ExpTime Reasoning for Contextualized $\mathcal{ALC}$

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Abstract

The combination of often heterogeneous and contradicting knowledge sources in the Semantic Web demands for methods to represent and reason with a notion of context. Among context representation frameworks, Contextualized Knowledge Repository (CKR) is a novel proposal grounded in the well-studied AI theories of context and aims at bringing the advantages of contextual representation to the Semantic Web. In this paper we show that reasoning in $\mathcal{ALC}$-based CKR is decidable and EXPTIME-complete. The result is presented trough a reduction from $\mathcal{ALC}$-based CKRs to $\mathcal{ALCO}(\sqcup)$ knowledge bases. This result is important, because it shows that the addition of the contextual layer does not come at the cost of increased complexity of reasoning.

1 Introduction

The enormous amount of semantic resources available on the Web is the result of the autonomous contribution of communities of people and applications. It is becoming clear that most of the knowledge represented in these resources is not universally valid, but it holds under certain circumstances, such as within a given time range, or geographical/cultural region, or only in a specific domain/community of interest. Sometimes, these circumstances are made explicit in the resource meta-data, in other cases they are hard-coded in URIs, and in extreme cases they are completely missing. Development of standard languages that enable to specify contextual meta-data, and formal semantics that explains how this additional meta-knowledge should be interpreted, is one of the important points in the agenda of the Semantic Web and Linked Open Data research communities. This interest is witnessed by the recent proposals of formalisms for the representation and reasoning about contextual knowledge in the Semantic Web (Klarman and Gutiérrez-Basulto 2011; Bao, Tao, and McGuinness 2010; Sahoo et al. 2010; Serafini and Homola 2012). Contextualized Knowledge Repository (CKR), introduced in the latter paper, is a logical framework that allows for encapsulation of description logic (DL) knowledge bases (KB) into contexts, and the specification of the contextual structure via meta language. (Serafini and Homola 2012) proves some basic logical properties for CKR where contextualized knowledge is expressed in the highly complex DL language $\mathcal{SROIQ}$. This work shows that reasoning with $\mathcal{SROIQ}$-based CKR is 2NEXPTIME-complete by a polynomial embedding of a CKR into a flat $\mathcal{SROIQ}$ knowledge base. The consequence of this being that the complexity of reasoning with $\mathcal{SROIQ}$-based CKR is same as for $\mathcal{SROIQ}$.

In many cases, however, the knowledge available in the Semantic Web is expressed in weaker/simpler languages, with lower complexity. Therefore it turns out to be of interest to investigate on the complexity of CKR based on simpler DL languages. In this paper we focus on $\mathcal{ALC}$, where the problem of KB consistency is EXPTIME-complete, and we want to check if it is possible to add a context structure on top of $\mathcal{ALC}$ without jumping to a higher complexity class.

Among other approaches for contextualized reasoning on top of $\mathcal{ALC}$, the lowest complexity of reasoning is known for the contextual DL $\mathcal{ALC}_{\mathcal{ALC}}$, which is 2EXPTIME-complete (Klarman and Gutiérrez-Basulto 2011). A trivial upper-bound for $\mathcal{ALC}$-based CKR can be obtained by applying the same embedding previously used in (Serafini and Homola 2012). However, such a translation (of $\mathcal{ALC}$-based CKR) ends up in $\mathcal{SROIQ}$, yielding a 2NEXPTIME upper bound.

The present paper shows that reasoning in $\mathcal{ALC}$ based CKR is indeed EXPTIME-complete, i.e., it provides a positive answer to the above question. Following the methodology proposed in (Serafini and Homola 2012) we design a polynomial reduction that translates an $\mathcal{ALC}$-based CKR into an equivalent $\mathcal{ALCO}(\sqcup)$ KB, in which reasoning is known to be EXPTIME-complete.

2 Contextualized Knowledge Repositories

A DL vocabulary Σ is composed on the three mutually disjoint subsets N_{C} of atomic concepts, N_{R} of roles, and N_{I} of individuals. In $\mathcal{ALC}$ (Schmidt-Schauß and Smolka 1991), concepts are inductively defined from atomic concepts and the constructors $\neg, \sqcap$ and $\exists$. A TBox $\mathcal{T}$ is a finite set of general concept inclusions (GCI) of the form $C \sqsubseteq D$. An ABox $\mathcal{A}$ is a finite set of axioms of the form $C(a)$ or $R(a,b)$. A knowledge base is a pair $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$. For precise semantics and other details on $\mathcal{ALC}$ please refer to (Baader et al. 2003). $\mathcal{ALCO}$ extends $\mathcal{ALC}$ with nominals, using the constructor $\{a\}$ for any individual $a \in N_{\mathrm{I}}$. In addition $\mathcal{ALCO}(\sqcup)$ adds the role union ($\sqcup$). Both $\mathcal{ALC}$ and $\mathcal{ALCO}$ are EXPTIME-complete (Baader et al. 2003; Tobies 2000). The role union is polynomially reducible (e.g., by replacing $\exists R_1 \sqcup \cdots \sqcup R_n.C$ with $\exists R_1.C \sqcup \cdots \sqcup$

$\exists R_n.C$) and hence also $\mathcal{ALCO}(\sqcup)$ is ExpTime-complete.

A CKR is defined on the basis of two languages: the *object language* specifies knowledge within contexts, and the *meta language* specifies knowledge about contexts. To keep high separation between these languages, as each is built on top of a distinct DL vocabulary.

Definition 1 (Meta vocabulary)**.** *A meta vocabulary Γ is a DL vocabulary that contains: (1) a finite set of individuals called* context identifiers*; (2) a finite set of roles $\mathbf{A}$ called* dimensions*; (3) for every dimension $A \in \mathbf{A}$, a finite set of individuals D_A, called* dimensional values*, and a role $\prec_A$, called* coverage relation*.*

The number of dimensions $k = |\mathbf{A}|$ is assumed to be a fixed constant. Intuitively, meta-assertions of the form $A(\mathcal{C}, d)$ for a context identifier $\mathcal{C}$ and $d \in D_A$, state that the value of the dimension A of the context $\mathcal{C}$ is d, while meta-assertions of the form $d \prec_A e$ state that the value d of the dimension A is covered by the value e. Depending on the dimension, the coverage relation has different intuitive meanings, e.g., if A is location then the coverage relation is topological containment.

Dimensional vectors are used to identify each context with a specific set of dimensional values. Given a meta-vocabulary Γ with dimensions $\mathbf{A} = \{A_1, \ldots, A_k\}$, a *dimensional vector* $\mathbf{d}$ is a (possibly empty) set of assignments $\{A_{i_1}{:=}d_{A_{i_1}}, \ldots, A_{i_m}{:=}d_{A_{i_m}}\}$ with $i_j \neq i_h$, whenever $1 \leq j \neq h \leq m$. A dimensional vector $\mathbf{d}$ is full if it assigns values to all dimensions (i.e., $m = k$), otherwise it is partial. If it is apparent which value belongs to which dimension, we simply write $\{d_{A_{i_1}}, \ldots, d_{A_{i_m}}\}$. By d_{A_i} we denote the actual value that $\mathbf{d}$ assigns to the dimension A_i (similarly e_{A_i} for vector $\mathbf{e}$, etc). The *dimensional space* of Γ (denoted $\mathfrak{D}_\Gamma$) is the set $\mathfrak{D}_\Gamma$ of all full dimensional vectors of Γ.

Example 1 (Dimensions)**.** Suppose a meta vocabulary Γ with three dimensions: time, location and topic, and values: $D_{\mathsf{topic}} = \{\mathsf{football}, \mathsf{fwc}, \mathsf{nfl}\}$ (fwc standing for FIFA WC, nfl for national football leagues), $D_{\mathsf{location}} = \{\mathsf{africa}, \mathsf{world}\}$, $D_{\mathsf{time}} = \{2010\}$. Coverage relations are as follows: $\mathsf{africa} \prec_{\mathsf{location}} \mathsf{world}$; $\mathsf{fwc} \prec_{\mathsf{topic}} \mathsf{football}$; $\mathsf{nfl} \prec_{\mathsf{topic}} \mathsf{football}$. Three dimensional vectors that will be used to identify the contexts of our running example are: $\mathbf{fb} = \{2010, \mathsf{world}, \mathsf{football}\}$, $\mathbf{wc10} = \{2010, \mathsf{africa}, \mathsf{fwc}\}$ and $\mathbf{nfl10} = \{2010, \mathsf{world}, \mathsf{nfl}\}$. $\diamond$

The knowledge inside contexts is build on top of an object vocabulary. Object vocabulary contains regular (unqualified) symbols and also qualified symbols of the form $X_{\mathbf{d}}$ where $\mathbf{d}$ is a dimensional vector of Γ.

Definition 2 (Object vocabulary)**.** *Let Γ be a meta-vocabulary, and $\Sigma^B = (N_{\mathrm{C}}^B, N_{\mathrm{R}}^B, N_{\mathrm{I}}^B)$ a DL vocabulary called, basic vocabulary, The object vocabulary Σ is the DL vocabulary defined on the following sets of symbols:*

$$N_{\mathrm{C}} = \{X_{\mathbf{d_B}} | X \in N_{\mathrm{C}}^B, \text{ and } \mathbf{d_B} \text{ is a dimensional vector of } \Gamma\}$$

$$N_{\mathrm{R}} = \{X_{\mathbf{d_B}} | X \in N_{\mathrm{R}}^B, \text{ and } \mathbf{d_B} \text{ is a dimensional vector of } \Gamma\}$$

$$N_{\mathrm{I}} = N_{\mathrm{I}}^B$$

For every $X_{\mathbf{d_B}} \in N_{\mathrm{C}} \cup N_{\mathrm{R}}$, if $\mathbf{B} = \emptyset$, then $X_\emptyset$ is called unqualified symbol, and it is simplified by X; if $\mathbf{B}$ is a partial dimensional vector, then $X_{\mathbf{d_B}}$ is called partially qualified symbol; if $\mathbf{d_B}$ is a full dimensional vector, then $X_{\mathbf{d_B}}$ is called fully qualified symbol.

Qualified symbols are used inside contexts to refer to the meaning of symbols w.r.t. some other context, e.g., intuitively speaking $\mathsf{Player}_{\mathbf{wc10}}$ (also written $\mathsf{Player}_{\{2010,\mathsf{africa},\mathsf{fwc}\}}$) represents the concept of player as defined in the context respective to $\mathbf{wc10}$. If the vector is partial and some dimensions are missing, the semantics always takes the respective values from the context where the symbol appears, e.g., the same symbol can be written as $\mathsf{Player}_{\{\mathsf{africa},\mathsf{fwc}\}}$ if it appears, say, in context $\mathbf{nfl10}$ as $\mathbf{nfl10}$ has the value of time set also to 2010. In this respect, if any non-qualified symbol X appears in the context of $\mathbf{d}$, it is understood as qualified by the empty dimensional vector $\{\}$ and all dimensional values are taken from the context, i.e. X equals to $X_{\mathbf{d}}$ when used in the context of $\mathbf{d}$. This will become apparent from the semantics.

Context is a DL KB over the object vocabulary located in the dimensional space by some full dimensional vector.

Definition 3 (Context)**.** *Given a pair of meta/object vocabularies $\langle\Gamma, \Sigma\rangle$, a context is a triple $\langle\mathcal{C}, \dim(\mathcal{C}), \mathrm{K}(\mathcal{C})\rangle$ where: (a) $\mathcal{C}$ is a context identifier of Γ; (b) $\dim(\mathcal{C})$ is a full dimensional vector of $\mathfrak{D}_\Gamma$; (c) $\mathrm{K}(\mathcal{C})$ is an $\mathcal{ALC}$ knowledge base over Σ.*

CKR is a collection of contexts located in a common dimensional space, together with some meta knowledge asserted in a separate DL KB.

Definition 4 (Contextualized Knowledge Repository)**.** *Given a pair of meta/object vocabularies $\langle\Gamma, \Sigma\rangle$, a CKR knowledge base (CKR) is a pair $\mathfrak{K} = \langle\mathfrak{M}, \mathfrak{C}\rangle$ such that:*

1. *$\mathfrak{C}$ is a set of contexts on $\langle\Gamma, \Sigma\rangle$;*
2. *$\mathfrak{M}$, called* meta knowledge*, is a DL knowledge base over Γ such that:*
   - *(a) for every $A \in \mathbf{A}$, and every $d, d' \in D_A$, if $\mathfrak{M} \models A(\mathcal{C}, d)$ and $\mathfrak{M} \models A(\mathcal{C}, d')$, then $\mathfrak{M} \models d = d'$;*
   - *(b) for every $\mathcal{C} \in \mathfrak{C}$ with $\dim(\mathcal{C}) = \mathbf{d}$ and for every $A \in \mathbf{A}$, we have $\mathfrak{M} \models A(\mathcal{C}, d_A)$;*
   - *(c) the relation $\{\langle d, d'\rangle \mid \mathfrak{M} \models \prec_A(d, d')\}$ is a strict partial order on D_A.*

For a CKR $\mathfrak{K}$, we will denote by $\mathcal{C}_{\mathbf{d}}$ a context with $\dim(\mathcal{C}) = \mathbf{d}$; for $\mathbf{d}, \mathbf{e} \in \mathfrak{D}_\Gamma$ and $\mathbf{B}, \mathbf{C} \subseteq \mathbf{A}$, $\mathbf{d_B} := \{(A{:=}d) \in \mathbf{d} \mid A \in \mathbf{B}\}$ is the projection of $\mathbf{d}$ w.r.t. $\mathbf{B}$; and $\mathbf{d_B}{+}\mathbf{e_C} := \mathbf{d_B} \cup \{(A{:=}d) \in \mathbf{e_C} \mid A \notin \mathbf{B}\}$ is the completion of $\mathbf{d_B}$ w.r.t. $\mathbf{e_C}$.

Important notion is the strict ($\prec$) and non-strict ($\preceq$) coverage between dimensional values: for $d, d' \in D_A$, $d \prec d'$ if $\mathfrak{M} \models \prec_A(d, d')$; and $d \preceq d'$ if either $d \prec d'$ or $\mathfrak{M} \models d = d'$.

Similarly, coverage for dimensional vectors: $\mathbf{d} \preceq_{\mathbf{B}} \mathbf{e}$ for some $\mathbf{B} \subseteq \mathbf{A}$ if $d_B \preceq e_B$ for each $B \in \mathbf{B}$; and $\mathbf{d} \prec_{\mathbf{B}} \mathbf{e}$ if $\mathbf{d} \preceq_{\mathbf{B}} \mathbf{e}$ and $d_B \prec e_B$ for at least one $B \in \mathbf{B}$. Also, $\mathbf{d} \preceq \mathbf{e}$ if $\mathbf{d} \preceq_{\mathbf{A}} \mathbf{e}$, and $\mathbf{d} \prec \mathbf{e}$ if $\mathbf{d} \prec_{\mathbf{A}} \mathbf{e}$.

Finally coverage for contexts: $\mathcal{C}_{\mathbf{d}} \preceq \mathcal{C}_{\mathbf{e}}$ if $\mathbf{d} \preceq \mathbf{e}$, and $\mathcal{C}_{\mathbf{d}} \prec \mathcal{C}_{\mathbf{e}}$ if $\mathbf{d} \prec \mathbf{e}$. Intuitively, if $\mathcal{C}_{\mathbf{d}} \prec \mathcal{C}_{\mathbf{e}}$, then $\mathcal{C}_{\mathbf{d}}$ is the narrower and $\mathcal{C}_{\mathbf{e}}$ is the broader context of these two.

Note that in CKR built on top of more expressive logics, conditions 2(a,c) of Definition 4 can be assured directly in the meta knowledge with respective axioms: each $A \in \mathbf{A}$ is declared functional, and each $\prec_A$ is declared irreflexive and transitive. In $\mathcal{ALC}$ we do not have this option, however this is not a problem, because the number of all dimensions is assumed to be finite as is the number of contexts in a CKR. Hence after the meta knowledge is modeled, conditions 2(a,c) can be verified extralogically (e.g., by some script). These conditions are needed to assure reasonable properties of contextual space, i.e., acyclicity, dimensional values uniquely determined (Serafini and Homola 2012).

Example 2 (Contextualized Knowledge Repository)**.** Using the meta knowledge specified in Example 1, CKR $\mathfrak{K}_{fb}$ will contain three contexts, identified by the previously defined vectors: $\mathcal{C}_{\mathbf{fb}}, \mathcal{C}_{\mathbf{wc10}}, \mathcal{C}_{\mathbf{nfl10}}$. The coverage ($\prec$) of $\mathfrak{K}_{fb}$ is defined as follows: $\mathcal{C}_{\mathbf{wc10}} \prec \mathcal{C}_{\mathbf{fb}}, \mathcal{C}_{\mathbf{nfl10}} \prec \mathcal{C}_{\mathbf{fb}}$. We consider these axioms to be included in local knowledge bases of $\mathfrak{K}_{fb}$:

$\mathrm{K}(\mathcal{C}_{\mathbf{fb}})$: $\mathsf{Player} \sqsubseteq \exists \mathsf{playsFor}.\mathsf{Team}$, $\mathsf{NationalTeam} \sqsubseteq \mathsf{Team}$, $\mathsf{ClubTeam} \sqsubseteq \mathsf{Team}$

$\mathrm{K}(\mathcal{C}_{\mathbf{wc10}})$: $\mathsf{Player} \sqsubseteq \exists \mathsf{playsFor}_{\mathbf{fb}}.\mathsf{NationalTeam}_{\mathbf{fb}}$

$\mathrm{K}(\mathcal{C}_{\mathbf{nfl10}})$: $\mathsf{Player} \sqsubseteq \exists \mathsf{playsFor}_{\mathbf{fb}}.\mathsf{ClubTeam}_{\mathbf{fb}}$

◇

Semantics of CKR relies on DL semantics inside each context while the compatibility between contexts is assured by some additional semantic restrictions. However, we have to slightly generalize the notion of DL interpretation by admitting also interpretation where some of the symbols are not defined and where the domain is empty.

Definition 5. *A partial DL interpretation of a DL vocabulary Σ is a pair $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$, where $\Delta^{\mathcal{I}}$ is a (possibly empty) set, and $\cdot^{\mathcal{I}}$ is a partial function, that is totally defined on N_{C} and N_{R}, with $C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ for every concept $C \in N_{\mathrm{C}}$ and $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ for $R \in N_{\mathrm{R}}$, and it is partially defined on N_{I}, with $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ for $a \in N_{\mathrm{I}}$ on which $\mathcal{I}$ is defined.*

Semantics of non-atomic concepts is defined from the usual interpretation of $\mathcal{ALC}$ constructors (Baader et al. 2003). Notice that for each concept C, if $\mathcal{I}$ is the interpretation on the empty set $\Delta_{\mathcal{I}}$, then $C^{\mathcal{I}} = \emptyset$. An interpretation $\mathcal{I}$ *satisfies* an axiom α (denoted $\mathcal{I} \models \alpha$) if it is defined on all the symbols occurring in α and the following holds: $\mathcal{I} \models C \sqsubseteq D$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$; $\mathcal{I} \models C(a)$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$; and $\mathcal{I} \models R(a,b)$ iff $\langle a^{\mathcal{I}}, b^{\mathcal{I}} \rangle \in R^{\mathcal{I}}$. $\mathcal{I}$ is a *model* of $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ (denoted $\mathcal{I} \models \mathcal{K}$) iff it satisfies all axioms in $\mathcal{T} \cup \mathcal{A}$; $\mathcal{K}$ is satisfiable if it has a model. Notice that, if $\mathcal{I}$ is the interpretation on the empty domain we have that $\mathcal{I} \models C \sqsubseteq D$ for every pair of $\mathcal{ALC}$ concepts C and D, and $\mathcal{I} \not\models \alpha$ for every assertion α of the form $C(a)$ and $R(a,b)$.

A concept C is *satisfiable* w.r.t. an $\mathcal{ALC}$ KB $\mathcal{K}$ iff there exists a model $\mathcal{I}$ of $\mathcal{K}$ s.t. $C^{\mathcal{I}}$ is non-empty. A subsumption formula $C \sqsubseteq D$ is *entailed* by an $\mathcal{ALC}$ KB $\mathcal{K}$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ in all models $\mathcal{I}$ of $\mathfrak{K}$. It is well known that these two problems are inter-reducible (Baader et al. 2003).

As apparent from the definition, there are some exceptions to the standard DL semantics. Partial interpretations need not necessarily provide denotations for all individuals of Σ, as some of them may not be meaningful in all contexts, this is just a technical issue resulting from the fact that all contexts rely on the same object vocabulary Σ but some element of Σ could be meaningless in a context. Interpretations with empty domains, also are useful to represent the situation in which some, but not all the contexts are inconsistent.

Definition 6 (CKR-Model)**.** *A model of a CKR $\mathfrak{K}$ is a collection $\mathfrak{I} = \{\mathcal{I}_{\mathbf{d}}\}_{\mathbf{d} \in \mathfrak{D}_\Gamma}$ of partial DL interpretations, called local interpretations, s.t. for all $\mathbf{d}, \mathbf{e}, \mathbf{f} \in \mathfrak{D}_\Gamma$, $A \in N_{\mathrm{C}}^B$, $R \in N_{\mathrm{R}}^B$, $X \in N_{\mathrm{C}}^B \cup N_{\mathrm{R}}^B$, $a \in N_{\mathrm{I}}^B$:*

1. $(\top_{\mathbf{d}})^{\mathcal{I}_{\mathbf{f}}} \subseteq (\top_{\mathbf{e}})^{\mathcal{I}_{\mathbf{f}}}$ *if* $\mathbf{d} \prec \mathbf{e}$
2. $(A_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} \subseteq (\top_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}}$
3. $(R_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} \subseteq (\top_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} \times (\top_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}}$
4. $a^{\mathcal{I}_{\mathbf{e}}} = a^{\mathcal{I}_{\mathbf{d}}}$ *if* $\mathbf{d} \prec \mathbf{e}$ *and, either* $a^{\mathcal{I}_{\mathbf{d}}}$ *is defined or,* $a^{\mathcal{I}_{\mathbf{e}}}$ *is defined and* $a^{\mathcal{I}_{\mathbf{e}}} \in \Delta_{\mathbf{d}}$
5. $(X_{\mathbf{d_B}})^{\mathcal{I}_{\mathbf{e}}} = (X_{\mathbf{d_B}+\mathbf{e}})^{\mathcal{I}_{\mathbf{e}}}$
6. $(X_{\mathbf{d}})^{\mathcal{I}_{\mathbf{e}}} = (X_{\mathbf{d}})^{\mathcal{I}_{\mathbf{d}}}$ *if* $\mathbf{d} \prec \mathbf{e}$
7. $(A_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} = (A_{\mathbf{f}})^{\mathcal{I}_{\mathbf{e}}} \cap \Delta_{\mathbf{d}}$ *if* $\mathbf{d} \prec \mathbf{e}$
8. $(R_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} = (R_{\mathbf{f}})^{\mathcal{I}_{\mathbf{e}}} \cap (\Delta_{\mathbf{d}} \times \Delta_{\mathbf{d}})$ *if* $\mathbf{d} \prec \mathbf{e}$
9. $\mathcal{I}_{\mathbf{d}} \models \mathrm{K}(\mathcal{C}_{\mathbf{d}})$

The semantics takes care that local domains respect the coverage hierarchy (condition 1). Given contexts $\mathcal{C}_{\mathbf{d}} \prec \mathcal{C}_{\mathbf{e}}$, if an individual a occurs in $\mathcal{C}_{\mathbf{d}}$ then it must be defined also in $\mathcal{C}_{\mathbf{e}}$ with the same meaning; if a only occurs in $\mathcal{C}_{\mathbf{e}}$ however, it does not have to be defined in $\mathcal{C}_{\mathbf{d}}$ (condition 4). The interpretation of any qualified symbol $X_{\mathbf{f}}$ is roofed under $(\top_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}}$ in any context $\mathcal{C}_{\mathbf{d}}$, regardless the relation between $\mathcal{C}_{\mathbf{f}}$ and $\mathcal{C}_{\mathbf{d}}$ (conditions $2, 3$). Semantics of qualified symbols is given by conditions 6–8: if $X_{\mathbf{d}}$ "comes" from a more specific context $C_{\mathbf{d}}$ than $\mathcal{C}_{\mathbf{e}}$, its interpretation in $\mathcal{C}_{\mathbf{e}}$ is exactly equal as in its home context $\mathcal{C}_{\mathbf{d}}$ (condition 6); if this is not the case then the interpretation of any $X_{\mathbf{f}}$ in $\mathcal{C}_{\mathbf{d}}$ and $\mathcal{C}_{\mathbf{e}}$ must be equal modulo the domain of the more specific context of these two (conditions 7 and 8). Finally, each $\mathcal{I}_{\mathbf{d}}$ is a DL-model of $\mathcal{C}_{\mathbf{d}}$ (condition 9). Note that condition 5 provides the meaning of partially qualified symbols, assuming that the values for attributes not specified are taken from the dimensions of the context in which the expression appears. From this reading, we can show that, albeit useful for modeling, in fact partially qualified vectors are a kind of syntactic sugar in this framework (Serafini and Homola 2012).

Given a CKR $\mathfrak{K}$ and $\mathbf{d} \in \mathfrak{D}_\Gamma$, a concept C is $\mathbf{d}$-*satisfiable* w.r.t. $\mathfrak{K}$ if there exists a CKR model $\mathfrak{I} = \{\mathcal{I}_{\mathbf{d}}\}_{\mathbf{d} \in \mathfrak{D}_\Gamma}$ of $\mathfrak{K}$ such that $C^{\mathcal{I}_{\mathbf{d}}} \neq \emptyset$; $\mathfrak{K}$ is $\mathbf{d}$-satisfiable if it has a CKR model $\mathfrak{I} = \{\mathcal{I}_{\mathbf{d}}\}_{\mathbf{d} \in \mathfrak{D}_\Gamma}$ such that $\Delta_{\mathbf{d}} \neq \emptyset$; $\mathfrak{K}$ is *globally satisfiable* if it has a CKR model $\mathfrak{I} = \{\mathcal{I}_{\mathbf{d}}\}_{\mathbf{d} \in \mathfrak{D}_\Gamma}$ such that $\Delta_{\mathbf{d}} \neq \emptyset$ for some $\mathbf{d} \in \mathfrak{D}_\Gamma$. An axiom α is $\mathbf{d}$-*entailed* by $\mathfrak{K}$ (denoted $\mathfrak{K} \models \mathbf{d} : \alpha$) if for every model $\mathfrak{I} = \{\mathcal{I}_{\mathbf{d}}\}_{\mathbf{d} \in \mathfrak{D}_\Gamma}$ of $\mathfrak{K}$ it holds $\mathcal{I}_{\mathbf{d}} \models \alpha$. As usual, $\mathbf{d}$-entailment can be reduced to $\mathbf{d}$-satisfiability: in particular $\mathfrak{K} \models \mathbf{d} : C \sqsubseteq D$ iff $C \sqcap \neg D$ is not $\mathbf{d}$-satisfiable w.r.t. $\mathfrak{K}$.

3 $\mathcal{ALC}$-based CKR is EXPTIME-complete

In this section we show how $\mathcal{ALC}$-based CKR reduces into a single KB in $\mathcal{ALCO}(\sqcup)$. In order to embed a CKR into a single DL knowledge base we need a vocabulary that keeps track of the different meaning that a symbol can have in different contexts, and of the constraints on their interpretations. For every fully qualified atomic concept, role, and individual $A_{\mathbf{d}}$, $R_{\mathbf{d}}$, and a we introduce the symbols $A^{\mathbf{e}}_{\mathbf{d}}$, $R^{\mathbf{e}}_{\mathbf{d}}$, and $a^{\mathbf{e}}$, for each context $\mathcal{C}_{\mathbf{e}}$.

Definition 7 (Transformed vocabulary $\#(\Gamma,\Sigma)$)**.** *Given a pair of meta/object vocabularies $\langle\Gamma,\Sigma\rangle$ with the respective dimensional space $\mathfrak{D}_\Gamma$, let $\Sigma^{\mathrm{B}} = N^{\mathrm{B}}_{\mathrm{C}} \uplus N^{\mathrm{B}}_{\mathrm{R}} \uplus N^{\mathrm{B}}_{\mathrm{I}}$ be the base-vocabulary of Σ and let* undef *be a new individual not appearing in Σ. Let us define a DL-vocabulary $\#(\Gamma,\Sigma) = \#N_{\mathrm{C}} \uplus \#N_{\mathrm{R}} \uplus \#N_{\mathrm{I}}$ such that:*

1. $\#N_{\mathrm{C}} = \{A^{\mathbf{e}}_{\mathbf{d}} \mid A \in N^{\mathrm{B}}_{\mathrm{C}} \wedge \mathbf{d},\mathbf{e} \in \mathfrak{D}_\Gamma\}$*;*
2. $\#N_{\mathrm{R}} = \{R^{\mathbf{e}}_{\mathbf{d}} \mid R \in N^{\mathrm{B}}_{\mathrm{R}} \wedge \mathbf{d},\mathbf{e} \in \mathfrak{D}_\Gamma\}$*;*
3. $\#N_{\mathrm{I}} = \{a^{\mathbf{e}} \mid a \in N^{\mathrm{B}}_{\mathrm{I}} \wedge \mathbf{e} \in \mathfrak{D}_\Gamma\} \cup \{\mathsf{undef}\}$.

For each full dimensional vector $\mathbf{d} \in \mathfrak{D}_\Gamma$, we define an operator $(\cdot)\#\mathbf{d}$ which embeds constructs in Σ into $\#(\Gamma,\Sigma)$.

Definition 8 ($\#\mathbf{d}$ operator)**.** *For every full dimensional vector $\mathbf{d} \in \mathfrak{D}_\Gamma$, $(\cdot)\#\mathbf{d}$ is defined as follows:*

- $a\#\mathbf{d} = a^{\mathbf{d}}$ *for every individual a;*
- $A_{\mathbf{f_B}}\#\mathbf{d} = A^{\mathbf{d}}_{\mathbf{f_B+d}}$ *for every atomic concept $A_{\mathbf{f_B}}$;*
- $R_{\mathbf{f_B}}\#\mathbf{d} = \bigsqcup_{\mathbf{d}' \preceq \mathbf{d}} R^{\mathbf{d}'}_{\mathbf{f_B+d}}$ *for every role $R_{\mathbf{f_B}}$.*

For complex complex concepts and axioms $(\cdot)\#\mathbf{d}$ is defined recursively:

- $(\neg C)\#\mathbf{d} = \top^{\mathbf{d}}_{\mathbf{d}} \sqcap \neg(C\#\mathbf{d})$
- $\bot\#\mathbf{d} = \bot$
- $(C \sqcap D)\#\mathbf{d} = C\#\mathbf{d} \sqcap D\#\mathbf{d}$
- $(\exists R.C)\#\mathbf{d} = \exists R\#\mathbf{d}.C\#\mathbf{d}$
- $C(a)\#\mathbf{d} = C\#\mathbf{d}(a\#\mathbf{d})$
- $R(a,b)\#\mathbf{d} = R\#\mathbf{d}(a\#\mathbf{d}, b\#\mathbf{d})$
- $(C \sqsubseteq D)\#\mathbf{d} = C\#\mathbf{d} \sqsubseteq D\#\mathbf{d}$

Note that each partially qualified symbol $X_{\mathbf{d_B}}$ occurring in $\mathcal{C}_{\mathbf{e}}$ is translated into the same symbol as the completed symbol $X_{\mathbf{d_B+e}}$. According to the semantics these symbols have equal meaning, so this is correct.

While the meaning of $A^{\mathbf{e}}_{\mathbf{d}}$ and $a^{\mathbf{e}}$ directly corresponds to the meaning of $A_{\mathbf{d}}$ and a in $\mathcal{C}_{\mathbf{e}}$, the translation to $\mathcal{ALCO}(\sqcup)$ requires a more complex encoding of roles. Each symbol $R^{\mathbf{e}}_{\mathbf{d}}$ will encode the exclusive portion of $R_{\mathbf{d}}$ which is present in $\mathcal{C}_{\mathbf{e}}$ but not in any of its sub-contexts. The overall meaning of $R_{\mathbf{d}}$ in $\mathcal{C}_{\mathbf{e}}$ is then obtained by means of union. This is necessary to support condition 8 of CKR models.

The recursive translation of complex concepts and axioms is straight forward apart from the case of $\bot$, which has the same meaning ($\emptyset$) in all contexts, and the complement constructor where we must reflect that the complement is always respective to the domain of the context in which it appears.

Using the $(\cdot)\#\mathbf{d}$ operators we now transform a CKR knowledge base $\mathfrak{K}$ into a DL KB $\#(\mathfrak{K})$ over $\#(\Gamma,\Sigma)$.

Definition 9 (Transformed CKR $\#(\mathfrak{K})$)**.** *For every CKR $\mathfrak{K}$ over $\langle\Gamma,\Sigma\rangle$, let $\#(\mathfrak{K})$ be a DL knowledge base over $\#(\Gamma,\Sigma)$ such that for every individual a, concept A, role R, atomic concept or role X, and for any full dimensional vectors $\mathbf{d},\mathbf{d}',\mathbf{e},\mathbf{f}$ it contains the following axioms:*

1. $\top^{\mathbf{f}}_{\mathbf{d}} \sqsubseteq \top^{\mathbf{f}}_{\mathbf{e}}$, *for* $\mathbf{d} \prec \mathbf{e}$*;*
2. $A^{\mathbf{d}}_{\mathbf{e}} \sqsubseteq \top^{\mathbf{d}}_{\mathbf{e}}$*;*
3. $\exists R^{\mathbf{d}}_{\mathbf{e}}.\top \sqsubseteq \top^{\mathbf{d}}_{\mathbf{e}}$ *and* $\top \sqsubseteq \forall R^{\mathbf{d}}_{\mathbf{e}}.\top^{\mathbf{d}}_{\mathbf{e}}$*;*
4. *a)* $\top^{\mathbf{d}}_{\mathbf{d}} \sqcap \{a^{\mathbf{e}}\} \sqsubseteq \{a^{\mathbf{d}}\}$, *if* $\mathbf{d} \prec \mathbf{e}$*;*
   b) $\{a^{\mathbf{d}}\} \sqsubseteq \{a^{\mathbf{e}}, \mathsf{undef}\}$, *if* $\mathbf{d} \prec \mathbf{e}$*;*
   c) $\neg\top^{\mathbf{d}}_{\mathbf{d}}(\mathsf{undef})$*;*
6. $\top^{\mathbf{d}}_{\mathbf{d}} \equiv \top^{\mathbf{e}}_{\mathbf{d}}$, *if* $\mathbf{d} \prec \mathbf{e}$*;*
7. $A^{\mathbf{d}}_{\mathbf{f}} \equiv A^{\mathbf{e}}_{\mathbf{f}} \sqcap \top^{\mathbf{d}}_{\mathbf{d}}$, *if* $\mathbf{d} \prec \mathbf{e}$*;*
8. $\exists R^{\mathbf{d}}_{\mathbf{f}}.\top^{\mathbf{d}'}_{\mathbf{d}'} \sqsubseteq \neg\top^{\mathbf{d}'}_{\mathbf{d}'}$, *if* $\mathbf{d}' \prec \mathbf{d}$*;*
9. $\phi\#\mathbf{d}$, *for all* $\phi \in \mathrm{K}(\mathcal{C})$ *and* $\mathbf{d} = \dim(\mathcal{C})$.

The formulae added to $\#(\mathfrak{K})$ in the previous definition correspond step-by-step with the conditions of Definition 6; in each step we add formulae to deal with the respective condition. Step 5 is missing, as we do not deal with incomplete symbols directly, as we previously explained.

Example 3 (CKR translation)**.** Given the CKR $\mathfrak{K}_{fb}$ of Example 2, we can now show how its axioms in the relative transformed CKR $(\mathfrak{K}_{fb})\#$ are defined. For simplicity we only present the transformation of the described axioms, without the additional axioms introduced by Definition 9:

$\mathsf{Player}^{\mathbf{fb}}_{\mathbf{fb}} \sqsubseteq \exists(\mathsf{playsFor}^{\mathbf{fb}}_{\mathbf{fb}} \sqcup \mathsf{playsFor}^{\mathbf{nfl10}}_{\mathbf{fb}} \sqcup \mathsf{playsFor}^{\mathbf{wc10}}_{\mathbf{fb}}).\mathsf{Team}^{\mathbf{fb}}_{\mathbf{fb}}$
$\mathsf{NationalTeam}^{\mathbf{fb}}_{\mathbf{fb}} \sqsubseteq \mathsf{Team}^{\mathbf{fb}}_{\mathbf{fb}}$
$\mathsf{ClubTeam}^{\mathbf{fb}}_{\mathbf{fb}} \sqsubseteq \mathsf{Team}^{\mathbf{fb}}_{\mathbf{fb}}$
$\mathsf{Player}^{\mathbf{wc10}}_{\mathbf{wc10}} \sqsubseteq \exists\mathsf{playsFor}^{\mathbf{wc10}}_{\mathbf{fb}}.\mathsf{NationalTeam}^{\mathbf{wc10}}_{\mathbf{fb}}$
$\mathsf{Player}^{\mathbf{nfl10}}_{\mathbf{nfl10}} \sqsubseteq \exists\mathsf{playsFor}^{\mathbf{nfl10}}_{\mathbf{fb}}.\mathsf{ClubTeam}^{\mathbf{nfl10}}_{\mathbf{fb}}$

We want to remark the provided translation of roles. Intuitively, the role PlaysFor in the axiom in $\mathcal{C}_{\mathbf{fb}}$ is translated in the union of its three disjoint interpretations in $\mathcal{C}_{\mathbf{fb}}$ and its sub-contexts, that is, the role relating players and their teams in the context for world cup (e.g. playsFor(buffon, team_italy)), in national leagues (e.g. playsFor(buffon, juventus)) and for relations not appearing in the two sub-contexts. In the case of axioms from lower contexts $\mathcal{C}_{\mathbf{wc10}}$ and $\mathcal{C}_{\mathbf{nfl10}}$ this is not evident because unions of sub-context roles only include the interpretation of the role in the given context. ◇

The transformation $\#(\cdot)$ embeds the input CKR into a single DL knowledge base. We will now show, that this embedding is proper, i.e., if we have two models, one of $\mathfrak{K}$ and the other one of $\#(\mathfrak{K})$ such that these models agree on the interpretation of atomic symbols, then they also agree on the interpretation of complex concepts and consequently also on all formulae.

Lemma 1 (Embedding lemma)**.** *Given a CKR $\mathfrak{K}$, let $\mathfrak{I}$ be a model of $\mathfrak{K}$ and $\mathcal{I}$ be a model of $\#(\mathfrak{K})$ such that for every $\mathbf{d} \in \mathfrak{D}_\Gamma$:*

1. $a^{\mathcal{I}_{\mathbf{d}}} = (a\#\mathbf{d})^{\mathcal{I}}$, *for each individual a of Σ for which $a^{\mathcal{I}_{\mathbf{d}}}$ is defined;*
2. $A^{\mathcal{I}_{\mathbf{d}}} = (A\#\mathbf{d})^{\mathcal{I}}$, *for every atomic concept $A \in \Sigma$;*
3. $R^{\mathcal{I}_{\mathbf{d}}} = (R\#\mathbf{d})^{\mathcal{I}}$, *for every role $R \in \Sigma$.*

Then for every $\mathbf{d} \in \mathfrak{D}_\Gamma$, every concept C, and every axiom ϕ such that $\mathcal{I}_\mathbf{d}$ is defined on all constants occurring in ϕ we have: (a) $C^{\mathcal{I}_\mathbf{d}} = (C\#\mathbf{d})^\mathcal{I}$; (b) $\mathcal{I}_\mathbf{d} \models \phi$ iff $\mathcal{I} \models \phi\#\mathbf{d}$.

Proof. Observe that the domain $\Delta_\mathbf{d}$ of $\mathcal{I}_\mathbf{d}$ is embedded into the domain $\Delta^\mathcal{I}$ of $\mathcal{I}$. Later in the proof we will denote this fact by (‡): $\Delta_\mathbf{d} = \top^{\mathcal{I}_\mathbf{d}} = (\top\#\mathbf{d})^\mathcal{I} = (\top^\mathbf{d}_\mathbf{d})^\mathcal{I} \subseteq \Delta^\mathcal{I}$. We prove that for every concept X it holds that $X^{\mathcal{I}_\mathbf{d}} = (X\#\mathbf{d})^\mathcal{I}$ by structural induction on X.

If $X = A$ is an atomic concept, then $A^{\mathcal{I}_\mathbf{d}} = (A\#\mathbf{d})^\mathcal{I}$ follows directly from the assumptions.

If $X = \neg C$, then $(\neg C)\#\mathbf{d}^\mathcal{I} = (\top^\mathbf{d}_\mathbf{d} \sqcap \neg(C\#\mathbf{d}))^\mathcal{I}$ = $(\top^\mathbf{d}_\mathbf{d})^\mathcal{I} \cap (\Delta^\mathcal{I} \setminus (C\#\mathbf{d})^\mathcal{I})$. By induction and (‡) this equals to $(\top^\mathbf{d}_\mathbf{d})^\mathcal{I} \setminus C^{\mathcal{I}_\mathbf{d}}$ and hence to $\neg C^{\mathcal{I}_\mathbf{d}}$.

If $X = C \sqcap D$, then $(C \sqcap D)\#\mathbf{d}^\mathcal{I} = ((C\#\mathbf{d}) \sqcap (D\#\mathbf{d}))^\mathcal{I} = (C\#\mathbf{d})^\mathcal{I} \cap (D\#\mathbf{d})^\mathcal{I} = C^{\mathcal{I}_\mathbf{d}} \cap D^{\mathcal{I}_\mathbf{d}} = C \sqcap D^{\mathcal{I}_\mathbf{d}}$ directly by induction.

If $X = \exists R.C$, then $(\exists R.C)\#\mathbf{d}^\mathcal{I} = (\exists R\#\mathbf{d}.C\#\mathbf{d})^\mathcal{I} = \{x \in \Delta^\mathcal{I} \mid \exists y\ (x,y) \in \exists R\#\mathbf{d}^\mathcal{I} \wedge y \in C\#\mathbf{d}^\mathcal{I}\}$. From the assumptions of the lemma, from induction, and due to $\Delta_\mathbf{d} \subseteq \Delta^\mathcal{I}$ this equals to $\{x \in \Delta_\mathbf{d} \mid \exists y\ (x,y) \in R^{\mathcal{I}_\mathbf{d}} \wedge y \in C^{\mathcal{I}_\mathbf{d}}\}$ and hence to $\exists R.C^{\mathcal{I}_\mathbf{d}}$.

We can now show that, given any $\mathbf{d} \in \mathfrak{D}_\Gamma$, for every axiom ϕ, $\mathcal{I}_\mathbf{d} \models \phi$ iff $\mathcal{I} \models \phi\#\mathbf{d}$. We proceed by cases on the form of ϕ.

If $\phi = C(a)$, $\phi\#\mathbf{d} = C\#\mathbf{d}(a\#\mathbf{d})$: since, as assumed, $a^{\mathcal{I}_\mathbf{d}}$ is defined, $a^{\mathcal{I}_\mathbf{d}} = a\#\mathbf{d}^\mathcal{I}$ follows from the assumptions. Also we have already proved that $C^{\mathcal{I}_\mathbf{d}} = C\#\mathbf{d}^\mathcal{I}$. Therefore $\mathcal{I}_\mathbf{d} \models C(a)$ iff $a^{\mathcal{I}_\mathbf{d}} \in C^{\mathcal{I}_\mathbf{d}}$ iff $a\#\mathbf{d}^\mathcal{I} \in C\#\mathbf{d}^\mathcal{I}$ iff $\mathcal{I} \models C\#\mathbf{d}(a\#\mathbf{d})$. The case for $\phi = R(a,b)$, can be proved analogously.

If $\phi = C \sqsubseteq D$, $\phi\#\mathbf{d} = C\#\mathbf{d} \sqsubseteq D\#\mathbf{d}$: we have already proved that $C^{\mathcal{I}_\mathbf{d}} = C\#\mathbf{d}^\mathcal{I}$ and $D^{\mathcal{I}_\mathbf{d}} = D\#\mathbf{d}^\mathcal{I}$ Therefore $\mathcal{I}_\mathbf{d} \models C \sqsubseteq D$ iff $C^{\mathcal{I}_\mathbf{d}} \subseteq B^{\mathcal{I}_\mathbf{d}}$ iff $C\#\mathbf{d}^\mathcal{I} \subseteq D\#\mathbf{d}^\mathcal{I}$ iff $\mathcal{I} \models C\#\mathbf{d} \sqsubseteq D\#\mathbf{d}$. □

We are now able to check $\mathbf{d}$-satisfiability of a CKR knowledge base $\mathfrak{K}$ by checking for satisfiability/entailment in $\#(\mathfrak{K})$. This is formally established by the following two lemmata.

Lemma 2. *If $\mathfrak{K}$ is $\mathbf{d}$-satisfiable then $\top^\mathbf{d}_\mathbf{d}$ is satisfiable with respect to $\#(\mathfrak{K})$.*

Proof. As $\mathfrak{K}$ is $\mathbf{d}$-satisfiable, it has a model $\mathfrak{I}$ with $\Delta_\mathbf{d} \neq \emptyset$. Let $\mathcal{I} = \langle \Delta, \cdot^\mathcal{I} \rangle$ be a DL interpretation s.t. for every individual a, atomic concept A, role R, and $\mathbf{d}, \mathbf{e} \in \mathfrak{D}_\Gamma$:

- $\Delta = \bigcup_{\mathbf{d} \in \mathfrak{D}_\Gamma} \Delta_\mathbf{d} \cup \{x_\mathsf{undef}\}$, and $x_\mathsf{undef} \notin \Delta_\mathbf{d}$ for $\mathbf{d} \in \mathfrak{D}_\Gamma$;
- $(a^\mathbf{d})^\mathcal{I} = a^{\mathcal{I}_\mathbf{d}}$ if $a^{\mathcal{I}_\mathbf{d}}$ is defined, otherwise $(a^\mathbf{d})^\mathcal{I} = x_\mathsf{undef}$, and $\mathsf{undef}^\mathcal{I} = x_\mathsf{undef}$;
- $(A^\mathbf{d}_\mathbf{e})^\mathcal{I} = A^{\mathcal{I}_\mathbf{d}}_\mathbf{e}$;
- $(R^\mathbf{d}_\mathbf{e})^\mathcal{I} = R^{\mathcal{I}_\mathbf{d}}_\mathbf{e} \setminus \bigcup_{\mathbf{d}' \prec \mathbf{d}} \Delta_{\mathbf{d}'} \times \Delta_{\mathbf{d}'}$.

We now have to prove that $\mathcal{I}$ satisfies all axioms of $\#(\mathfrak{K})$ as given in Definition 9. For items items 1-2, 6-7 follows directly from the construction and from the corresponding condition of $\mathcal{I}$ being a CKR model (Definition 6).

As for item 3, since $\mathfrak{I}$ is a model of $\mathfrak{K}$, from Definition 6 we have $R^{\mathcal{I}_\mathbf{d}}_\mathbf{e} \subseteq \top^{\mathcal{I}_\mathbf{d}}_\mathbf{e} \times \top^{\mathcal{I}_\mathbf{d}}_\mathbf{e}$. From the construction of $\mathcal{I}$, we get $(R^\mathbf{d}_\mathbf{e})^\mathcal{I} = R^{\mathcal{I}_\mathbf{d}}_\mathbf{e} \setminus \bigcup_{\mathbf{d}' \prec \mathbf{d}} \Delta_{\mathbf{d}'} \times \Delta_{\mathbf{d}'} \subseteq R^{\mathcal{I}_\mathbf{d}}_\mathbf{e} \subseteq \top^{\mathcal{I}_\mathbf{d}}_\mathbf{e} \times \top^{\mathcal{I}_\mathbf{d}}_\mathbf{e}$. The last inclusion follows from $\mathcal{I}$ being a CKR-model (Definition 6, condition 3). Hence both the domain and range of $(R^\mathbf{d}_\mathbf{e})^\mathcal{I}$ are under $\top^{\mathcal{I}_\mathbf{d}}_\mathbf{e}$ and axioms from item 3 are satisfied.

The first type of axioms added in item 4 is $\top^\mathbf{d}_\mathbf{d} \sqcap \{a^\mathbf{e}\} \sqsubseteq \{a^\mathbf{d}\}$ for any individual a and for any two $\mathbf{d}, \mathbf{e} \in \mathfrak{D}_\Gamma$. Let $x \in (\top^\mathbf{d}_\mathbf{d} \sqcap \{a^\mathbf{e}\})^\mathcal{I}$, that is, $x \in (\top^\mathbf{d}_\mathbf{d})^\mathcal{I}$ and $x = (a^\mathbf{e})^\mathcal{I}$. Notice that $x \neq x_\mathsf{undef}$ because $x \in (\top^\mathbf{d}_\mathbf{d})^\mathcal{I} = \top^{\mathcal{I}_\mathbf{d}}_\mathbf{d} = \Delta_\mathbf{d}$ and the construction implies $x_\mathsf{undef} \notin \Delta_\mathbf{d}$. Altogether this implies that $a^{\mathcal{I}_\mathbf{e}} = (a^\mathbf{e})^\mathcal{I}$ is defined in $\mathfrak{I}$ and also $a^{\mathcal{I}_\mathbf{e}} \in \Delta_\mathbf{d} = (\top^\mathbf{d}_\mathbf{d})^\mathcal{I}$. From Condition 4 of the CKR model this implies that $x = a^{\mathcal{I}_\mathbf{e}} = a^{\mathcal{I}_\mathbf{d}}$ and therefore $x \in (\{a^\mathbf{d}\})^\mathcal{I}$.

The second type of axioms added in item 4 is $\{a^\mathbf{d}\} \sqsubseteq \{a^\mathbf{e}, \mathsf{undef}\}$ for any individual a and for any two $\mathbf{d}, \mathbf{e} \in \mathfrak{D}_\Gamma$. That is, we have to show that either $(a^\mathbf{d})^\mathcal{I} = (a^\mathbf{e})^\mathcal{I}$ of $(a^\mathbf{d})^\mathcal{I} = \mathsf{undef}^\mathcal{I}$. If $a^{\mathcal{I}_\mathbf{d}}$ is defined, then from Condition 4 of CKR models we have $a^{\mathcal{I}_\mathbf{d}} = a^{\mathcal{I}_\mathbf{e}}$ and hence from the construction $(a^\mathbf{d})^\mathcal{I} = (a^\mathbf{e})^\mathcal{I}$. If $a^{\mathcal{I}_\mathbf{d}}$ is undefined, then directly from the construction $(a^\mathbf{d})^\mathcal{I} = \mathsf{undef}^\mathcal{I}$.

The last type of axioms added in item 4 is $\neg\top^\mathbf{d}_\mathbf{d}(\mathsf{undef})$ for any $\mathbf{d} \in \mathfrak{D}_\Gamma$. This follows directly from the construction of $\mathcal{I}$ as $\mathsf{undef}^\mathcal{I} = x_\mathsf{undef} \notin (\top^\mathbf{d}_\mathbf{d})^\mathcal{I} = \Delta_\mathbf{d}$.

The axioms added in item 8 are of the form $\exists R^\mathbf{d}_\mathbf{f}.\top^{\mathbf{d}'}_{\mathbf{d}'} \sqsubseteq \neg\top^{\mathbf{d}'}_{\mathbf{d}'}$ where $\mathbf{d}' \prec \mathbf{d}$. Assume $x \in (\exists R^\mathbf{d}_\mathbf{f}.\top^{\mathbf{d}'}_{\mathbf{d}'})^\mathcal{I}$. This implied that there must be $y \in (\top^{\mathbf{d}'}_{\mathbf{d}'})^\mathcal{I} = \Delta_{\mathbf{d}'}$ such that $\langle x, y \rangle \in (R^\mathbf{d}_\mathbf{f})^\mathcal{I}$. Due to construction of $\mathcal{I}$, the sets $(R^\mathbf{d}_\mathbf{f})^\mathcal{I}$ and $\Delta_{\mathbf{d}'} \times \Delta_{\mathbf{d}'}$ are disjoint because $\mathbf{d}' \prec \mathbf{d}$. Since $y \in \Delta_{\mathbf{d}'}$ and $\langle x, y \rangle \in (R^\mathbf{d}_\mathbf{f})^\mathcal{I}$, this implies $x \notin \Delta_{\mathbf{d}'} = (\top^{\mathbf{d}'}_{\mathbf{d}'})^\mathcal{I}$ and therefore $x \in (\neg\top^{\mathbf{d}'}_{\mathbf{d}'})^\mathcal{I}$. Therefore the axiom holds.

For item 9 we have to prove that $\mathcal{I}$ satisfies all axioms of the form $\phi\#\mathbf{d}$ where $\phi \in \mathrm{K}(\mathcal{C}_\mathbf{d})$. We will obtain this as a consequence of Lemma 1 and the fact that $\mathcal{I}_\mathbf{d} \models \phi$. Observe that Lemma 1 is indeed applicable: $a^{\mathcal{I}_\mathbf{d}} = (a^\mathbf{d})^\mathcal{I} = (a\#\mathbf{d})^\mathcal{I}$ and $A^{\mathcal{I}_\mathbf{d}}_\mathbf{f} = (A^\mathbf{d}_\mathbf{f})^\mathcal{I} = (A_\mathbf{f}\#\mathbf{d})^\mathcal{I}$ directly from the construction for all individuals a (s.t. $a^{\mathcal{I}_\mathbf{d}}$ is defined) and for every atomic concept A.

For roles we have to prove $(R_\mathbf{f}\#\mathbf{d})^\mathcal{I} = R^{\mathcal{I}_\mathbf{d}}_\mathbf{f}$ by structural induction: if there is no $\mathbf{d}' \prec \mathbf{d}$ then this follows from the construction of $\mathcal{I}$. In all other cases we have: $(R_\mathbf{f}\#\mathbf{d})^\mathcal{I} = (\bigsqcup_{\mathbf{d}' \preceq \mathbf{d}} R^{\mathbf{d}'}_\mathbf{f})^\mathcal{I} = (R^\mathbf{d}_\mathbf{f})^\mathcal{I} \cup (\bigsqcup_{\mathbf{d}' \prec \mathbf{d}} R^{\mathbf{d}'}_\mathbf{f})^\mathcal{I}$. By induction and due to construction it can be shown that this equals to $R^{\mathcal{I}_\mathbf{d}}_\mathbf{f}$. Let us denote with $\mathbf{d} \lessdot \mathbf{e}$ the *direct coverage relation*, that is $\mathbf{d} \lessdot \mathbf{e}$ iff $\mathbf{d} \prec \mathbf{e}$ and $(\forall \mathbf{f})\ \mathbf{d} \not\prec \mathbf{f} \vee \mathbf{f} \not\prec \mathbf{e}$. Thus we have that $(\bigsqcup_{\mathbf{d}' \prec \mathbf{d}} R^{\mathbf{d}'}_\mathbf{f})^\mathcal{I} = (\bigsqcup_{\mathbf{d}' \lessdot \mathbf{d}} R_\mathbf{f}\#\mathbf{d}')^\mathcal{I} = \bigcup_{\mathbf{d}' \lessdot \mathbf{d}} (R_\mathbf{f}\#\mathbf{d}')^\mathcal{I}$. By induction hypothesis $\bigcup_{\mathbf{d}' \lessdot \mathbf{d}} (R_\mathbf{f}\#\mathbf{d}')^\mathcal{I} = \bigcup_{\mathbf{d}' \lessdot \mathbf{d}} R^{\mathcal{I}_{\mathbf{d}'}}_\mathbf{f}$. Moreover we have $\bigcup_{\mathbf{d}' \lessdot \mathbf{d}} R^{\mathcal{I}_{\mathbf{d}'}}_\mathbf{f} = \bigcup_{\mathbf{d}' \lessdot \mathbf{d}} (R^{\mathcal{I}_\mathbf{d}}_\mathbf{f} \cap \Delta_{\mathbf{d}'} \times \Delta_{\mathbf{d}'})$ which in fact equals $R^{\mathcal{I}_\mathbf{d}}_\mathbf{f} \cap \bigcup_{\mathbf{d}' \lessdot \mathbf{d}} (\Delta_{\mathbf{d}'} \times \Delta_{\mathbf{d}'})$. Going back to the first equation, we also have that $(R^\mathbf{d}_\mathbf{f})^\mathcal{I} = (R^{\mathcal{I}_\mathbf{d}}_\mathbf{f} \setminus \bigcup_{\mathbf{d}' \prec \mathbf{d}} \Delta_{\mathbf{d}'} \times \Delta_{\mathbf{d}'}) = (R^{\mathcal{I}_\mathbf{d}}_\mathbf{f} \setminus \bigcup_{\mathbf{d}' \lessdot \mathbf{d}} \Delta_{\mathbf{d}'} \times \Delta_{\mathbf{d}'})$. Summing up we have that $(R_\mathbf{f}\#\mathbf{d})^\mathcal{I} = (R^{\mathcal{I}_\mathbf{d}}_\mathbf{f} \setminus \bigcup_{\mathbf{d}' \lessdot \mathbf{d}} \Delta_{\mathbf{d}'} \times \Delta_{\mathbf{d}'}) \cup (R^{\mathcal{I}_\mathbf{d}}_\mathbf{f} \cap \bigcup_{\mathbf{d}' \lessdot \mathbf{d}} (\Delta_{\mathbf{d}'} \times \Delta_{\mathbf{d}'})) = R^{\mathcal{I}_\mathbf{d}}_\mathbf{f}$ thus the assertion is verified.

Thus we have showed that Lemma 1 is applicable, hence for any axioms $\phi \in \mathrm{K}(\mathcal{C}_\mathbf{d})$ we have $\mathcal{I}_\mathbf{d} \models \phi$ because $\mathfrak{I}$ is a model of $\mathfrak{K}$, and by Lemma 1 we then have $\mathcal{I} \models \phi\#\mathbf{d}$. Note that if constants appear in ϕ then $\mathcal{I}_\mathbf{d}$ certainly is defined on these constants because in such a case ϕ appears directly in the ABox of $\mathrm{K}(\mathcal{C}_\mathbf{d})$.

Summing up, we have showed that $\mathcal{I}$ is a model of $\#(\mathfrak{K})$. Finally, as $\mathfrak{K}$ is $\mathbf{d}$-satisfiable, we have $\emptyset \neq \Delta_\mathbf{d} \subseteq \Delta$ hence $\Delta \neq \emptyset$ and hence $\#(\mathfrak{K})$ is satisfiable. □

Lemma 3. *If $\top^\mathbf{d}_\mathbf{d}$ is satisfiable with respect to $\#(\mathfrak{K})$, then $\mathfrak{K}$ is $\mathbf{d}$-satisfiable.*

Proof. Given a model $\mathcal{I}$ of $\#(\mathfrak{K})$ with $(\top^\mathbf{d}_\mathbf{d})^\mathcal{I} \neq \emptyset$, let us construct a CKR model $\mathfrak{I} = \{\mathcal{I}_\mathbf{d}\}_{\mathbf{d}\in\mathfrak{D}_\Gamma}$ s.t. for every individual a, atomic concept $A_{\mathbf{f_B}}$, role $R_{\mathbf{f_B}}$ (both partially qualified), and for all $\mathbf{d} \in \mathfrak{D}_\Gamma$, $\mathcal{I}_\mathbf{d} = \langle \Delta_\mathbf{d}, \mathcal{I}_\mathbf{d}\rangle$ is as follows:

1. $\Delta_\mathbf{d} = (\top^\mathbf{d}_\mathbf{d})^\mathcal{I}$;
2. $a^{\mathcal{I}_\mathbf{d}} = (a^\mathbf{d})^\mathcal{I}$ if $(a^\mathbf{d})^\mathcal{I} \neq \mathsf{undef}^\mathcal{I}$ else $a^{\mathcal{I}_\mathbf{d}}$ is undefined;
3. $(A_{\mathbf{f_B}})^{\mathcal{I}_\mathbf{d}} = (A^\mathbf{d}_{\mathbf{f_B}+\mathbf{d}})^\mathcal{I}$;
4. $(R_{\mathbf{f_B}})^{\mathcal{I}_\mathbf{d}} = (\bigsqcup_{\mathbf{d}'\preceq\mathbf{d}} R^{\mathbf{d}'}_{\mathbf{f_B}+\mathbf{d}})^\mathcal{I}$.

To show that that $\mathfrak{I}$ is a model of $\mathfrak{K}$, we have to show that conditions of Definition 6 are satisfied.

1. By Definition 9, if $\mathbf{d} \prec \mathbf{e}$, then $\mathcal{I} \models \top^\mathbf{f}_\mathbf{d} \sqsubseteq \top^\mathbf{f}_\mathbf{e}$. This implies that $(\top^\mathbf{f}_\mathbf{d})^\mathcal{I} \subseteq (\top^\mathbf{f}_\mathbf{e})^\mathcal{I}$ and by construction $(\top_\mathbf{d})^{\mathcal{I}_\mathbf{f}} \subseteq (\top_\mathbf{e})^{\mathcal{I}_\mathbf{f}}$.
2. By Definition 9 we have that $\mathcal{I} \models A^\mathbf{d}_\mathbf{f} \sqsubseteq \top^\mathbf{d}_\mathbf{f}$. Hence $(A^\mathbf{d}_\mathbf{f})^\mathcal{I} \subseteq (\top^\mathbf{d}_\mathbf{f})^\mathcal{I}$, which implies $(A_\mathbf{f})^{\mathcal{I}_\mathbf{d}} \subseteq (\top_\mathbf{f})^{\mathcal{I}_\mathbf{d}}$.
3. By construction we have that $(R_\mathbf{f})^{\mathcal{I}_\mathbf{d}} = (\bigsqcup_{\mathbf{d}'\preceq\mathbf{d}} R^{\mathbf{d}'}_\mathbf{f})^\mathcal{I}$. By Definition 9, we have that, for every $\mathbf{d}' \preceq \mathbf{d}$, $\mathcal{I} \models \exists R^{\mathbf{d}'}_\mathbf{f}.\top \sqsubseteq \top^{\mathbf{d}'}_\mathbf{f}$ and $\mathcal{I} \models \top \sqsubseteq \forall R^{\mathbf{d}'}_\mathbf{f}.\top^{\mathbf{d}'}_\mathbf{f}$ which implies $(R^{\mathbf{d}'}_\mathbf{f})^\mathcal{I} \subseteq (\top^{\mathbf{d}'}_\mathbf{f})^\mathcal{I} \times (\top^{\mathbf{d}'}_\mathbf{f})^\mathcal{I}$. By Definition 9, for every $\mathbf{d}' \preceq \mathbf{d}$ we have that $(\top^{\mathbf{d}'}_\mathbf{f})^\mathcal{I} \subseteq (\top^\mathbf{d}_\mathbf{f})^\mathcal{I}$. Hence $(R^\mathbf{d}_\mathbf{f})^\mathcal{I} \subseteq (\top^\mathbf{d}_\mathbf{f})^\mathcal{I} \times (\top^\mathbf{d}_\mathbf{f})^\mathcal{I}$ which implies that $(R_\mathbf{f})^{\mathcal{I}_\mathbf{d}} \subseteq (\top_\mathbf{f})^{\mathcal{I}_\mathbf{d}} \times (\top_\mathbf{f})^{\mathcal{I}_\mathbf{d}}$.
4. Suppose $\mathbf{d} \prec \mathbf{e}$ and $a^{\mathcal{I}_\mathbf{d}}$ is defined. From the construction of $\#(\mathfrak{K})$ we have $\mathcal{I} \models \{a^\mathbf{d}\} \sqsubseteq \{a^\mathbf{e}, \mathsf{undef}\}$, that is, either $(a^\mathbf{d})^\mathcal{I} = (a^\mathbf{e})^\mathcal{I}$ or $(a^\mathbf{d})^\mathcal{I} = \mathsf{undef}^\mathcal{I}$. However, since $a^{\mathcal{I}_\mathbf{d}}$ is defined, due to the construction of $\mathfrak{I}$ it must be the case that $(a^\mathbf{d})^\mathcal{I} \neq \mathsf{undef}^\mathcal{I}$ and hence $(a^\mathbf{d})^\mathcal{I} = (a^\mathbf{e})^\mathcal{I}$. From the construction of $\mathfrak{I}$ we have $a^{\mathcal{I}_\mathbf{d}} = (a^\mathbf{d})^\mathcal{I} = (a^\mathbf{e})^\mathcal{I} = a^{\mathcal{I}_\mathbf{e}}$.
   Suppose the other case, i.e., $\mathbf{d} \prec \mathbf{e}$ and $a^{\mathcal{I}_\mathbf{e}}$ is defined and $a^{\mathcal{I}_\mathbf{e}} \in \Delta_\mathbf{d}$. From the construction of $\#(\mathfrak{K})$ we have $\mathcal{I} \models \top^\mathbf{d}_\mathbf{d} \sqcap \{a^\mathbf{e}\} \sqsubseteq \{a^\mathbf{d}\}$, that is, $(\top^\mathbf{d}_\mathbf{d})^\mathcal{I} \cap \{(a^\mathbf{e})^\mathcal{I}\} \subseteq \{(a^\mathbf{d})^\mathcal{I}\}$. We have assumed $a^{\mathcal{I}_\mathbf{e}} \in \Delta_\mathbf{d}$ and hence the construction of $\mathfrak{I}$ this implies $(a^\mathbf{e})^\mathcal{I} \in (\top^\mathbf{d}_\mathbf{d})^\mathcal{I}$ and hence the above inclusion reduces into $\{(a^\mathbf{e})^\mathcal{I}\} \subseteq \{(a^\mathbf{d})^\mathcal{I}\}$ which implies $(a^\mathbf{e})^\mathcal{I} = (a^\mathbf{d})^\mathcal{I}$ and from the construction of $\mathfrak{I}$ also $a^{\mathcal{I}_\mathbf{d}} = a^{\mathcal{I}_\mathbf{e}}$.
5. If X is an atomic concept A, by definition we directly have that $(A_{\mathbf{d_B}})^{\mathcal{I}_\mathbf{e}} = (A^\mathbf{e}_{\mathbf{d_B}+\mathbf{e}})^\mathcal{I} = (A_{\mathbf{d_B}+\mathbf{e}})^{\mathcal{I}_\mathbf{e}}$. On the other hand, if X is a role R, we have $(R_{\mathbf{d_B}})^{\mathcal{I}_\mathbf{e}} = (\bigsqcup_{\mathbf{e}'\preceq\mathbf{e}} R^{\mathbf{e}'}_{\mathbf{d_B}+\mathbf{e}})^\mathcal{I} = (R_{\mathbf{d_B}+\mathbf{e}})^{\mathcal{I}_\mathbf{e}}$.
6. We must prove that $(X_\mathbf{d})^{\mathcal{I}_\mathbf{e}} = (X_\mathbf{d})^{\mathcal{I}_\mathbf{d}}$ for $\mathbf{d} \prec \mathbf{e}$ and any atomic concept or role $X_\mathbf{d}$. Since steps 2, 3, 7, and 8 were already proved, we know that the respective conditions of CKR models hold for $\mathfrak{I}$.
   Let $X_\mathbf{d} = A_\mathbf{d}$ be an atomic concept. From condition 2 od CKR models and form the construction we have $A^{\mathcal{I}_\mathbf{e}}_\mathbf{d} \subseteq \top^{\mathcal{I}_\mathbf{e}}_\mathbf{d} = (\top^\mathbf{e}_\mathbf{d})^\mathcal{I} = (\top^\mathbf{d}_\mathbf{d})^\mathcal{I} = \Delta_\mathbf{d}$ where one of the equation follows from the fact that $\mathcal{I} \models \top^\mathbf{d}_\mathbf{d} \equiv \top^\mathbf{e}_\mathbf{d}$. Then from condition 7 of CKR models we get $A^{\mathcal{I}_\mathbf{d}}_\mathbf{d} = A^{\mathcal{I}_\mathbf{e}}_\mathbf{d} \cap \Delta_\mathbf{d}$. But since $A^{\mathcal{I}_\mathbf{e}}_\mathbf{d} \subseteq \Delta_\mathbf{d}$, then $A^{\mathcal{I}_\mathbf{d}}_\mathbf{d} = A^{\mathcal{I}_\mathbf{e}}_\mathbf{d}$.
   The case when $X_\mathbf{d} = R_\mathbf{d}$ is a role is analogical, only we have to use conditions 3 and 8 of CKR models instead of conditions 2 and 7.
7. If $\mathbf{d} \prec \mathbf{e}$, by Definition 9 we have that $\mathcal{I} \models A^\mathbf{d}_\mathbf{f} \equiv A^\mathbf{e}_\mathbf{f} \sqcap \top^\mathbf{d}_\mathbf{d}$. This implies that $(A^\mathbf{d}_\mathbf{f})^\mathcal{I} = (A^\mathbf{e}_\mathbf{f})^\mathcal{I} \cap (\top^\mathbf{d}_\mathbf{d})^\mathcal{I}$, which implies $(A_\mathbf{f})^{\mathcal{I}_\mathbf{d}} = (A_\mathbf{f})^{\mathcal{I}_\mathbf{e}} \cap \Delta_\mathbf{d}$;
8. By construction, we have that $(R_\mathbf{f})^{\mathcal{I}_\mathbf{d}} = (\bigsqcup_{\mathbf{d}'\preceq\mathbf{d}} R^{\mathbf{d}'}_\mathbf{f})^\mathcal{I}$ and $(R_\mathbf{f})^{\mathcal{I}_\mathbf{e}} = (\bigsqcup_{\mathbf{e}'\preceq\mathbf{e}} R^{\mathbf{e}'}_\mathbf{f})^\mathcal{I}$. By Definition 9, we have that for every $\mathbf{e}' \in \mathfrak{D}_\Gamma$ such that $\mathbf{d} \prec \mathbf{e}' \preceq \mathbf{e}$, $\mathcal{I} \models \exists R^{\mathbf{e}'}_\mathbf{f}.\top^\mathbf{d}_\mathbf{d} \sqsubseteq \neg\top^\mathbf{d}_\mathbf{d}$. This implies that $(R^{\mathbf{e}'}_\mathbf{f})^\mathcal{I} \cap (\top^\mathbf{d}_\mathbf{d})^\mathcal{I} \times (\top^\mathbf{d}_\mathbf{d})^\mathcal{I}$ is necessarily empty. Hence, given $(R_\mathbf{f})^{\mathcal{I}_\mathbf{e}} \cap (\Delta_\mathbf{d} \times \Delta_\mathbf{d}) = (\bigsqcup_{\mathbf{e}'\preceq\mathbf{e}} R^{\mathbf{e}'}_\mathbf{f})^\mathcal{I} \cap (\top^\mathbf{d}_\mathbf{d})^\mathcal{I} \times (\top^\mathbf{d}_\mathbf{d})^\mathcal{I}$, this equals to $(\bigsqcup_{\mathbf{e}'\preceq\mathbf{d}} R^{\mathbf{e}'}_\mathbf{f})^\mathcal{I} = (R_\mathbf{f})^{\mathcal{I}_\mathbf{d}}$. Thus we have that $(R_\mathbf{f})^{\mathcal{I}_\mathbf{d}} = (R_\mathbf{f})^{\mathcal{I}_\mathbf{e}} \cap (\Delta_\mathbf{d} \times \Delta_\mathbf{d})$.
9. Let $\mathbf{d} = \dim(\mathcal{C})$. For every $\phi \in \mathrm{K}(\mathcal{C})$, we have that $\mathcal{I} \models \phi\#\mathbf{d}$. To apply the result of Lemma 1, we have to prove that $\mathfrak{I}$ satisfies the preconditions on interpretations. For every individual $a^\mathbf{d} = a\#\mathbf{d}$, we have that $a^{\mathcal{I}_\mathbf{d}} = (a^\mathbf{d})^\mathcal{I}$ if $(a^\mathbf{d})^\mathcal{I} \neq \mathsf{undef}^\mathcal{I}$, otherwise $a^{\mathcal{I}_\mathbf{d}}$ is undefined: thus, for any individual a with $a^{\mathcal{I}_\mathbf{d}}$ defined, it holds that $a^{\mathcal{I}_\mathbf{d}} = (a^\mathbf{d})^\mathcal{I}$. For every atomic concept $A_{\mathbf{f_B}}\#\mathbf{d} = A^\mathbf{d}_{\mathbf{f_B}+\mathbf{d}}$, we directly have that $(A_{\mathbf{f_B}})^{\mathcal{I}_\mathbf{d}} = (A^\mathbf{d}_{\mathbf{f_B}+\mathbf{d}})^\mathcal{I}$. In the case of roles, we have that $(R_{\mathbf{f_B}}\#\mathbf{d})^\mathcal{I} = (\bigsqcup_{\mathbf{d}'\preceq\mathbf{d}} R^{\mathbf{d}'}_{\mathbf{f_B}+\mathbf{d}})^\mathcal{I} = (R_{\mathbf{f_B}})^{\mathcal{I}_\mathbf{d}}$. Thus Lemma 1 is applicable and $\mathcal{I}_\mathbf{d} \models \phi$, which proves the assertion.

Hence $\mathfrak{I}$ is a model of $\mathfrak{K}$ and by the construction $\Delta_\mathbf{d} = (\top^\mathbf{d}_\mathbf{d})^\mathcal{I}$, which is nonempty. And so $\mathfrak{K}$ is $\mathbf{d}$-satisfiable. □

Finally, building on the findings of this section, we are able to analyze the complexity of this reduction approach.

Theorem 1. *The problem of satisfiability checking of $\mathcal{ALC}$-based CKR $\mathfrak{K}$ is* EXPTIME*-complete.*

Proof. The decidability follows from Lemmata 2 and 3, as the problem of checking $\mathfrak{K} \models \mathbf{d} : \phi$ can be rewritten into the problem of checking if $\#(\mathfrak{K}) \models \phi\sharp\mathbf{d}$.

We will show that the transformation $\#(\cdot)$ is polynomial. Since $\#(\mathfrak{K})$ is $\mathcal{ALCO}(\sqcup)$ knowledge base and deciding entailment is EXPTIME-hard for $\mathcal{ALCO}(\sqcup)$ (Baader et al. 2003) it follows that checking if $\mathfrak{K} \models \mathbf{d} : \phi$ is possible within the upper bound of EXPTIME worst case complexity.

Without loss of generality, we will consider the size of the input to be the total number of occurrences of all symbols from Σ and Γ in both $\mathfrak{K}$ and ϕ summed together with the total number of all DL constructors in $\mathfrak{K}$ and ϕ and with the number of formulae in $\mathfrak{K}$ and ϕ. We will denote this number by m. The real size of input to be processed depends on the encoding of symbols. As the number of symbols used in any particular knowledge base is always finite, suitable encoding can always be found such that the real size of input is $c \times m$ for some constant c (Tobies 2001).

As explained before, the number of contextual dimensions is assumed to be a fixed constant k. While in theory the number of possible dimensional values may be large, in practice the number of contexts n is always smaller than m. This is because whenever a new context $\mathcal{C}$ is initialized, also k new formulae are added in the meta knowledge, by which the dimensional values are associated with $\mathcal{C}$.

Let us now determine the size of $\#(\mathfrak{K})$. We will go through the construction in Definition 9 point by point:

1. one axiom $\top^{\mathbf{f}}_{\mathbf{d}} \sqsubseteq \top^{\mathbf{f}}_{\mathbf{e}}$ for any three initialized contexts $\mathcal{C}_{\mathbf{d}}$, $\mathcal{C}_{\mathbf{e}}$ and $\mathcal{C}_{\mathbf{f}}$, with $\mathbf{d} \prec \mathbf{e}$. These are maximum n^3 of axioms of size 3, i.e., with total size bounded with $3 \times n^3$;
2. one axiom $A^{\mathbf{d}}_{\mathbf{e}} \sqsubseteq \top^{\mathbf{d}}_{\mathbf{e}}$ for any two initialized contexts $\mathcal{C}_{\mathbf{d}}$, $\mathcal{C}_{\mathbf{e}}$ and for any $A_{\mathbf{e}}$ occurring in $\mathfrak{K}$. Note that $A_{\mathbf{e}}$ occurs in $\mathfrak{K}$ whenever $A_{\mathbf{e_B}}$ occurs in $\mathcal{C}_{\mathbf{g}}$ with $\mathbf{e} = \mathbf{e_B}+\mathbf{g}$ (below this sense will be also used w.r.t. roles). This means that for each such occurrence of $A_{\mathbf{e}}$ in $\mathfrak{K}$ there is at least one actual occurrence of some $A_{\mathbf{e_B}}$ with possibly incomplete dimensional vector. Therefore at most m atomic symbols (concepts, roles and individuals) in total occur in $\mathfrak{K}$ in this sense. This implies that most $m \times n^2$ axioms of size 3 are added in this step, with total size bounded with $3 \times m \times n^2$;
3. a pair of axioms $\exists R^{\mathbf{d}}_{\mathbf{e}}.\top \sqsubseteq \top^{\mathbf{d}}_{\mathbf{e}}$ and $\top \sqsubseteq \forall R^{\mathbf{d}}_{\mathbf{e}}.\top^{\mathbf{d}}_{\mathbf{e}}$; for any two initialized contexts $\mathcal{C}_{\mathbf{d}}$, $\mathcal{C}_{\mathbf{e}}$ and for any $R_{\mathbf{e}}$ occurring in $\mathfrak{K}$. Similarly to the previous step this yields at most $2 \times m \times n^2$ axioms of size 5, i.e., with total size bounded with $10 \times m \times n^2$;
4. two axioms $\top^{\mathbf{d}}_{\mathbf{d}} \sqcap \{a^{\mathbf{e}}\} \sqsubseteq \{a^{\mathbf{d}}\}$ and $\{a^{\mathbf{d}}\} \sqsubseteq \{a^{\mathbf{e}}, \mathsf{undef}\}$ for any two initialized contexts $\mathcal{C}_{\mathbf{d}}$, $\mathcal{C}_{\mathbf{e}}$ with $\mathbf{d} \prec \mathbf{e}$ and for any constant a, and one axiom $\neg\top^{\mathbf{d}}_{\mathbf{d}}(\mathsf{undef})$ for any initialized context $\mathcal{C}_{\mathbf{d}}$ This leads to maximum of $m \times n^2$ axioms of size 7, maximum of $m \times n^2$ axioms of size 9, and maximum of n axioms of size 3. The total sum of all these axioms is bounded under $16 \times m \times n^2 + 3 \times n$;
7. $\top^{\mathbf{d}}_{\mathbf{d}} \equiv \top^{\mathbf{e}}_{\mathbf{d}}$ for any two initialized contexts $\mathcal{C}_{\mathbf{d}}$, $\mathcal{C}_{\mathbf{e}}$ with $\mathbf{d} \preceq \mathbf{e}$. This leads to the maximum of n^2 axioms of size 3, i.e., with total size bounded with $3 \times n^2$;
8. one axiom $\exists R^{\mathbf{d}}_{\mathbf{f}}.\top^{\mathbf{d}'}_{\mathbf{d}'} \sqsubseteq \neg\top^{\mathbf{d}'}_{\mathbf{d}'}$, for every role $R_{\mathbf{f}}$ and any three initialized contexts $\mathcal{C}_{\mathbf{d}'}$, $\mathcal{C}_{\mathbf{d}}$ and $\mathcal{C}_{\mathbf{f}}$ with $\mathbf{d}' \prec \mathbf{d}$. This yields at most $m \times n^3$ axioms of size 6, i.e., with total size bounded with $6 \times m \times n^3$;
9. one axiom $\phi\#\mathbf{d}$ for every axiom ϕ occurring in any context $\mathrm{K}(\mathcal{C})$ of $\mathfrak{K}$. Not that the original size of these axioms combined is smaller then m (precisely, it is $m - |\mathfrak{M}|$). However, application of the $\#(\cdot)$ operator yields to a blow up in the axiom size because each symbol may be replaced by up to $2 \times n - 1$ new symbols (the worst case is for roles). Therefore the total size of the axioms added in this step is bounded with $(2 \times n - 1) \times m$.

Summing up, the transformed knowledge base $\#(\mathfrak{K})$ is bounded in size with $6 \times m \times n^3 + 29 \times m \times n^2 + 3 \times n^3 + 3 \times n^2 + (2 \times n + 2) \times m$ which is under $O(m^4)$ since $n \leq m$.

□

4 Related Works

The main complexity result for the CKR framework is described in (Serafini and Homola 2012) and says that reasoning in $\mathcal{SROIQ}$-based CKR is 2NExpTime-complete. In other words this proves that, in the case of $\mathcal{SROIQ}$ CKR, the contextualization of knowledge provided by the framework does not imply a complexity jump in the reasoning. This non-jumping property, however, is not guaranteed for CKR based on weaker languages. Indeed the complexity of $\mathcal{ALC}$-based CKR, presented in this paper, cannot be obtained by directly applying the reduction proposed in (Serafini and Homola 2012). This translation indeed introduces role chain axioms in the target knowledge base, which implies that the resulting knowledge base is strictly in $\mathcal{SROIQ}$ with its high complexity. In order to stay in same complexity as $\mathcal{ALC}$, we had to adapt the reduction (which now goes to $\mathcal{ALCO}(\sqcup)$) without using role chains and role inclusion axioms.

A related proposal for the contextualization of the $\mathcal{ALC}$ DL is $\mathcal{ALC}_{\mathcal{ALC}}$ (Klarman and Gutiérrez-Basulto 2010). This proposal is a multi-modal extension of $\mathcal{ALC}$ which shares with CKR the possibility to formalize the contextual structure in a meta-language separated in the language used to represent the domain knowledge. The two frameworks differ from the expressivity of the contextual assertions: in $\mathcal{ALC}_{\mathcal{ALC}}$ it is possible to qualify knowledge with respect to context classes rather than the more simple qualification with respect to individual contexts of CKR. One effect of this rich contextualization is that complexity of reasoning in $\mathcal{ALC}_{\mathcal{ALC}}$ jumps to 2ExpTime from the ExpTime complexity of the object language $\mathcal{ALC}$. On the other hand, in the CKR framework, the complexity result presented in this paper proves that contextualization of the $\mathcal{ALC}$ DL can be obtained without such a complexity jump.

Complexity results in CKR open the possibility to prove other upper-bound complexity on other ontology modularization formalisms, such as Distributed Description Logics (Ghidini and Serafini 2008), Package based Description Logics (Bao et al. 2009), and $\mathcal{E}$-connections (Kutz et al. 2004). A formal comparison of these approaches with CKR is out of the scope of this paper, however, from preliminary investigations we notice that these formalisms can be rewritten into an equivalent CKR composed of a top (global) context, used to formalize the integrated vision that covers a set of contexts each of which encapsulates one of the ontology modules. For instance the bridge rule $i : A \xrightarrow{\sqsubseteq} j : B$ can be represented in the global context g with the axioms $A_i \sqsubseteq \forall R_{ij} B_j$, where $i \prec g$ and $j \prec g$.

5 Conclusions

With its semantics rooted in the well studied principles of contextual representation, Contextualized Knowledge

Repository (CKR) is a knowledge representation framework that provides an effective contextual layer for SW and LOD knowledge resources. CKR knowledge bases can be built on top of the highly expressive $\mathcal{SROIQ}$ or any of its fragments. As knowledge in the SW can be available in simpler languages, it is interesting to study the properties of the framework in such relevant fragments. One example is the $\mathcal{ALC}$ DL, for which reasoning is known to be EXPTIME-complete, in contrast to $\mathcal{SROIQ}$ which is 2NEXPTIME-complete.

In this paper we showed that reasoning in $\mathcal{ALC}$-based CKR is EXPTIME-complete: this proves that CKR contextualization over $\mathcal{ALC}$ does not yield a jump in complexity compared to the underlying object language. This result extends to $\mathcal{ALC}$-based CKR the similar property previously proved for CKR based on $\mathcal{SROIQ}$ (Serafini and Homola 2012) and allows the comparison to similar approaches that allow for contextualization of $\mathcal{ALC}$ (Klarman and Gutiérrez-Basulto 2010).

The presented study opens possible further directions of investigation. The most direct evolution to this work regards the extension of the "non-jumping" result to other relevant fragments of $\mathcal{SROIQ}$ (e.g., $\mathcal{SHIQ}$). In a more general scope, as already mentioned, an interesting direction regards the extension of the complexity results to other ontology modularization formalisms, as DDLs (Ghidini and Serafini 2008) or PDLs (Bao et al. 2009).

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