Delta'$. Hence, according to Definition 6, Δ' defends α.

Let $\alpha \in \mathit{Defence}_{\mathcal{P}}(\Delta')$ and $\rho = (\alpha, D, \alpha) \in \mathcal{P}$ such that $\mathit{cons}(D) = \overline{\alpha}$. Then there is an assumption $\beta \in \mathit{ctx}(\rho)$ and a derivation D' such that $(\beta, D', \beta) \in \mathcal{P}$, where $\mathit{cons}(D') = \overline{\beta}$ and $\mathit{prem}(D') \subseteq \Delta'$. Therefore Δ' defends α with respect to [5,10] and from [5,10]-completeness of Δ' we have $\alpha \in \Delta'$. So $\Delta' \subset \Delta$ is a complete context of $\mathcal{F}$ with respect to Definition 7, what contradicts the assumption that Δ is the least complete context. Therefore, Δ is the least complete context and $\mathcal{E} = \mathit{Cn}_{\mathcal{R}}(\Delta)$ the least complete extension of $\mathcal{F}$ with respect to [5,10].

Let Δ be the grounded context and $\mathcal{E} = \mathit{Cn}_{\mathcal{R}}(\Delta)$ the grounded extension of $\mathcal{F}$ as in [5,10]. Lemma 4 and the first case of Proposition 2 implies, Δ is a complete context of $\mathcal{F}$ with respect to Definition 7. We show it is the least one. Assume by contradiction there is a context $\Delta' \subset \Delta$ such that Δ' is complete with respect to Definition 7. We show Δ' is also a complete context with respect to [5,10]. Since $\mathcal{F}$ is flat, Δ' is closed.

Let $\alpha \in \Delta'$ and Δ'' be a context such that $\overline{\alpha} \in \mathit{Cn}_{\mathcal{R}}(\Delta'')$. Then there is a derivation D such that $(\alpha, D, \alpha) \in \mathcal{P}$ and $\mathit{cons}(D) = \overline{\alpha}$. Since, according to Definition 7, Δ' is complete, there is an assumption $\beta \in \mathit{ctx}(\rho)$ and a derivation D' such that $(\beta, D', \beta) \in \mathcal{P}$, $\mathit{cons}(D') = \overline{\beta}$ and $\mathit{prem}(D') \subseteq \Delta'$. Furthermore, $\beta \notin \Delta'$ follows from attack-freeness of Δ'. Therefore Δ' attacks $\Delta'' \setminus \Delta$ and consequently Δ' is admissible with respect to [5,10].

Let α be defended by Δ' with respect to [5,10] and Δ'' be a context such that $\overline{\alpha} \in \mathit{Cn}_{\mathcal{R}}(\Delta'')$. Then there is a conflict resolution $\rho = (\alpha, D', \alpha) \in \mathcal{P}$ such that $\mathit{cons}(D') = \overline{\alpha}$ and $\mathit{prem}(D') \subseteq \Delta''$. Since Δ' defends α with respect to [5,10], there is an assumption $\beta \in \mathit{ctx}(\rho)$ and a derivation D such that $\mathit{cons}(D) = \overline{\beta}$ and $\mathit{prem}(D) \subseteq \Delta'$. From the construction of $\mathcal{P}$, $(\beta, D, \beta) \in \mathcal{P}$ and consequently $\alpha \in \mathit{Defence}_{\mathcal{P}}(\Delta')$ according to Definition 6. Since Δ' is complete with respect to Definition 7, $\alpha \in \Delta'$. Therefore $\Delta' \subset \Delta$ is a complete context of $\mathcal{F}$ with respect to [5,10], what contradicts the assumption that Δ is the grounded context. Therefore, Δ is the least complete context and $\mathcal{E} = \mathit{Cn}_{\mathcal{R}}(\Delta)$ the least complete extension of $\mathcal{F}$ with respect to Definition 7.

3. $\Sigma = \mathit{preferred}$.

Let Δ be a preferred context of $\mathcal{F}$ and $\mathcal{E} = \mathit{Cn}_{\mathcal{R}}(\Delta)$ a preferred conflict-free extension of $\mathcal{F}$. We need to show that Δ is a subset-maximal complete context with respect to [5,10]. Since a preferred context is also a complete context, the first case of Proposition 2 implies, Δ is a complete context with respect to [5,10]. We show it is a maximal one. Assume by contradiction there is a context $\Delta' \supset \Delta$ such that Δ' is complete with respect to [5,10]. We show Δ' is also a complete context with respect to Definition 7.

Attack-freeness of Δ' follows from the fact that Δ' does not attack Δ' with respect to [5,10]. Let $\alpha \in \Delta'$ and $\rho = (\alpha, D, \alpha) \in \mathcal{P}$ such that $\mathit{cons}(D) = \overline{\alpha}$. Then $\mathit{ctx}(\rho)$ attacks Δ' and Δ' attacks $\mathit{ctx}(\rho) \setminus \Delta'$ with respect to [5,10]. Therefore there is an assumption $\beta \in \mathit{ctx}(\rho)$ and a derivation D' such that $(\beta, D', \beta) \in \mathcal{P}$, where $\mathit{cons}(D') = \overline{\beta}$ and $\mathit{prem}(D') \subseteq \Delta'$. Hence, according to Definition 6, Δ' defends α.

Let $\alpha \in \mathit{Defence}_{\mathcal{P}}(\Delta')$ and $\rho = (\alpha, D, \alpha) \in \mathcal{P}$ such that $\mathit{cons}(D) = \overline{\alpha}$. Then there is an assumption $\beta \in \mathit{ctx}(\rho)$ and a derivation D' such that $(\beta, D', \beta) \in \mathcal{P}$, where $\mathit{cons}(D') = \overline{\beta}$ and $\mathit{prem}(D') \subseteq \Delta'$. Therefore Δ' defends α with respect to [5,10] and from [5,10]-completeness of Δ' we have $\alpha \in \Delta'$. So $\Delta' \supset \Delta$ is a complete context of $\mathcal{F}$ with respect to Definition 7, what contradicts the assumption that Δ is a maximal complete context. Therefore, Δ is a maximal complete context and $\mathcal{E} = \mathit{Cn}_{\mathcal{R}}(\Delta)$ a maximal complete extension of $\mathcal{F}$ with respect to [5,10].

Let Δ be a preferred context and $\mathcal{E} = \mathit{Cn}_{\mathcal{R}}(\Delta)$ a preferred extension of $\mathcal{F}$ as in [5,10]. Definition 7 and the first case of Proposition 2 implies, Δ is a complete context of $\mathcal{F}$ with respect to Definition 7. We show it is a maximal one. Assume by contradiction there is a context $\Delta' \supset \Delta$ such that Δ' is complete with respect to Definition 7. We show Δ' is also a complete context with respect to [5,10]. Since $\mathcal{F}$ is flat, Δ' is closed.

Let $\alpha \in \Delta'$ and Δ'' be a context such that $\overline{\alpha} \in \mathit{Cn}_{\mathcal{R}}(\Delta'')$. Then there is a derivation D such that $(\alpha, D, \alpha) \in \mathcal{P}$ and $\mathit{cons}(D) = \overline{\alpha}$. Since, according to Definition 7, Δ' is complete, there is an assumption $\beta \in \mathit{ctx}(\rho)$ and a derivation D' such that $(\beta, D', \beta) \in \mathcal{P}$, $\mathit{cons}(D') = \overline{\beta}$ and $\mathit{prem}(D') \subseteq \Delta'$. Furthermore, $\beta \notin \Delta'$ follows from attack-freeness of Δ'. Therefore Δ' attacks $\Delta'' \setminus \Delta$ and consequently Δ' is admissible with respect to [5,10].

Let α be defended by Δ' with respect to [5,10] and Δ'' be a context such that $\overline{\alpha} \in \mathit{Cn}_{\mathcal{R}}(\Delta'')$. Then there is a conflict resolution $\rho = (\alpha, D', \alpha) \in \mathcal{P}$ such that $\mathit{cons}(D') = \overline{\alpha}$ and $\mathit{prem}(D') \subseteq \Delta''$. Since Δ' defends α with respect to [5,10], there is an assumption $\beta \in \mathit{ctx}(\rho)$ and a derivation D such that $\mathit{cons}(D) = \overline{\beta}$ and $\mathit{prem}(D) \subseteq \Delta'$. From the construction of $\mathcal{P}$, $(\beta, D, \beta) \in \mathcal{P}$ and consequently $\alpha \in \mathit{Defence}_{\mathcal{P}}(\Delta')$ according to Definition 6. Since Δ' is complete with respect to Definition 7, $\alpha \in \Delta'$. Therefore $\Delta' \supset \Delta$ is a complete context of $\mathcal{F}$ with respect to [5,10], what contradicts the assumption that Δ is a preferred context. Therefore, Δ is a maximal complete context and $\mathcal{E} = \mathit{Cn}_{\mathcal{R}}(\Delta)$ a maximal complete extension of $\mathcal{F}$ with respect to Definition 7.

4. $\Sigma = \mathit{ideal}$.

Let Δ be an ideal context of $\mathcal{F}$ and $\mathcal{E} = \mathit{Cn}_{\mathcal{R}}(\Delta)$ an ideal conflict-free extension of $\mathcal{F}$. We need to show that Δ is a maximal complete context contained by every preferred context with respect to [5,10]. Since an ideal context is also a complete context (Lemma 5), the first case of Proposition 2 implies, Δ is a complete context with respect to [5,10]. Let Δ_p be a preferred context with respect [5,10]. Let $\alpha \in \Delta$. Since Δ is an ideal context with respect to Definition 7, the third case of Proposition 2 implies $\alpha \in \Delta_p$. Assume by contradiction there is a complete context Δ' contained by every preferred context of $\mathcal{F}$ with respect to [5,10] such that $\Delta' \supset \Delta$. The first and the third cases of Proposition 2 implies, Δ' is an ideal context with respect to Definition 7, what contradicts the assumption that Δ is an ideal context of $\mathcal{F}$ with respect to Definition 7. Therefore Δ is an ideal context and $\mathcal{E} = \mathit{Cn}_{\mathcal{R}}(\Delta)$ an ideal extension of $\mathcal{F}$ with respect to [5,10].

Let Δ be an ideal context of $\mathcal{F}$ and $\mathcal{E} = Cn_{\mathcal{R}}(\Delta)$ an ideal conflict-free extension of $\mathcal{F}$ as in [5,10]. We need to show that Δ is a maximal complete context contained by every preferred context with respect to Definition 7. Since an ideal context is also a complete context (Lemma 5), the first case of Proposition 2 implies, Δ is a complete context with respect to Definition 7. Let Δ_p be a preferred context with respect Definition 7. Let $\alpha \in \Delta$. Since Δ is an ideal context with respect to [5,10], the third case of Proposition 2 implies $\alpha \in \Delta_p$. Assume by contradiction there is a complete context Δ' contained by every preferred context of $\mathcal{F}$ with respect to Definition 7 such that $\Delta' \supset \Delta$. The first and the third cases of Proposition 2 implies, Δ' is an ideal context with respect to [5,10], what contradicts the assumption that Δ is an ideal context of $\mathcal{F}$ with respect to [5,10]. Therefore Δ is an ideal context and $\mathcal{E} = Cn_{\mathcal{R}}(\Delta)$ an ideal extension of $\mathcal{F}$ with respect to Definition 7.

5. $\Sigma = stable$.
Let Δ be a stable context of $\mathcal{F}$ and $\mathcal{E} = Cn_{\mathcal{R}}(\Delta)$ a stable conflict-free extension of $\mathcal{F}$. We need to show that Δ is a stable context of $\mathcal{F}$ with respect to [5,10]. Lemma 6 and the first case of Proposition 2 implies, Δ is a complete context of $\mathcal{F}$ with respect to [5,10]. Therefore Δ does not attack itself with respect to [5,10]. We need to show that Δ attacks (with respect to [5,10]) every assumption not belonging to Δ. Let $\alpha \in \mathcal{A} \setminus \Delta$. Since Δ is stable with respect to Definition 7, there is a conflict resolution $\rho \in \mathcal{P}$ such that $\rho = (\alpha, D, \alpha)$, $cons(D) = \overline{\alpha}$ and $prem(D) \subseteq \Delta$. From the previous we have, Δ attacks α with respect to [5,10]. Hence Δ is a stable context and $\mathcal{E} = Cn_{\mathcal{R}}(\Delta)$ a stable extension of $\mathcal{F}$ with [5,10].
Let Δ be a stable context of $\mathcal{F}$ and $\mathcal{E} = Cn_{\mathcal{R}}(\Delta)$ a stable conflict-free extension of $\mathcal{F}$ as in [5,10]. We need to show that Δ is a stable context of $\mathcal{F}$ with respect to Definition 7. Lemma 6 and the first case of Proposition 2 implies, Δ is a complete context of $\mathcal{F}$ with respect to Definition 7. Therefore Δ is attack-free with respect to Definition 5. We need to show that Δ attacks (with respect to Definition 5) every assumption not belonging to Δ. Let $\alpha \in \mathcal{A} \setminus \Delta$. Since Δ is stable with respect to [5,10], $\overline{\alpha} \in Cn_{\mathcal{R}}(\Delta)$. From the previous and the construction of $\mathcal{P}$ we have, there is a derivation D such that $(\alpha, D, \alpha) \in \mathcal{P}$, $cons(D) = \overline{\alpha}$ and $prem(D) \subseteq \Delta$. Consequently Δ attacks α with respect to Definition 5. Hence Δ is a stable context and $\mathcal{E} = Cn_{\mathcal{R}}(\Delta)$ a stable extension of $\mathcal{F}$ with respect to Definition 7.

□

Proposition 3. *If $\mathcal{P}$ is total then each $\Sigma_{\mathcal{P}}$-extension is conflict-free.*

Proof. Let $\mathcal{P}$ be a total set of conflict resolutions and Δ be an attack-free context. Let Δ be not conflict-free, i.e. there exist a conflict (D_1, D_2) such that $prem(D_1) \subseteq \Delta$ and $prem(D_2) \subseteq \Delta$. Since $\mathcal{P}$ is total, there exists a conflict resolution $\rho = (D_1, D_2, \alpha) \in \mathcal{P}$. Then $ctx(\rho) \subseteq prem(D_1) \cup prem(D_2) \subseteq \Delta$. i.e. Δ is not attack-free and we have a contradiction. Because each $\Sigma_{\mathcal{P}}$-extension equals to $Cn_{\mathcal{R}}(\Delta)$ for some attack-free context Δ, it is also conflict free. □

Proposition 4. *If $\emptyset$ is not conflict-free then each $\Sigma_{\mathcal{P}}$-extension is not conflict-free.*

Proof. Because $Cn_{\mathcal{R}}$ is monotonic, $Cn_{\mathcal{R}}(\emptyset) \subseteq Cn_{\mathcal{R}}(\Delta)$ for each context Δ. Therefore if $\mathcal{E} = Cn_{\mathcal{R}}(\Delta) \supseteq Cn_{\mathcal{R}}(\emptyset)$ is an $\Sigma_{\mathcal{P}}$-extension, $\mathcal{E}$ is not conflict-free. $\square$

Proposition 5. *Each $\Sigma_{\mathcal{P}}$-extension is closed.*

Proof. According to Definition 7, $\mathcal{E} = Cn_{\mathcal{R}}(\Delta)$ for some context Δ, i.e. $\mathcal{E}$ is closed. $\square$

Proposition 6. *A context $\mathcal{E}$ is a $\Sigma_{\mathcal{P}}$-extension iff $\mathcal{E}$ is a $\Sigma_{\lfloor\mathcal{P}\rfloor}$-extension.*

Proof. Let Δ be a context and α be an assumption. We show that Δ attacks resp. defends α with respect to $\mathcal{P}$ iff Δ attacks resp. defends α with respect to $\lfloor\mathcal{P}\rfloor$.

Let Δ attacks α w.r.t. $\mathcal{P}$. According to Defintion 5, there is $\rho = (D_1, D_2, \alpha) \in \mathcal{P}$ with $ctx(\rho) \subseteq \Delta$. Since both D_1 and D_2 are finite, there exist only finitely many proper subconflicts of (D_1, D_2). If there does not exist $(D_1', D_2') \sqsubset (D_1, D_2)$ such that $\rho' = (D_1', D_2', \alpha) \in \mathcal{P}$, according to Definition 11, $\rho \in \lfloor\mathcal{P}\rfloor$ and Δ attacks α w.r.t. $\lfloor\mathcal{P}\rfloor$. Let (D_1', D_2') be a minimal proper subconflict of (D_1, D_2) with $\rho' = (D_1', D_2', \alpha) \in \mathcal{P}$. According to Definition 11, $\rho' \in \lfloor\mathcal{P}\rfloor$. If $\alpha \in prem(D_1) \cap prem(D_2)$, then $ctx(\rho') \subseteq prem(D_1) \cup prem(D_2) = ctx(\rho)$. If $\alpha \notin prem(D_1') \cap prem(D_2')$, then $ctx(\rho') = prem(D_1') \cup prem(D_2') \setminus \{\alpha\} \subseteq ctx(\rho)$. Therefore $ctx(\rho') \subseteq \Delta$ and according to Definition 5, Δ attacks α w.r.t. $\lfloor\mathcal{P}\rfloor$. Because $\lfloor\mathcal{P}\rfloor \subseteq \mathcal{P}$, the inverse implication is straightforward.

Let Δ defends α w.r.t. $\lfloor\mathcal{P}\rfloor$. According to Definition 6, for all $\rho = (D_1, D_2, \alpha) \in \lfloor\mathcal{P}\rfloor$ holds $ctx(\rho) \cap Attack_{\lfloor\mathcal{P}\rfloor}(\Delta) = ctx(\rho) \cap Attack_{\mathcal{P}}(\Delta) \neq \emptyset$. Let $\rho = (D_1, D_2, \alpha) \in \mathcal{P}$. Similarly as in the previous paragraph, either $\rho \in \lfloor\mathcal{P}\rfloor$ or there exists $\rho' = (D_1', D_2', \alpha) \in \lfloor\mathcal{P}\rfloor$ with $(D_1', D_2') \sqsubset (D_1, D_2)$ such that $ctx(\rho') \subseteq ctx(\rho)$. Since $ctx(\rho') \cap Attack_{\mathcal{P}}(\Delta) \neq \emptyset$, also $ctx(\rho) \cap Attack_{\mathcal{P}}(\Delta) \neq \emptyset$ and according to Definition 6, Δ defends α with respect to $\mathcal{P}$. Because $\lfloor\mathcal{P}\rfloor \subseteq \mathcal{P}$, the inverse implication is straightforward.

Proposition 7. *The bottom of a downward closed set $\mathcal{P}$ of conflict resolutions contains only $\sqsubseteq$-minimal conflicts.*

Proof. Let $\mathcal{P}$ be downward closed and $(D_1, D_2, \alpha) \in \lfloor\mathcal{P}\rfloor \subseteq \mathcal{P}$. According to Definition 11, for all $(D_1', D_2') \sqsubset (D_1, D_2)$ holds if $\alpha \in prem(D_1') \cup prem(D_2')$ then $(D_1', D_2', \alpha) \notin \mathcal{P}$. According to Definition 12, there exists a minimal $(D_1', D_2') \sqsubseteq (D_1, D_2)$ with $\alpha \in prem(D_1') \cup prem(D_2')$ such that $(D_1', D_2', \alpha) \in \mathcal{P}$. Therefore $(D_1', D_2') = (D_1, D_2)$, i.e. (D_1, D_2) is a minimal conflict. $\square$