# Tableaux Algorithm for Reasoning with Contextualized Knowledge

## Technical Report
## TR-FBK-DKM-2011-1

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### Abstract

The growth of the Semantic Web, a space where heterogeneous and contradicting knowledge sources coexist and are often combined, demands for effective methods to consider also the context within which the knowledge sources are intended to be valid. Contextualized Knowledge Repository (CKR) is a novel representation framework grounded in the well studied AI theories of context, whose aim is to bring the advantages of contextual representation to the Semantic Web. In this paper we present a sound and complete tableaux reasoning procedure for $\mathcal{ALC}$-based CKR. We study the complexity of the algorithm and discuss possible optimization. In fact, this is the first direct tableaux algorithm (i.e., not obtained by reduction) for any kind of CKR. It constitutes an important step towards practical decision algorithms for CKR based on $\mathcal{ALC}$ and more expressive DL, as currently the only practical algorithm is a forward chaining algorithm for CKR build on top of OWL Horst.

## 1 Introduction

With the recent growth of the resources published under the Semantic Web (SW) and Linked Open Data (LOD) initiatives, it becomes clear that most of this knowledge is not universally valid. A more accurate vision would consider this knowledge to hold under certain circumstances, such as specific time range, geo-political or cultural region, topic of interest, source of provenance, etc. Sometimes, these circumstances are made explicit in the metadata of the resource, but many times they are left implicit. This is partly because no general agreement on how this contextual metadata should be specified, but also because no common understanding on how they should affect the semantics of the data represented in the first place has yet been reached. As this problem becomes more and more apparent a number of formalisms (Klarman and Gutiérrez-Basulto 2011; Bao, Tao, and McGuinness 2010; Sahoo et al. 2010) for context dependent knowledge representation suitable for SW and LOD data have been recently proposed.

One of these formal frameworks is also Contextualized Knowledge Repository (CKR). It was developed on top of the $\mathcal{SROIQ}$ description logic (DL) with the aim to provide a formal contextual layer for languages as expressive as OWL 2 (Serafini and Homola 2011; 2012), however it can be adopted for any of its fragments. For instance an OWL Horst-based version is available (Joseph and Serafini 2011). Multiple knowledge resources can be integrated into a unique CKR repository as units called *contexts*. For instance, a CKR about football can be obtained by integrating the following four resources: a general ontology about sports, a more specific football ontology, and two specific knowledge resources concerned with the FIFA World Cup 2010 and National Football Leagues of 2010, in four contexts, say $\mathbf{sp}$, $\mathbf{fb}$, $\mathbf{wc10}$ and $\mathbf{nfl10}$. Contextual metadata can be specified for each context in form of *dimensional attributes*, e.g., the context $\mathbf{sp}$, which contains a generic resource about sports, is assigned with a $\mathsf{topic}$ dimension with value equal to $\mathsf{sports}$; while the $\mathsf{topic}$ dimension value of the context $\mathbf{wc10}$ will be equal to $\mathsf{fifa\_wc}$. The $\mathsf{time}$ dimension of contexts $\mathbf{wc10}$ and $\mathbf{nfl10}$ will be 2010, while for the context $\mathbf{fb}$ it can be possibly very general, such as 1800–2012, but also very specific, such as 2010 – this will depend on the perspective of the knowledge contained in this context. The values of each dimension are hierarchically ordered, e.g., $\mathsf{fifa\_wc} \prec \mathsf{football} \prec \mathsf{sports}$. This order is formally represented and imposes a hierarchical organization of contexts in the repository. Knowledge represented locally can be shared between contexts, for example, if a concept $\mathsf{TopSportsman}$ is defined in $\mathbf{sp}$, in more specific context $\mathbf{wc10}$ we are able to access a suitable restriction of this concept.

The design of CKR is grounded to the well known *locality* and *compatibility* principles of contextual knowledge representations (Ghidini and Giunchiglia 2001) and to the context as a box vision (Benerecetti, Bouquet, and Ghidini 2000; Lenat 1998). On the other hand, to a achieve practical applicability of the framework it is important to provide an effective reasoning algorithm capable to compute logical consequence. To date, the only practical reasoning algorithm is a forward chaining algorithm for the OWL Horst-based CKR (Joseph and Serafini 2011). For DL-based CKR, there is a natural deduction calculus and a translation procedure that translates a CKR repository into single $\mathcal{SROIQ}$ knowledge base (Serafini and Homola 2012). The former approach offers only a theoretical forward chaining method requiring to compute all logical consequences of each context which are then combined, the latter is more practical, however the translation introduces significant overhead and results into one big knowledge base which offers little space for additional optimization of reasoning driven by organization of knowledge inside contexts.

For $\mathcal{ALC}$ and more expressive DL, tableaux-based approaches have proven to be among the most practical reasoning methods (Baader et al. 2003). In this paper we present a sound and complete tableaux reasoning algorithm for $\mathcal{ALC}$-based CKR. The algorithm extends the well known $\mathcal{ALC}$ algorithm (Schmidt-Schauß and Smolka 1991; Buchheit, Donini, and Schaerf 1993) which is used for local reasoning inside contexts. A number of additional tableaux rules are added to ensure compatibility (i.e., knowledge propagation) between contexts w.r.t. the CKR semantics. Even if reasoning in $\mathcal{ALC}$ is EXPTIME-complete, the algorithm which we chose to extend is in NEXPTIME, and we show that the resulting tableaux algorithm for CKR is also in NEXPTIME. Note that this does not imply that the problem of reasoning with $\mathcal{ALC}$-based CKR in general is in NEXPTIME, this is still an open issue. We have chosen to extend this algorithm as also the known tableaux reasoning approaches for more expressive DL (Horrocks, Sattler, and Tobies 1999; Horrocks, Kutz, and Sattler 2006) can be seen as its extensions. We have thus made an important first step towards direct tableaux reasoning with CKR.

In the next section, we formally introduce $\mathcal{ALC}$-based CKR. In Sect. 3 we formally define CKR tableaux, then we develop the tableaux algorithm and show that it is sound and complete. Section 4 is dedicated to complexity results and preliminary thoughts on optimization. We discuss related work in Sect. 5 and conclusions follow in Sect. 6. Proofs of our statements can be found in the Appendix.

## 2 Contextualized Knowledge Repositories

In the following we briefly introduce the DL $\mathcal{ALC}$ and the basic definitions of CKR.

A DL vocabulary $\Sigma = N_\mathrm{C} \uplus N_\mathrm{R} \uplus N_\mathrm{I}$ is composed of three mutually disjoint subsets: $N_\mathrm{C}$ of atomic concepts (including the top concept $\top$ and the bottom concept $\bot$), $N_\mathrm{R}$ of roles, and $N_\mathrm{I}$ of individuals. Concept descriptions for $\mathcal{ALC}$ are inductively defined as follows:

$$C, D \ ::= \ A \mid \neg C \mid C \sqcap D \mid C \sqcup D \mid \exists R.C \mid \forall R.C$$

with $A \in N_\mathrm{C}$ and $R \in N_\mathrm{R}$. A concept $A$ is atomic iff $A \in N_\mathrm{C}$, otherwise $A$ is complex. A concept is in negation normal form (NNF) if the $\neg$ operator occurs only in front of atomic concepts. Every concept can be reduced in an equivalent concept in NNF (Baader et al. 2003). A TBox $\mathcal{T}$ of $\mathcal{ALC}$ is a finite set of general concept inclusions (CGI) which are expressions of the form $C \sqsubseteq D$, where $C, D$ are concepts. We write $C \equiv D$ as a shorthand for the pair of GCI $C \sqsubseteq D$ and $D \sqsubseteq C$. An ABox $\mathcal{A}$ of $\mathcal{ALC}$ is a finite set of axioms of the form $C(a)$ (concept assertion) or $R(a, b)$ (role assertion), where $a, b \in N_\mathrm{I}$, $R \in N_\mathrm{R}$ and $C$ is a concept. A knowledge base is a pair $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ where $\mathcal{T}$ is a TBox and $\mathcal{A}$ is an ABox.

An ($\mathcal{ALC}$-)interpretation is a pair $\mathcal{I} = \langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \rangle$, where $\Delta^\mathcal{I}$ is a non empty set, and $\cdot^\mathcal{I}$ is a function such that: $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$ for any atomic concept $A \in N_\mathrm{C}$, $R^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$ for all $R \in N_\mathrm{R}$, $a^\mathcal{I} \in \Delta^\mathcal{I}$, for all $a \in N_\mathrm{I}$. In addition, for all concepts $C, D$ and for all roles $R$, the following constraints are satisfied:

- $(\neg C)^\mathcal{I} = \Delta^\mathcal{I} \setminus C^\mathcal{I}$
- $(C \sqcap D)^\mathcal{I} = C^\mathcal{I} \cap D^\mathcal{I}$ and $(C \sqcup D)^\mathcal{I} = C^\mathcal{I} \cup D^\mathcal{I}$
- $(\exists R.C)^\mathcal{I} = \{x \in \Delta^\mathcal{I} \mid \exists y \in \Delta^\mathcal{I}, (x, y) \in R^\mathcal{I} \wedge y \in C^\mathcal{I}\}$
- $(\forall R.C)^\mathcal{I} = \{x \in \Delta^\mathcal{I} \mid \forall y \in \Delta^\mathcal{I}, (x, y) \in R^\mathcal{I} \implies y \in C^\mathcal{I}\}$

An interpretation $\mathcal{I}$ *satisfies* an axiom $\alpha$ (denoted $\mathcal{I} \models \alpha$) as follows: $\mathcal{I} \models C \sqsubseteq D$ iff $C^\mathcal{I} \subseteq D^\mathcal{I}$; $\mathcal{I} \models C(a)$ iff $a^\mathcal{I} \in C^\mathcal{I}$; and $\mathcal{I} \models R(a, b)$ iff $\langle a^\mathcal{I}, b^\mathcal{I} \rangle \in R^\mathcal{I}$. $\mathcal{I}$ is a *model* of $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ (denoted $\mathcal{I} \models \mathcal{K}$) iff it satisfies all axioms in $\mathcal{T} \cup \mathcal{A}$. We say that $\mathcal{K}$ is satisfiable if it has a model.

The two classic decision problems for $\mathcal{ALC}$ are *satisfiability* of concepts and subsumption *entailment*: a concept $C$ is satisfiable w.r.t. an $\mathcal{ALC}$ KB $\mathcal{K}$ iff there exists a model $\mathcal{I}$ of $\mathcal{K}$ s.t. $C^\mathcal{I}$ is non-empty; a subsumption formula $C \sqsubseteq D$ is entailed by an $\mathcal{ALC}$ KB $\mathcal{K}$ iff $C^\mathcal{I} \subseteq D^\mathcal{I}$ in all models $\mathcal{I}$ of $\mathfrak{K}$. It is well known that these two problems are inter-reducible (Baader et al. 2003).

We now briefly introduce the basic definition of CKR. For a more detailed description see (Serafini and Homola 2012). The logical formalization of contextual knowledge relies on two different languages: the *object-language*, used to specify knowledge within contexts, and the *meta-language*, used to specify knowledge about contexts. CKR maintains high level of separation between these two languages, as each is built on top of a distinct DL vocabulary.

**Definition 1** (Meta vocabulary)**.** *A meta vocabulary* $\Gamma$ *is a DL vocabulary that contains:*

1. *a set of individuals called* context identifiers*;*
2. *a finite set of roles* $\mathbf{A}$ *called* dimensions*;*
3. *for every dimension* $A \in \mathbf{A}$*, a set of individuals* $D_A$*, called* dimensional values*, and a role* $\prec_A$*, called* coverage relation*.*

The number of dimensions $k = |\mathbf{A}|$ is assumed to be a fixed constant. Intuitively, meta-assertions of the form $A(\mathcal{C}, d)$ for a context identifier $\mathcal{C}$ and $d \in D_A$, state that the value of the dimension $A$ of the context $\mathcal{C}$ is $d$, while meta-assertions of the form $d \prec_A e$ state that the value $d$ of the dimension $A$ is covered by the value $e$. Depending on the dimension, the coverage relation has different intuitive meanings, e.g., if $A$ is $\mathsf{location}$ then the coverage relation is topological containment.

Dimensional vectors will be used to identify each context with a specific set of dimensional values. Given a meta vocabulary $\Gamma$ with dimensions $\mathbf{A} = \{A_1, \dots, A_k\}$, a *dimensional vector* $\mathbf{d}$ is a set of assignments $\{A_{i_1}{:=}d_{A_{i_1}}, \dots, A_{i_m}{:=}d_{A_{i_m}}\}$ s.t. $0 \leq m \leq |\mathbf{A}|$, $d_{A_{i_n}} \in D_{A_{i_n}}$ for $1 \leq n \leq m$, and $i_j \neq i_m$ for $j \neq m$. A dimensional vector $\mathbf{d}$ is full if it assigns vlaues to all dimensions (i.e., $m = k$), otherwise it is partial. If it is apparent which value belongs to which dimension, we simply write $\{d_{A_{i_1}}, \dots, d_{A_{i_m}}\}$. By $d_{A_i}$ we denote the actual value that $\mathbf{d}$ assigns to the dimension $A_i$ (similarly $e_{A_i}$ for vector $\mathbf{e}$, etc). The *dimensional space* of $\Gamma$ (denoted $\mathfrak{D}_\Gamma$) is the set $\mathfrak{D}_\Gamma$ of all full dimensional vectors of $\Gamma$.

**Example 1** (Dimensions)**.** Suppose a meta vocabulary $\Gamma$ with three dimensions: $\mathsf{time}$, $\mathsf{location}$ and $\mathsf{topic}$, and values: $D_\mathsf{topic} = \{\mathsf{sports}, \mathsf{football}, \mathsf{fwc}, \mathsf{nfl}\}$ (fwc standing for

FIFA WC, nfl for national football leagues), $D_{\mathsf{location}} = \{\mathsf{africa}, \mathsf{world}\}$, $D_{\mathsf{time}} = \{2010\}$. Coverage relations are as follows: africa $\prec_{\mathsf{location}}$ world; football $\prec_{\mathsf{topic}}$ sports; fwc $\prec_{\mathsf{topic}}$ football; nfl $\prec_{\mathsf{topic}}$ football. Four dimensional vectors that will be used in our running example can be seen in Fig. 1. ◇

The knowledge inside contexts is build on top of an object vocabulary. Object vocabulary contains regular (unqualified) symbols and also qualified symbols of the form $X_{\mathbf{d}}$ where $\mathbf{d}$ is a dimensional vector of $\Gamma$.

**Definition 2** (Object vocabulary)**.** *Let $\Gamma$ be a meta vocabulary. An object vocabulary $\Sigma$ is any DL vocabulary closed under concept/role qualification, i.e., for any dimensional vector $\mathbf{d}$ of $\Gamma$ (full or partial), and for any unqualified concept/role $X \in \Sigma$, $\Sigma$ also contains $X_{\mathbf{d}}$.*

Qualified symbols are used inside contexts to refer to the meaning of symbols w.r.t. some other context, e.g., intuitively speaking $\mathsf{WinnerTeam}_{\mathbf{wc10}}$ (also written $\mathsf{WinnerTeam}_{\{\mathsf{2010,africa,fwc}\}}$) means the winner team as defined in the context respective to $\mathbf{wc10}$. If the vector is partial and some dimensions are missing, the semantics always takes the respective values from the context where the symbol appears, e.g., the same symbol can be written as $\mathsf{WinnerTeam}_{\{\mathsf{africa,fwc}\}}$ if it appears, say, in context $\mathbf{nfl10}$ as $\mathbf{nfl10}$ has the value of time set also to 2010. In this respect, if any non-qualified symbol $X$ appears in the context of $\mathbf{d}$, it is understood as qualified by the empty dimensional vector $\{\}$ and all dimensional values are taken from the context, i.e. $X$ equals to $X_{\mathbf{d}}$ when used in the context of $\mathbf{d}$.

**Example 2** (Qualified symbols)**.** In order to reuse information from the context $\mathbf{wc10}$ we would state in $\mathbf{fb}$ that the "champion team" is exactly the winner of FIFA WC, using the axiom: $\mathsf{WorldChampionTeam} \equiv \mathsf{WinnerTeam}_{\mathbf{wc10}}$. Also, in the context of $\mathbf{wc10}$ we want to state that Buffon plays for team Italy, but at the national level he plays for Juventus. We can use the ABox axioms $\mathsf{playsFor}(\mathsf{buffon}, \mathsf{team\_italy})$ and $\mathsf{playsFor}_{\mathbf{nfl10}}(\mathsf{buffon}, \mathsf{juventus})$. ◇

Contexts and CKR knowledge bases are formally defined as follows.

**Definition 3** (Context)**.** *Given a pair of meta/object vocabularies $\langle \Gamma, \Sigma \rangle$, a context is a triple $\langle \mathcal{C}, \mathrm{dim}(\mathcal{C}), \mathrm{K}(\mathcal{C}) \rangle$ where:*

1. *$\mathcal{C}$ is a context identifier of $\Gamma$;*
2. *$\mathrm{dim}(\mathcal{C})$ is a full dimensional vector of $\mathfrak{D}_\Gamma$;*
3. *$\mathrm{K}(\mathcal{C})$ is an $\mathcal{ALC}$ knowledge base over $\Sigma$.*

**Definition 4** (Contextualized Knowledge Repository)**.** *Given a pair of meta/object vocabularies $\langle \Gamma, \Sigma \rangle$, a CKR knowledge base (CKR) is a pair $\mathfrak{K} = \langle \mathfrak{M}, \mathfrak{C} \rangle$ such that:*

1. *$\mathfrak{C}$ is a set of contexts on $\langle \Gamma, \Sigma \rangle$;*
2. *$\mathfrak{M}$, called* meta knowledge*, is a DL knowledge base over $\Gamma$ such that:*
   (a) *every $A \in \mathbf{A}$ is a role;*
   (b) *for every $\mathcal{C} \in \mathfrak{C}$ with $\mathrm{dim}(\mathcal{C}) = \mathbf{d}$ and for every $A \in \mathbf{A}$, we have $\mathfrak{M} \models A(\mathcal{C}, d_A)$;*
   (c) *for every $A \in \mathbf{A}$, there is just one $d \in D_A$ s.t. $\mathfrak{M} \models A(\mathcal{C}, d)$, and the relation $\{d \prec_A d' \mid \mathfrak{M} \models \prec_A(d, d')\}$ is a strict partial order on $D_A$.*

$$\begin{tikzcd}
 & \mathbf{sp} = \langle 2010, \mathsf{world}, \mathsf{sports} \rangle & \\
 & \mathbf{fb} = \langle 2010, \mathsf{world}, \mathsf{football} \rangle \arrow[u, "\prec"'] & \\
\mathbf{wc10} = \langle 2010, \mathsf{africa}, \mathsf{fwc} \rangle \arrow[ur, "\prec"] & & \mathbf{nfl10} = \langle 2010, \mathsf{world}, \mathsf{nfl} \rangle \arrow[ul, "\prec"']
\end{tikzcd}$$

Figure 1: Overview of $\mathfrak{K}_{fb}$ coverage structure

In the rest of the paper we assume that CKR knowledge bases are defined over some suitable vocabulary $\langle \Gamma, \Sigma \rangle$, and all axioms and concepts are in NNF. For a CKR $\mathfrak{K}$, we will denote by $\mathcal{C}_{\mathbf{d}}$ a context with $\mathrm{dim}(\mathcal{C}) = \mathbf{d}$.

Coverage between context is defined as follows. Given $\mathbf{d}, \mathbf{e} \in \mathfrak{D}_\Gamma$ we say that $\mathbf{e}$ *covers* $\mathbf{d}$ w.r.t. a set of dimensions $\mathbf{B} \subseteq \mathbf{A}$ (denoted $\mathbf{d} \prec_{\mathbf{B}} \mathbf{e}$) if $d_A \prec_A e_A$ for all $A \in \mathbf{B}$. For convenience we write $\mathbf{d} \prec \mathbf{e}$ if $\mathbf{d} \prec_{\mathbf{A}} \mathbf{e}$; also $\mathcal{C}_{\mathbf{d}} \prec \mathcal{C}_{\mathbf{e}}$ if $\mathbf{d} \prec \mathbf{e}$. Intuitively, if one context covers another, its perspective is broader. Two auxiliary operations with dimensional vectors: $\mathbf{d}_{\mathbf{B}} := \{(A{:=}d) \in \mathbf{d} \mid A \in \mathbf{B}\}$ is the projection of $\mathbf{d}$ w.r.t. $\mathbf{B} \subseteq \mathbf{A}$; and $\mathbf{d}_{\mathbf{B}} + \mathbf{e}_{\mathbf{C}} := \mathbf{d}_{\mathbf{B}} \cup \{(A{:=}d) \in \mathbf{e}_{\mathbf{C}} \mid A \notin \mathbf{B}\}$ is the completion of $\mathbf{d}_{\mathbf{B}}$ w.r.t. $\mathbf{e}_{\mathbf{C}}$.

Note that in CKR built on top of more expressive logics, condition (2c) of Definition 4 can be assured directly in the meta knowledge with respective axioms: each $A \in \mathbf{A}$ is declared functional, and each $\prec_A$ is declared irreflexive and transitive. In $\mathcal{ALC}$ we do not have this option, however this is not a problem, because the number of all dimensions is assumed to be finite as is the number of context in a CKR. Hence after the meta knowledge is modeled, the condition (2c) can be verified even without a reasoner (e.g., by some script). These conditions are needed to assure reasonable properties of contextual space, i.e., acyclicity, dimensional values uniquely determined (Serafini and Homola 2012).

**Example 3** (Contextualized Knowledge Repository)**.** Using the meta knowledge specified in Example 1, CKR $\mathfrak{K}_{fb}$ will contain four contexts: $\mathcal{C}_{\mathbf{sp}}$, $\mathcal{C}_{\mathbf{fb}}$, $\mathcal{C}_{\mathbf{wc10}}$, $\mathcal{C}_{\mathbf{nfl10}}$. The coverage ($\prec$) of $\mathfrak{K}_{fb}$ is depicted in Fig. 1. We consider these axioms to be included in local knowledge bases of $\mathfrak{K}_{fb}$:

$\mathrm{K}(\mathcal{C}_{\mathbf{sp}})$: $\mathsf{Team} \sqsubseteq \mathsf{Organization}$, $\mathsf{Sportsman} \sqsubseteq \mathsf{Person}$,
$\mathsf{TopSportsman} \sqsubseteq \mathsf{Sportsman}$,

$\mathrm{K}(\mathcal{C}_{\mathbf{fb}})$: $\mathsf{Team} \sqsubseteq \mathsf{Team}_{\mathbf{sp}}$, $\mathsf{Player} \sqsubseteq \mathsf{Sportsman}_{\mathbf{sp}}$,
$\mathsf{Player} \sqsubseteq \forall \mathsf{playsFor}.\mathsf{Team}$,
$\mathsf{WorldChampionPlayer} \equiv \mathsf{ChampionPlayer}_{\mathbf{wc10}}$,
$\mathsf{TopSportsman}_{\mathbf{sp}} \equiv \mathsf{WorldChampionPlayer}$,

$\mathrm{K}(\mathcal{C}_{\mathbf{wc10}})$: $\mathsf{WinnerTeam} \equiv \mathsf{WinnerFinal}$,
$\mathsf{RunnerUpTeam} \equiv \mathsf{LoserFinal}$,
$\mathsf{ChampionPlayer} \equiv \forall \mathsf{playsFor}.\mathsf{WinnerTeam}$

$\mathrm{K}(\mathcal{C}_{\mathbf{nfl10}})$: $\mathsf{WinnerTeam} \equiv \mathsf{LeagueWinner}$,
$\mathsf{WinnerSerieA} \sqsubseteq \mathsf{LeagueWinner}$,
$\mathsf{WinnerPremierLeague} \sqsubseteq \mathsf{LeagueWinner}, \ldots$

◇

Semantics of CKR relies on DL semantics inside each context while the compatibility between contexts is assured by some additional semantic restrictions. Local interpretation need not necessarily provide denotations for all individuals of $\Sigma$, as some of them may not be meaningful in all contexts. Also, some of the local domains may be empty.

**Definition 5** (Local Interpretation)**.** *Local interpretation of a context $\mathcal{C}_{\mathbf{d}}$ is a pair $\mathcal{I}_{\mathbf{d}} = \langle \Delta_{\mathbf{d}}, \cdot^{\mathcal{I}_{\mathbf{d}}} \rangle$ s.t.: (a) either $\Delta_{\mathbf{d}} = \emptyset$; (b) or there is $N'_{\mathrm{I}} \subseteq N_{\mathrm{I}}$ s.t. $\mathcal{I}_{\mathbf{d}}$ is an $\mathcal{ALC}$-interpretation over $\Sigma' = N_{\mathrm{C}} \uplus N_{\mathrm{R}} \uplus N'_{\mathrm{I}}$.*

**Definition 6** (CKR-Model)**.** *A model of a CKR $\mathfrak{K}$ is a collection $\mathfrak{I} = \{\mathcal{I}_{\mathbf{d}}\}_{\mathbf{d}\in\mathfrak{D}_\Gamma}$ of local interpretations s.t. for all $\mathbf{d}, \mathbf{e}, \mathbf{f} \in \mathfrak{D}_\Gamma$, $A \in N_{\mathrm{C}}$, $R \in N_{\mathrm{R}}$, $X \in N_{\mathrm{C}} \cup N_{\mathrm{R}}$, $a \in N_{\mathrm{I}}$:*

*1.* $(\top_{\mathbf{d}})^{\mathcal{I}_{\mathbf{f}}} \subseteq (\top_{\mathbf{e}})^{\mathcal{I}_{\mathbf{f}}}$ *if* $\mathbf{d} \prec \mathbf{e}$

*2.* $(A_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} \subseteq (\top_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}}$

*3.* $(R_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} \subseteq (\top_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} \times (\top_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}}$

*4.* $a^{\mathcal{I}_{\mathbf{e}}} = a^{\mathcal{I}_{\mathbf{d}}}$ *given* $\mathbf{d} \prec \mathbf{e}$*, either* $a^{\mathcal{I}_{\mathbf{d}}}$ *is defined*
*or* $a^{\mathcal{I}_{\mathbf{e}}}$ *is defined and* $a^{\mathcal{I}_{\mathbf{e}}} \in \Delta_{\mathbf{d}}$

*5.* $(X_{\mathbf{d}_{\mathbf{B}}})^{\mathcal{I}_{\mathbf{e}}} = (X_{\mathbf{d}_{\mathbf{B}+\mathbf{e}}})^{\mathcal{I}_{\mathbf{e}}}$

*6.* $(X_{\mathbf{d}})^{\mathcal{I}_{\mathbf{e}}} = (X_{\mathbf{d}})^{\mathcal{I}_{\mathbf{d}}}$ *if* $\mathbf{d} \prec \mathbf{e}$

*7.* $(A_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} = (A_{\mathbf{f}})^{\mathcal{I}_{\mathbf{e}}} \cap \Delta_{\mathbf{d}}$ *if* $\mathbf{d} \prec \mathbf{e}$

*8.* $(R_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} = (R_{\mathbf{f}})^{\mathcal{I}_{\mathbf{e}}} \cap (\Delta_{\mathbf{d}} \times \Delta_{\mathbf{d}})$ *if* $\mathbf{d} \prec \mathbf{e}$

*9.* $\mathcal{I}_{\mathbf{d}} \models \mathrm{K}(\mathcal{C}_{\mathbf{d}})$

The semantics takes care that local domains respect the coverage hierarchy (condition 1). Given contexts $\mathcal{C}_{\mathbf{d}} \prec \mathcal{C}_{\mathbf{e}}$, if an individual $a$ occurs in $\mathcal{C}_{\mathbf{d}}$ then it must be defined also in $\mathcal{C}_{\mathbf{e}}$ with the same meaning; if $a$ only occurs in $\mathcal{C}_{\mathbf{e}}$ however, it does not have to be defined in $\mathcal{C}_{\mathbf{d}}$ (condition 4). The interpretation of any qualified symbol $X_{\mathbf{f}}$ is roofed under $(\top_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}}$ in any context $\mathcal{C}_{\mathbf{d}}$, regardless the relation between $\mathcal{C}_{\mathbf{f}}$ and $\mathcal{C}_{\mathbf{d}}$ (conditions 2, 3). Semantics of qualified symbols are given by conditions 6–8: if $X_{\mathbf{d}}$ "comes" from a more specific context $C_{\mathbf{d}}$ than $\mathcal{C}_{\mathbf{e}}$, its interpretation in $\mathcal{C}_{\mathbf{e}}$ is exactly equal as in its home context $\mathcal{C}_{\mathbf{d}}$ (condition 6); if this is not the case then the interpretation of any $X_{\mathbf{f}}$ in $\mathcal{C}_{\mathbf{d}}$ and $\mathcal{C}_{\mathbf{e}}$ must be equal modulo the domain of the more specific context of these two (conditions 7 and 8). Finally, each $\mathcal{I}_{\mathbf{d}}$ is a DL-model of $\mathcal{C}_{\mathbf{d}}$ (condition 9).

Given a CKR $\mathfrak{K}$ and $\mathbf{d} \in \mathfrak{D}_\Gamma$, a concept $C$ is $\mathbf{d}$-*satisfiable* w.r.t. $\mathfrak{K}$ if there exists a CKR model $\mathfrak{I} = \{\mathcal{I}_{\mathbf{d}}\}_{\mathbf{d}\in\mathfrak{D}_\Gamma}$ of $\mathfrak{K}$ such that $C^{\mathcal{I}_{\mathbf{d}}} \neq \emptyset$; $\mathfrak{K}$ is $\mathbf{d}$-satisfiable if it has a CKR model $\mathfrak{I} = \{\mathcal{I}_{\mathbf{d}}\}_{\mathbf{d}\in\mathfrak{D}_\Gamma}$ such that $\Delta_{\mathbf{d}} \neq \emptyset$; $\mathfrak{K}$ is *globally satisfiable* if it has a CKR model $\mathfrak{I} = \{\mathcal{I}_{\mathbf{d}}\}_{\mathbf{d}\in\mathfrak{D}_\Gamma}$ such that $\Delta_{\mathbf{d}} \neq \emptyset$ for every $\mathbf{d} \in \mathfrak{D}_\Gamma$. An axiom $\alpha$ is $\mathbf{d}$-*entailed* by $\mathfrak{K}$ (denoted $\mathfrak{K} \models \mathbf{d} : \alpha$) if for every model $\mathfrak{I} = \{\mathcal{I}_{\mathbf{d}}\}_{\mathbf{d}\in\mathfrak{D}_\Gamma}$ of $\mathfrak{K}$ it holds $\mathcal{I}_{\mathbf{d}} \models \alpha$. As usual, $\mathbf{d}$-entailment can be reduced to $\mathbf{d}$-satisfiability: in particular $\mathfrak{K} \models \mathbf{d} : C \sqsubseteq D$ iff $C \sqcap \neg D$ is not $\mathbf{d}$-satisfiable w.r.t. $\mathfrak{K}$.

**Example 4** (CKR model)**.** Given a CKR model $\mathfrak{I}$ of our example CKR $\mathfrak{K}_{fb}$, by condition 1 domains are interpreted as expected also in the scope of other contexts: thus for example, it holds that $\top_{\mathbf{fb}}^{\mathcal{I}_{\mathbf{nfl10}}} \subseteq \top_{\mathbf{sp}}^{\mathcal{I}_{\mathbf{nfl10}}}$. By condition 2 also qualified symbols are interpreted accordingly: e.g., $\mathsf{ChampionPlayer}_{\mathbf{wc10}}^{\mathcal{I}_{\mathbf{fb}}} \subseteq \top_{\mathbf{wc10}}^{\mathcal{I}_{\mathbf{fb}}}$. Since $\mathbf{wc10} \prec \mathbf{fb}$, by condition 5 $\mathsf{ChampionPlayer}_{\mathbf{wc10}}^{\mathcal{I}_{\mathbf{fb}}} = \mathsf{ChampionPlayer}^{\mathcal{I}_{\mathbf{wc10}}}$. On the other hand, by condition 6, since $\mathbf{wc10} \not\prec \mathbf{nfl10}$ and $\mathbf{wc10} \not\prec \mathbf{nfl10}$ but $\mathbf{nfl10} \prec \mathbf{fb}$, the semantics only implies that $\mathsf{ChampionPlayer}_{\mathbf{wc10}}^{\mathcal{I}_{\mathbf{nfl10}}} = \mathsf{ChampionPlayer}_{\mathbf{wc10}}^{\mathcal{I}_{\mathbf{fb}}} \cap \Delta_{\mathbf{nfl10}}$. $\diamond$

Albeit useful for modeling, in fact partially qualified vectors are a kind of syntactic sugar in this framework (Serafini and Homola 2012). Hence to simplify the tableaux algorithm we may assume without loss of generality that the CKR on the input is always fully qualified (in a preprocessing step all qualified symbols are completed w.r.t. the context in which they appear). In the rest of the paper we will make this assumption. The only exception are examples where non-qualified symbols are used to improve readability.

## 3 Tableaux Algorithm for CKR

In this section we describe a tableaux algorithm $C_{\mathcal{T}}$, show its termination and that it is sound and complete w.r.t. the CKR semantics. This is done in two steps, first we consider CKR with TBox axioms only. In the second step we then extend the algorithm for full CKR knowledge bases with TBoxes and ABoxes as well. Before we do this we first formally define CKR tableaux and show their correspondence with CKR models.

### 3.1 Tableaux for CKR

Given any complex concept $C$, we denote by $\mathrm{clos}(C)$ the set of all syntactically correct atomic and complex concepts that occur in $C$. Observe that $C$ occurs in $\mathrm{clos}(C)$ and that the size of $\mathrm{clos}(C)$ is bounded by the the length of the string representation of $C$ (Baader et al. 2003).

The closure of a concept $C$ w.r.t. a CKR $\mathfrak{K}$, and the set of roles appearing in $C$ and $\mathfrak{K}$ are defined respectively as follows:

$$\begin{aligned}
\mathrm{clos}_{\mathfrak{K}}(C) &= \mathrm{clos}(C) \cup \{\mathrm{clos}(\neg D \sqcup E) \mid D \sqsubseteq E \in K\} \\
&\quad \cup \{\mathrm{clos}(D) \mid D(a) \in K\} \\
&\quad \cup \{\mathrm{clos}(\neg \top_{\mathbf{e}} \sqcup \top_{\mathbf{f}}) \mid \mathbf{e} \prec \mathbf{f}\} \\
\mathcal{R}_{\mathfrak{K},C} &= \{R \in N_{\mathrm{R}} \mid R \text{ occurs in } \mathfrak{K} \text{ or } C\}
\end{aligned}$$

These two sets contain all possible concepts and roles that are relevant in order to verify $\mathbf{d}$-satisfiability of $C$ w.r.t. $\mathfrak{K}$. The size of $\mathrm{clos}_{\mathfrak{K}}(C)$ is bounded by the total string-length of $\mathfrak{K}$ and $C$. We now proceed by introducing CKR tableaux.

**Definition 7** (CKR tableau)**.** *Given a CKR $\mathfrak{K}$ with no ABoxes, a concept $C$, and $\mathbf{d} \in \mathfrak{D}_\Gamma$, a $\mathbf{d}$-tableau for $C$ is a structure $\mathbf{T} = \{\mathbf{T}_{\mathbf{e}}\}_{\mathbf{e}\in\mathfrak{D}_\Gamma}$ such that for all $\mathbf{e} \in \mathfrak{D}_\Gamma$, $\mathbf{T}_{\mathbf{e}} = \langle \mathcal{S}_{\mathbf{e}}, \mathcal{E}_{\mathbf{e}}, \mathcal{L}_{\mathbf{e}} \rangle$ where:*

- *$\mathcal{S}_{\mathbf{e}}$ is a possibly empty set;*
- *$\mathcal{E}_{\mathbf{e}} : \mathcal{R}_{\mathfrak{K},C} \to 2^{\mathcal{S}_{\mathbf{e}} \times \mathcal{S}_{\mathbf{e}}}$ is a function mapping each role name in $\mathcal{R}_{\mathfrak{K},C}$ to a set of pairs of elements in $\mathcal{S}_{\mathbf{e}}$;*
- *$\mathcal{L}_{\mathbf{e}} : \mathcal{S}_{\mathbf{e}} \to 2^{\mathrm{clos}_{\mathfrak{K}}(C)}$ is a function mapping each element of $\mathcal{S}_{\mathbf{e}}$ to a subset of the concept closure $\mathrm{clos}_{\mathfrak{K}}(C)$;*

*in addition there is a node $s_0 \in \mathcal{S}_{\mathbf{d}}$ such that $C \in \mathcal{L}_{\mathbf{d}}(s_0)$; and the following conditions are satisfied for every $\mathbf{e}, \mathbf{f}, \mathbf{g} \in \mathfrak{D}_\Gamma$, $s, t \in \mathcal{S}_{\mathbf{e}}$, atomic concept $A \in \mathrm{clos}_{\mathfrak{K}}(C)$, possibly complex concepts $C_1, C_2 \in \mathrm{clos}_{\mathfrak{K}}(C)$, and $R \in \mathcal{R}_{\mathfrak{K},C}$:*

*(1) if $A \in \mathcal{L}_{\mathbf{e}}(s)$, then $\neg A \notin \mathcal{L}_{\mathbf{e}}(s)$;*

*(2) if $C_1 \sqcap C_2 \in \mathcal{L}_{\mathbf{e}}(s)$, then $C_1 \in \mathcal{L}_{\mathbf{e}}(s)$ and $C_2 \in \mathcal{L}_{\mathbf{e}}(s)$;*

*(3) if $C_1 \sqcup C_2 \in \mathcal{L}_{\mathbf{e}}(s)$, then $C_1 \in \mathcal{L}_{\mathbf{e}}(s)$ or $C_2 \in \mathcal{L}_{\mathbf{e}}(s)$;*

*(4) if $\exists R.C_1 \in \mathcal{L}_{\mathbf{e}}(s)$, then there is some $t \in \mathcal{S}_{\mathbf{e}}$ such that $\langle s,t\rangle \in \mathcal{E}_{\mathbf{e}}(R)$ and $C_1 \in \mathcal{L}_{\mathbf{e}}(t)$;*

*(5) if $\forall R.C_1 \in \mathcal{L}_{\mathbf{e}}(s)$ and $\langle s,t\rangle \in \mathcal{E}_{\mathbf{e}}(R)$, then $C_1 \in \mathcal{L}_{\mathbf{e}}(t)$;*

*(6) if $C_1 \sqsubseteq C_2 \in \mathrm{K}(\mathcal{C}_{\mathbf{e}})$, then $\mathrm{nnf}(\neg C_1 \sqcup C_2)$ in $\mathcal{L}_{\mathbf{e}}(s)$;*

*(7) if $\mathbf{e} \prec \mathbf{f}$, then $s \in \mathcal{S}_{\mathbf{f}}$;*

*(8) if $\mathbf{e} \succ \mathbf{f}$ and $\top_{\mathbf{f}} \in \mathcal{L}_{\mathbf{e}}(s)$, then $s \in \mathcal{S}_{\mathbf{f}}$;*

*(9) if $\mathbf{f} \prec \mathbf{g}$ then $(\neg\top_{\mathbf{f}} \sqcup \top_{\mathbf{g}}) \in \mathcal{L}_{\mathbf{e}}(s)$;*

*(10) if $A_{\mathbf{f}} \in \mathcal{L}_{\mathbf{e}}(s)$, then $\top_{\mathbf{f}} \in \mathcal{L}_{\mathbf{e}}(s)$;*

*(11) if $\langle s,t\rangle \in \mathcal{E}_{\mathbf{e}}(R_{\mathbf{f}})$, then $\top_{\mathbf{f}} \in \mathcal{L}_{\mathbf{e}}(s)$ and $\top_{\mathbf{f}} \in \mathcal{L}_{\mathbf{e}}(t)$;*

*(12) if $s \in \mathcal{S}_{\mathbf{e}} \cap \mathcal{S}_{\mathbf{f}}$ with $\mathbf{e} \prec \mathbf{f}$ or $\mathbf{f} \prec \mathbf{e}$ and $A_{\mathbf{g}} \in \mathcal{L}_{\mathbf{e}}(s)$, then $A_{\mathbf{g}} \in \mathcal{L}_{\mathbf{f}}(s)$;*

*(13) if $s,t \in \mathcal{S}_{\mathbf{e}} \cap \mathcal{S}_{\mathbf{f}}$ with $\mathbf{e} \prec \mathbf{f}$ or $\mathbf{f} \prec \mathbf{e}$ and $\langle s,t\rangle \in \mathcal{E}_{\mathbf{e}}(R_{\mathbf{g}})$, then $\langle s,t\rangle \in \mathcal{E}_{\mathbf{f}}(R_{\mathbf{g}})$.*

There is one-to-one correspondence between CKR tableaux and CKR models. Intuitively speaking, the elements of a local domain $\Delta_{\mathbf{e}}$ are represented by $\mathcal{S}_{\mathbf{e}}$ in the tableau. The set of nodes of $\mathcal{S}_{\mathbf{e}}$ which contain a concept $C$ in their $\mathcal{L}_{\mathbf{e}}$-labels represents the interpretation assigned to $C$ by $\mathcal{I}_{\mathbf{e}}$, and similarly $\mathcal{E}_{\mathbf{e}}(R)$ represents the interpretation assigned by $\mathcal{I}_{\mathbf{d}}$ to the role $R$.

**Lemma 1.** *Given a CKR $\mathfrak{K}$ with no ABoxes and some $\mathbf{d} \in \mathfrak{D}_\Gamma$, a concept $C$ is $\mathbf{d}$-satisfiable w.r.t. $\mathfrak{K}$ iff there exists a $\mathbf{d}$-tableau for $C$.*

While CKR models and tableaux are possibly cyclic, they can be unravelled into a tree. This is a well known result for $\mathcal{ALC}$ (Baader et al. 2003) and it also holds for CKR built on top of $\mathcal{ALC}$.

**Lemma 2** (Tree model property)**.** *Given a CKR $\mathfrak{K}$ in without ABoxes, a concept $C$ and $\mathbf{d} \in \mathfrak{D}_\Gamma$. If there is a CKR model $\mathfrak{I}$ of $\mathfrak{K}$ with $C^{\mathcal{I}_{\mathbf{d}}} \neq \emptyset$, then there also exists a CKR model $\mathfrak{I}'$ of $\mathfrak{K}$ such that $C^{\mathcal{I}'_{\mathbf{d}}} \neq \emptyset$ and $\mathfrak{I}'$ is tree-shaped.*

## 3.2 Reasoning with TBoxes

The tableaux algorithm for CKR in this section is partly based on the well known $\mathcal{ALC}$ tableaux algorithm (Schmidt-Schauß and Smolka 1991; Buchheit, Donini, and Schaerf 1993) which is extended in order to deal with multiple contexts. Classic $\mathcal{ALC}$ tableaux rules will be used inside each context, while additional tableaux rules will be introduced in order to deal with the cross-context semantics. The algorithm works on a structure called completion tree. It is a partial representation of a CKR tableau that the algorithm incrementally constructs.

**Definition 8** (Completion tree)**.** *Given a CKR $\mathfrak{K}$ without ABoxes, a completion tree is a triple $T = \langle V, E, \mathcal{L}\rangle$ s.t.:*

1. *$V$ is an ordered set of elements with order $<_V$;*
2. *$\langle V, E\rangle$ is a tree;*
3. *there is a collection $\{V_{\mathbf{d}}\}_{\mathbf{d}\in\mathfrak{D}_\Gamma}$ of sets such that $V_{\mathbf{d}} \subseteq V$;*
4. *$E_{\mathbf{d}} = \{\langle x,y\rangle \in E \mid x,y \in V_{\mathbf{d}}\}$, for each $\mathbf{d} \in \mathfrak{D}_\Gamma$;*
5. *$\mathcal{L} = \{\mathcal{L}_{\mathbf{d}}\}_{\mathbf{d}\in\mathfrak{D}_\Gamma}$ is a collection of labeling functions such that for each $\mathbf{d} \in \mathfrak{D}_\Gamma$: (a) $\mathcal{L}_{\mathbf{d}}(x) \subseteq \mathrm{clos}_{\mathfrak{K}}(C)$, for each $x \in V_{\mathbf{d}}$; (b) $\mathcal{L}_{\mathbf{d}}(\langle x,y\rangle) \subseteq \mathcal{R}_{\mathfrak{K},C}$, for each $\langle x,y\rangle \in E_{\mathbf{d}}$.*

In order to verify $\mathbf{d}$-satisfiability of a concept $C$ w.r.t. a CKR $\mathfrak{K}$, the algorithm initializes and then iteratively expands the tree by applying the tableaux expansion rules listed in Table 1.

To avoid infinite looping, a blocking policy is required. We adapt the blocking technique introduced for by Buchheit et al. (Buchheit, Donini, and Schaerf 1993). We assume, that the algorithms always add nodes into the completion tree respecting the order $<_V$ (i.e., whenever a new $x$ node is added, $y <_V x$ holds for all $y$ already in $V$).

**Definition 9** (Blocking)**.** *Given a CKR $\mathfrak{K}$, a completion tree $T = \langle V, E, \mathcal{L}\rangle$, we say that a node $w \in V$ is the witness for $x \in V$, if the following three conditions hold:*

1. *$\mathcal{L}_{\mathbf{d}}(x) = \mathcal{L}_{\mathbf{d}}(w)$ for all $\mathbf{d} \in \mathfrak{D}_\Gamma$;*
2. *$w <_V x$;*
3. *there is no $y \in V$ such that $y <_V w$ and $\mathcal{L}_{\mathbf{d}}(x) = \mathcal{L}_{\mathbf{d}}(y)$ for all $\mathbf{d} \in \mathfrak{D}_\Gamma$.*

*We say that $x \in V$ is blocked by $w \in V$ if $w$ is the witness for $x$.*

Note that this policy is *global*, that is, in order to determine if one node blocks another, all labels must be compared. Each of the tableaux rules listed in Table 1 has preconditions which determine if it can be applied on some particular node or not (the if-part of the rule). We say that a tableaux rule is applicable if all of its preconditions are satisfied for some node $x \in V$ or a pair of nodes $x, y \in V$. Applicability of rules helps us to realize when the completion is over. A completion tree $T$ is complete, if none of the tableaux expansion rules in Table 1 is applicable on any of its nodes.

Inconsistencies in the completion tree are called clashes. A completion tree $T = \langle V, E, L\rangle$ contains a clash in a node $x \in V$, if for some $\mathbf{d} \in \mathfrak{D}_\Gamma$ and for some concept $C$ both $C \in \mathcal{L}_{\mathbf{d}}(x)$ and $\neg C \in \mathcal{L}_{\mathbf{d}}(x)$, or if $\bot \in \mathcal{L}_{\mathbf{d}}(x)$. We say that $T$ is clash-free if no clash occurs in any of its nodes.

Formally the algorithm is defined as follows:

**Definition 10** (Algorithm $C_{\mathcal{T}}$)**.** *Given as input a CKR $\mathfrak{K}$ with no ABoxes, $\mathbf{d} \in \mathfrak{D}_\Gamma$, and a concept $C$ in NNF, the algorithm $C_{\mathcal{T}}$ verifies the $\mathbf{d}$-satisfiability of $C$ w.r.t. $\mathfrak{K}$ in the following steps:*

1. *initialize a new completion tree $T$ as follows:*
   *(a) $V_{\mathbf{e}} := \emptyset$, $E := \emptyset$, $L_{\mathbf{e}} := \emptyset$, for every $\mathbf{e} \in \mathfrak{D}_\Gamma$;*
   *(b) $V_{\mathbf{d}} := \{s_0\}$; $L_{\mathbf{d}}(s_0) := \{C\}$;*
2. *exhaustively apply the completion rules of Table 1 on $T$;*
3. *once $T$ is complete, answer "$C$ is $\mathbf{d}$-satisfiable w.r.t. $\mathfrak{K}$" if $T$ is clash-free, answer "$C$ is not $\mathbf{d}$-satisfiable w.r.t. $\mathfrak{K}$" otherwise.*

The first five rules used by the algorithm (from $\sqcap$- to $\mathcal{T}$-rule) are the usual $\mathcal{ALC}$ tableaux rules (Baader et al. 2003) that are responsible for expansions corresponding to local reasoning inside each context.

The $\Delta\!\uparrow$- and $\Delta\!\downarrow$-rules are responsible for propagation of nodes between the local parts of the completion tree (respective to each context) according to the semantics: if $\mathbf{d} \prec \mathbf{e}$, all nodes from $V_{\mathbf{d}}$ are propagated to $V_{\mathbf{e}}$ ($\Delta\!\uparrow$-rule), but only

| | |
|---|---|
| $\sqcap$-rule: | **if** $x \in V_{\mathbf{d}}$, $C_1 \sqcap C_2 \in \mathcal{L}_{\mathbf{d}}(x), \{C_1, C_2\} \not\subseteq \mathcal{L}_{\mathbf{d}}(x)$,<br>and $x$ is not blocked,<br>**then** $\mathcal{L}_{\mathbf{d}}(x) := \mathcal{L}_{\mathbf{d}}(x) \cup \{C_1, C_2\}$ |
| $\sqcup$-rule: | **if** $x \in V_{\mathbf{d}}$, $C_1 \sqcup C_2 \in \mathcal{L}_{\mathbf{d}}(x), \{C_1, C_2\} \cap \mathcal{L}_{\mathbf{d}}(x) = \emptyset$,<br>and $x$ is not blocked,<br>**then** $\mathcal{L}_{\mathbf{d}}(x) := \mathcal{L}_{\mathbf{d}}(x) \cup \{C_1\}$ or $\mathcal{L}_{\mathbf{d}}(x) := \mathcal{L}_{\mathbf{d}}(x) \cup \{C_2\}$ |
| $\exists$-rule: | **if** $x \in V_{\mathbf{d}}$, $\exists R.C \in \mathcal{L}_{\mathbf{d}}(x)$, $x$ is not blocked, and<br>there is no $R$-successor $y \in V_{\mathbf{d}}$ of $x$ s.t. $C \in \mathcal{L}_{\mathbf{d}}(y)$<br>**then** $V_{\mathbf{d}} := V_{\mathbf{d}} \cup \{z\}$, $E_{\mathbf{d}} := E_{\mathbf{d}} \cup \{\langle x, z\rangle\}$<br>$\mathcal{L}_{\mathbf{d}}(\langle x, z\rangle) := \{R\}$, $\mathcal{L}_{\mathbf{d}}(z) := \{C\}$ |
| $\forall$-rule: | **if** $x \in V_{\mathbf{d}}$, $\forall R.C \in \mathcal{L}_{\mathbf{d}}(x)$, $x$ is not blocked and<br>there exists $R$-successor $y \in V_{\mathbf{d}}$ of $x$ s.t. $C \notin \mathcal{L}_{\mathbf{d}}(y)$<br>**then** $\mathcal{L}_{\mathbf{d}}(y) := \mathcal{L}_{\mathbf{d}}(y) \cup \{C\}$ |
| $\mathcal{T}$-rule: | **if** $x \in V_{\mathbf{d}}$, $C \sqsubseteq D \in \mathrm{K}(\mathcal{C}_{\mathbf{d}}), \mathrm{nnf}(\neg C \sqcup D) \notin \mathcal{L}_{\mathbf{d}}(x)$,<br>and $x$ is not blocked,<br>**then** $\mathcal{L}_{\mathbf{d}}(x) := \mathcal{L}_{\mathbf{d}}(x) \cup \{\mathrm{nnf}(\neg C \sqcup D)\}$ |
| $\Delta\!\uparrow$-rule: | **if** $x \in V_{\mathbf{d}}, \mathbf{d} \prec \mathbf{e}, x \notin V_{\mathbf{e}}$, and $x$ is not blocked<br>**then** $V_{\mathbf{e}} := V_{\mathbf{e}} \cup \{x\}$ |
| $\Delta\!\downarrow$-rule: | **if** $x \in V_{\mathbf{e}}, \mathbf{d} \prec \mathbf{e}, \top_{\mathbf{d}} \in \mathcal{L}_{\mathbf{e}}(x), x \notin V_{\mathbf{d}}$,<br>and $x$ is not blocked<br>**then** $V_{\mathbf{d}} = V_{\mathbf{d}} \cup \{x\}$ |
| $A$-rule: | **if** $x \in V_{\mathbf{d}} \cap V_{\mathbf{e}}, \mathbf{d} \prec \mathbf{e}$ or $\mathbf{d} \succ \mathbf{e}, A_{\mathbf{f}} \in \mathcal{L}_{\mathbf{d}}(x)$,<br>$A_{\mathbf{f}} \notin \mathcal{L}_{\mathbf{e}}(x)$, and $x$ is not blocked<br>**then** $\mathcal{L}_{\mathbf{e}}(x) := \mathcal{L}_{\mathbf{e}}(x) \cup \{A_{\mathbf{f}}\}$ |
| $R$-rule: | **if** $x, y \in V_{\mathbf{d}} \cap V_{\mathbf{e}}, \langle x, y\rangle \in E, \mathbf{d} \prec \mathbf{e}$ or $\mathbf{d} \succ \mathbf{e}$,<br>$R_{\mathbf{f}} \in \mathcal{L}_{\mathbf{d}}(\langle x, y\rangle), R_{\mathbf{f}} \notin \mathcal{L}_{\mathbf{e}}(\langle x, y\rangle), x$ not blocked<br>**then** $\mathcal{L}_{\mathbf{e}}(\langle x, y\rangle) := \mathcal{L}_{\mathbf{e}}(\langle x, y\rangle) \cup \{R_{\mathbf{f}}\}$ |
| $\top_A$-rule: | **if** $x \in V_{\mathbf{e}}, A_{\mathbf{d}} \in \mathcal{L}_{\mathbf{e}}(x), \top_{\mathbf{d}} \notin \mathcal{L}_{\mathbf{e}}(x)$,<br>and $x$ is not blocked<br>**then** $\mathcal{L}_{\mathbf{e}}(x) := \mathcal{L}_{\mathbf{e}}(x) \cup \{\top_{\mathbf{d}}\}$ |
| $\top_R$-rule: | **if** $x, y \in V_{\mathbf{e}}, \langle x, y\rangle \in E, R_{\mathbf{d}} \in \mathcal{L}_{\mathbf{e}}(\langle x, y\rangle)$,<br>$\top_{\mathbf{d}} \notin \mathcal{L}_{\mathbf{e}}(x) \cap \mathcal{L}_{\mathbf{e}}(y)$, and $x$ is not blocked<br>**then** $\mathcal{L}_{\mathbf{e}}(x) := \mathcal{L}_{\mathbf{e}}(x) \cup \{\top_{\mathbf{d}}\}$, $\mathcal{L}_{\mathbf{e}}(y) := \mathcal{L}_{\mathbf{e}}(y) \cup \{\top_{\mathbf{d}}\}$ |
| $\top_{\sqsubseteq}$-rule: | **if** $x \in V_{\mathbf{d}}$, $\mathbf{e} \prec \mathbf{f}$, $\neg\top_{\mathbf{e}} \sqcup \top_{\mathbf{f}} \notin \mathcal{L}_{\mathbf{d}}(x)$,<br>and $x$ is not blocked,<br>**then** $\mathcal{L}_{\mathbf{d}}(x) := \mathcal{L}_{\mathbf{d}}(x) \cup \{\neg\top_{\mathbf{e}} \sqcup \top_{\mathbf{f}}\}$ |

Table 1: CKR completion rules

those nodes are propagated from $V_{\mathbf{e}}$ to $V_{\mathbf{d}}$ for which it was already derived that they belong to $\top_{\mathbf{d}}$ ($\Delta\!\downarrow$-rule).

About $A$-rule and $R$-rule: note that given contexts $\mathcal{C}_{\mathbf{d}}$ and $\mathcal{C}_{\mathbf{e}}$, with $\mathbf{d} \prec \mathbf{e}$, the conditions 6 and 7 of CKR-models require that the interpretations of any concept $A_{\mathbf{f}}$ (role $R_{\mathbf{f}}$) in these two contexts agree on all domain elements shared by their interpretation domains. Hence, if a node (pair of nodes) belongs to both $V_{\mathbf{d}}$ and $V_{\mathbf{e}}$, that is, it belongs to both local tableaux, then its concept (role) labels are propagated from one local tableaux to another, in both ways.

The last three tableaux rules serve to maintain the first three semantic conditions of CKR-models. According to conditions 2 and 3, interpretation of any qualified concept $A_{\mathbf{d}}$ (role $R_{\mathbf{d}}$) is always roofed under $\top_{\mathbf{d}}$ in any context $\mathcal{C}_{\mathbf{e}}$. The $\top_A$- and $\top_R$-rules take care of this: if a qualified concept $A_{\mathbf{d}}$ (role $R_{\mathbf{d}}$) is found in the $\mathcal{L}_{\mathbf{e}}$-label of some node (edge) in $V_{\mathbf{e}}$, then $\top_{\mathbf{d}}$ is added to the $\mathcal{L}_{\mathbf{e}}$-label of this node (both nodes connected by this edge). Finally, if $\mathbf{e} \prec \mathbf{f}$ then the subsumption $\top_{\mathbf{e}} \sqsubseteq \top_{\mathbf{f}}$ must hold in any context, hence the $\top_{\sqsubseteq}$-rule adds $\neg\top_{\mathbf{e}} \sqcup \top_{\mathbf{f}}$ to the label of every node similarly to the way how the $\mathcal{T}$-rule deals with GCI axioms.

**Example 5** (Tableaux algorithm)**.** Using the algorithm and our example CKR $\mathfrak{K}_{fb}$, let us show the proof for the following subsumption:

$$\mathfrak{K}_{fb} \models \mathbf{nfl10} : \mathsf{WorldChampionPlayer}_{\mathbf{fb}} \sqsubseteq \forall\mathsf{playsFor}_{\mathbf{wc10}}.\mathsf{WinnerTeam}_{\mathbf{wc10}}$$

The algorithm first initializes $V_{\mathbf{nfl10}} = \{s_0\}$ and $\mathcal{L}_{\mathbf{nfl10}}(s_0) = \{\mathsf{WorldChampionPlayer}_{\mathbf{fb}} \sqcap \exists\mathsf{playsFor}_{\mathbf{wc10}}.\neg\mathsf{WinnerTeam}_{\mathbf{wc10}}\}$. Then it applies the tableaux rules as follows:

(1) $\mathcal{L}_{\mathbf{nfl10}}(s_0) := \mathcal{L}_{\mathbf{nfl10}}(s_0) \cup \{\mathsf{WorldChampionPlayer}_{\mathbf{fb}}, \exists\mathsf{playsFor}_{\mathbf{wc10}}.\neg\mathsf{WinnerTeam}_{\mathbf{wc10}}\}$ by $\sqcap$-rule;

(2) $V_{\mathbf{nfl10}} := V_{\mathbf{nfl10}} \cup \{s_1\}$, $E_{\mathbf{nfl10}} = \{\langle s_0, s_1\rangle\}$, $\mathcal{L}_{\mathbf{nfl10}}(\langle s_0, s_1\rangle) = \{\mathsf{playsFor}_{\mathbf{wc10}}\}$, $\mathcal{L}_{\mathbf{nfl10}}(s_1) = \{\neg\mathsf{WinnerTeam}_{\mathbf{wc10}}\}$ by $\exists$-rule;

(3) $V_{\mathbf{fb}} = \{s_0, s_1\}$, $\mathcal{L}_{\mathbf{fb}}(s_0) = \{\mathsf{WorldChampionPlayer}\}$, $\mathcal{L}_{\mathbf{fb}}(\langle s_0, s_1\rangle) = \{\mathsf{playsFor}_{\mathbf{wc10}}\}$ by $\Delta\!\uparrow$-, $A$- and $R$-rules;

(4) $\mathcal{L}_{\mathbf{fb}}(s_0) \cup \{\mathsf{ChampionPlayer}_{\mathbf{wc10}}\}$ by $\mathcal{T}$- and $\sqcup$-rules;

(5) $\mathcal{L}_{\mathbf{fb}}(s_0) := \mathcal{L}_{\mathbf{fb}}(s_0) \cup \{\top_{\mathbf{wc10}}\}$, $\mathcal{L}_{\mathbf{fb}}(s_1) := \mathcal{L}_{\mathbf{fb}}(s_1) \cup \{\top_{\mathbf{wc10}}\}$ by $\top_R$-rule;

(6) $V_{\mathbf{wc10}} = \{s_0, s_1\}$, $\mathcal{L}_{\mathbf{wc10}}(s_0) = \{\mathsf{ChampionPlayer}\}$, $\mathcal{L}_{\mathbf{wc10}}(\langle s_0, s_1\rangle) = \{\mathsf{playsFor}\}$ by $\Delta\!\downarrow$-, $A$- and $R$-rules;

(7) $\mathcal{L}_{\mathbf{wc10}}(s_0) := \mathcal{L}_{\mathbf{wc10}}(s_0) \cup \{\forall\mathsf{playsFor}.\mathsf{WinnerTeam}\}$ by $\mathcal{T}$- and $\sqcup$-rules;

(8) $\mathcal{L}_{\mathbf{wc10}}(s_1) := \mathcal{L}_{\mathbf{wc10}}(s_1) \cup \{\mathsf{WinnerTeam}\}$ by $\forall$-rule;

(9) $\mathcal{L}_{\mathbf{fb}}(s_1) := \mathcal{L}_{\mathbf{fb}}(s_1) \cup \{\mathsf{WinnerTeam}_{\mathbf{wc10}}\}$, $\mathcal{L}_{\mathbf{nfl10}}(s_1) := \mathcal{L}_{\mathbf{nfl10}}(s_1) \cup \{\mathsf{WinnerTeam}_{\mathbf{wc10}}\}$ by $A$-rule;

The application of last rule creates a clash, since $\mathcal{L}_{\mathbf{nfl10}}(s_1) = \{\neg\mathsf{WinnerTeam}_{\mathbf{wc10}}, \mathsf{WinnerTeam}_{\mathbf{wc10}}\}$. Note that in the non-deterministic choices asked in steps 4 and 7 (due to $\sqcup$-rule), all other choices immediately lead to a clash. Hence no clash-free completion tree can be constructed and the algorithm answers that the input concept is $\mathbf{nfl10}$-unsatisfiable w.r.t. $\mathfrak{K}_{fb}$, which implies that the subsumption in question is entailed due to the reduction.

Note the inter-contextual knowledge propagation that was required to proof this subsumption and executed by the algorithm: we first had to propagate nodes and their labels from $V_{\mathbf{nfl10}}$ to $V_{\mathbf{fb}}$ and finally to $V_{\mathbf{wc10}}$ by tracking the context coverage structure (steps from 3 to 6). Then with the application of last rule (step 9), we propagate back the derived concepts to the label $\mathcal{L}_{\mathbf{nfl10}}$ and finally detect the clash. $\diamond$

The algorithm $C_{\mathcal{T}}$ is correct: it terminates on any input and it is sound and complete. That is, it always provides correct answers. The proof of the following statement is available in the Appendix.

**Theorem 1** (Correctness)**.** *Given a CKR $\mathfrak{K}$ without ABoxes, $\mathbf{d} \in \mathfrak{D}_\Gamma$, and a concept $C$ in NNF on the input, the tableaux algorithm $C_\mathcal{T}$ always terminates and it is sound and complete.*

### 3.3 Reasoning with TBoxes and ABoxes

The algorithm is easily extended to handle ABoxes by encoding the ABox data into the completion tree during the initialization step. This technique is well known for logics like $\mathcal{ALC}$ (Baader et al. 2003), however, since in CKR different individuals may possibly have different meanings, we must further extend this construction. In the completion tree, individuals will be represented by elements of the form $a^\mathbf{g}$ where $a$ stands for an individual $a \in N_\mathrm{I}$ and $\mathbf{g} \in \mathfrak{D}_\Gamma$ identifies the context in which the individual was first introduced. To implement condition 4 of CKR-models we will sometimes merge nodes when needed.

**Definition 11** (Merging)**.** *Given a completion tree $T = \langle V, E, \mathcal{L}\rangle$ and two nodes $x, y \in V$, executing* $\mathrm{merge}(x, y)$ *transforms $T$ as follows:*

1. *for all $\mathbf{e} \in \mathfrak{D}_\Gamma$ if $y \in V_\mathbf{e}$, $V_\mathbf{e} := V_\mathbf{e} \cup \{x\}$ and $\mathcal{L}_\mathbf{e}(x) := \mathcal{L}_\mathbf{e}(x) \cup \mathcal{L}_\mathbf{e}(y)$;*
2. *for all $z \in V$ s.t. $\langle y, z\rangle \in E$, $E := E \cup \{\langle x, z\rangle\}$, and for all $\mathbf{e} \in \mathfrak{D}_\Gamma$, $\mathcal{L}_\mathbf{e}(\langle x, z\rangle) := \mathcal{L}_\mathbf{e}(\langle x, z\rangle) \cup \mathcal{L}_\mathbf{e}(\langle y, z\rangle)$;*
3. *for all $z \in V$ s.t. $\langle z, y\rangle \in E$, $E := E \cup \{\langle z, x\rangle\}$, and for all $\mathbf{e} \in \mathfrak{D}_\Gamma$, $\mathcal{L}_\mathbf{e}(\langle z, x\rangle) := \mathcal{L}_\mathbf{e}(\langle z, x\rangle) \cup \mathcal{L}_\mathbf{e}(\langle z, y\rangle)$;*
4. *remove $y$ from $V_\mathbf{e}$ for all $\mathbf{e} \in \mathfrak{D}_\Gamma$ and remove its incoming and outgoing edges from $E$;*

The algorithm will be extended by a new tableaux rule called M-rule that will be responsible for node merging. This rule is listed in Table 2.

| | |
|---|---|
| M-rule: | **if** $a^\mathbf{g} \in V_\mathbf{d}$, $a^\mathbf{h} \in V_\mathbf{e}$, $\mathbf{d} \preceq \mathbf{e}$, |
| | **then** $\mathrm{merge}(a^\mathbf{g}, a^\mathbf{h})$ |

Table 2: Merging rule

Finally, the algorithm itself is extended in order to handle full CKR knowledge bases with TBoxes and ABoxes.

**Definition 12** ($C_\mathcal{T}$ extended with ABoxes)**.** *Given as input a CKR $\mathfrak{K}$, $\mathbf{d} \in \mathfrak{D}_\Gamma$, and a concept $C$ in NNF, the algorithm $C_\mathcal{T}$ verifies the $\mathbf{d}$-satisfiability of $C$ w.r.t. $\mathfrak{K}$ in the following steps:*

1. *for all $\mathbf{e} \in \mathfrak{D}_\Gamma$, initialize $V_\mathbf{e}$, $E$, and $\mathcal{L}_\mathbf{e}$ as follows:*
   - *(a)* $V_\mathbf{e} := \{a^\mathbf{e} \mid C(a) \in \mathrm{K}(\mathcal{C}_\mathbf{e})\} \cup \{a^\mathbf{e}, b^\mathbf{e} \mid R(a, b) \in \mathrm{K}(\mathcal{C}_\mathbf{e})\}$;
     $E := \{\langle a^\mathbf{e}, b^\mathbf{e}\rangle \mid R(a, b) \in \mathrm{K}(\mathcal{C}_\mathbf{e}), \mathbf{e} \in \mathfrak{D}_\Gamma\}$;
     $\mathcal{L}_\mathbf{e}(a^\mathbf{e}) := \{C \mid C(a) \in \mathrm{K}(\mathcal{C}_\mathbf{e})\}$;
     $\mathcal{L}_\mathbf{e}(\langle a^\mathbf{e}, b^\mathbf{e}\rangle) := \{R \mid R(a, b) \in \mathrm{K}(\mathcal{C}_\mathbf{e})\}$;
   - *(b)* $V_\mathbf{d} := V_\mathbf{d} \cup \{s_0\}$, *where $s_0$ is a new constant in $V_\mathbf{d}$;* $\mathcal{L}_\mathbf{d}(s_0) := \{C\}$;
2. *exhaustively apply all completion rules of Tables 1 and 2 on $T$;*
3. *once $T$ is complete, answer "$C$ is $\mathbf{d}$-satisfiable w.r.t. $\mathfrak{K}$" if $T$ is clash-free, answer "$C$ is not $\mathbf{d}$-satisfiable w.r.t. $\mathfrak{K}$" otherwise.*

There may be cyclic relations in the ABox, and hence $T$ may now possibly be a cyclic graph. For convenience we keep calling it completion tree. As much as in $\mathcal{ALC}$, also in CKR this is not a problem, because the part that encodes the ABoxes is finite. Also the extended version of the algorithm is correct.

**Theorem 2** (Correctness with ABoxes)**.** *Given a CKR $\mathfrak{K}$, $\mathbf{d} \in \mathfrak{D}_\Gamma$, and a concept $C$ in NNF on the input, the tableaux algorithm $C_\mathcal{T}$ always terminates and it is sound and complete.*

## 4 Complexity and Preliminary Optimization

Reasoning in $\mathcal{ALC}$ with TBoxes is known to be EXPTIME-complete (Schild 1994). However, the $\mathcal{ALC}$ algorithm which we extended for CKR in this paper is in NEXPTIME (Buchheit, Donini, and Schaerf 1993), and this is the case also for the resulting tableaux algorithm for CKR.

**Theorem 3** (Complexity)**.** *The complexity of the $C_\mathcal{T}$ algorithm is* NEXPTIME *with respect to the combined size of the input.*

To obtain an optimal algorithm for CKR on top of $\mathcal{ALC}$ we would have to extend one of the EXPTIME algorithms for $\mathcal{ALC}$; either the EXPTIME tableaux algorithm (Donini and Massacci 2000), or the approach based by translation into propositional dynamic logic (De Giacomo and Lenzerini 1996): we plan to do this in the future. On the other hand, the algorithm presented in this paper is an important first step towards the algorithmic support for CKR based on more expressive DL, starting from $\mathcal{SHIQ}$ (Horrocks, Sattler, and Tobies 1999) and up to $\mathcal{SROIQ}$ (Horrocks, Kutz, and Sattler 2006). This is because the tableaux algorithms used by these logics can be seen as extension of the basic $\mathcal{ALC}$ algorithm on top of which we have built in this paper.

We conclude this section with our preliminary thoughts on optimization of the algorithm. As contextualization of a knowledge base splits the whole data set into a number of contexts, it makes sense to try to execute the local reasoning in each of the contexts in parallel, using parallel processors or a distributed system. In this case the time complexity will be divided between the contexts and will be bounded by the context which requires the longest execution time. Of course, there is propagation of knowledge between the contexts which increases the time needed by each local context. Therefore it is desired to limit the propagation of knowledge between the contexts and not propagate what is not necessary. Using a technique similar to lazy unfolding, a well known optimization method for DL (Baader et al. 2003), we were able to optimize the three tableaux rules responsible for propagation of the $\top_\mathbf{e}$ symbols as follows:

$\top^*_A$-rule: **if** $x \in V_{\mathbf{f}}$, $\mathbf{d} \preceq \mathbf{e}$, $A_{\mathbf{d}} \in \mathcal{L}_{\mathbf{f}}(x), \top_{\mathbf{e}} \notin \mathcal{L}_{\mathbf{f}}(x)$,
and $x$ is not blocked,
**then** $\mathcal{L}_{\mathbf{f}}(x) := \mathcal{L}_{\mathbf{f}}(x) \cup \{\top_{\mathbf{e}}\}$

$\top^*_R$-rule: **if** $x, y \in V_{\mathbf{f}}$, $\mathbf{d} \preceq \mathbf{e}$, $R_{\mathbf{d}} \in \mathcal{L}_{\mathbf{f}}(\langle x, y\rangle)$,
$\top_{\mathbf{e}} \notin \mathcal{L}_{\mathbf{f}}(x) \cap \mathcal{L}_{\mathbf{f}}(y)$, and $x$ is not blocked,
**then** $\mathcal{L}_{\mathbf{f}}(x) := \mathcal{L}_{\mathbf{f}}(x) \cup \{\top_{\mathbf{e}}\}$, $\mathcal{L}_{\mathbf{f}}(y) := \mathcal{L}_{\mathbf{f}}(y) \cup \{\top_{\mathbf{e}}\}$

$\top^*_{\sqsubseteq}$-rule: **if** $x \in V_{\mathbf{f}}$, $\mathbf{d} \prec \mathbf{e}$, $\neg\top_{\mathbf{e}} \in \mathcal{L}_{\mathbf{f}}(x), \neg\top_{\mathbf{d}} \notin \mathcal{L}_{\mathbf{f}}(x)$,
and $x$ is not blocked,
**then** $\mathcal{L}_{\mathbf{f}}(x) := \mathcal{L}_{\mathbf{f}}(x) \cup \{\neg\top_{\mathbf{d}}\}$

The main idea of these optimized rules is to avoid the introduction of a number of disjunctive concept expressions of the form $\neg\top_{\mathbf{d}} \sqcup \top_{\mathbf{e}}$ caused by the $\top_{\sqsubseteq}$-rule which could possibly cause unnecessary non-deterministic branching. Instead, we apply each disjunction only after one of the disjuncts is proven untrue (and hence the other one must hold).

Normally the $\top_A$-rule would add $\top_{\mathbf{d}}$ into the label of any node $x$ in which $A_{\mathbf{d}}$ was found. Consequently, the $\top_{\sqsubseteq}$-rule would be fired once for each $\mathbf{e} \succ \mathbf{d}$ and add $\neg\top_{\mathbf{d}} \sqcup \top_{\mathbf{e}}$ every time. This eventually results into adding $\top_{\mathbf{e}}$ into the same label for each such $\mathbf{e}$ over the run of the algorithm. The optimized $\top^*_A$-rule skips the introduction of these disjunctions, and directly adds the $\top_{\mathbf{e}}$ symbol for all such $\mathbf{e}$.

The $\top_R$-rule is also optimized in the very same fashion. Hence the two optimized rules $\top^*_A$- and $\top^*_R$-rule do the work previously done by the $\top_A$- and $\top_R$-rules but in addition they take care of the first part of the disjunction $\neg\top_{\mathbf{d}} \sqcup \top_{\mathbf{e}}$ (i.e., the one which adds $\top_{\mathbf{e}}$ if $\top_{\mathbf{d}}$ was found). We still have to take care of the second part, and this is done by the $\top^*_{\sqsubseteq}$-rule which adds $\neg\top_{\mathbf{d}}$ to any label in which $\neg\top_{\mathbf{e}}$ was found for $\mathbf{d} \prec \mathbf{e}$.

In order to assure the correctness of the new rules with respect to CKR interpretations, we also have to adapt the definition of tableaux: the definition is modified by simply replacing the conditions (9)–(11) with the following points.

(9') if $A_{\mathbf{f}} \in \mathcal{L}_{\mathbf{e}}(s)$, $\mathbf{f} \preceq \mathbf{g}$ then $\top_{\mathbf{g}} \in \mathcal{L}_{\mathbf{e}}(s)$;

(10') if $\langle s, t\rangle \in \mathcal{E}_{\mathbf{e}}(R_{\mathbf{f}})$, $\mathbf{f} \preceq \mathbf{g}$ then $\top_{\mathbf{g}} \in \mathcal{L}_{\mathbf{e}}(s)$ and $\top_{\mathbf{g}} \in \mathcal{L}_{\mathbf{e}}(t)$;

(11') if $\neg\top_{\mathbf{g}} \in \mathcal{L}_{\mathbf{e}}(s)$, $\mathbf{f} \prec \mathbf{g}$ then $\neg\top_{\mathbf{f}} \in \mathcal{L}_{\mathbf{e}}(s)$;

The version of the algorithm $C_{\mathcal{T}}$ that uses the $\top^*_A$-, $\top^*_R$-, and $\top^*_{\sqsubseteq}$-rules instead of the $\top_A$-, $\top_R$-, and $\top_{\sqsubseteq}$-rules respectively, will be denoted by $C_{\mathcal{T}}{}^*$.

**Lemma 3** (Correctness of optimized rules)**.** *Given a CKR $\mathfrak{K}$, $\mathbf{d} \in \mathfrak{D}_\Gamma$, and a concept $C$ in NNF on the input, the algorithm $C_{\mathcal{T}}{}^*$ always terminates and it is sound and complete.*

Let us now compare the original tableaux rules with the optimized rules on the following example.

**Example 6** (Optimized tablueaux rules)**.** We exemplify the use of the new rules with the following deduction:

$$\mathfrak{K}_{fb} \models \mathbf{sp} : \mathsf{TopSportsman} \sqsubseteq \forall\mathsf{playsFor}_{\mathbf{wc10}}.\mathsf{WinnerTeam}_{\mathbf{wc10}}$$

The algorithm initializes the completion tree with $V_{\mathbf{sp}} = \{s_0\}$ and $\mathcal{L}_{\mathbf{sp}} = \{\mathsf{TopSportsman} \sqcap \exists\mathsf{playsFor}_{\mathbf{wc10}}.\neg\mathsf{WinnerTeam}_{\mathbf{wc10}}\}$. The original algorithm $C_{\mathcal{T}}$ proceeds as follows:

(1) $\mathcal{L}_{\mathbf{sp}}(s_0) := \mathcal{L}_{\mathbf{sp}}(s_0) \cup \{\mathsf{TopSportsman}, \exists\mathsf{playsFor}_{\mathbf{wc10}}.\neg\mathsf{WinnerTeam}_{\mathbf{wc10}}\}$ by $\sqcap$-rule;

(2) $V_{\mathbf{sp}} := V_{\mathbf{sp}} \cup \{s_1\}$, $E_{\mathbf{sp}} = \{\langle s_0, s_1\rangle\}$, $\mathcal{L}_{\mathbf{sp}}(\langle s_0, s_1\rangle) = \{\mathsf{playsFor}_{\mathbf{wc10}}\}$, $\mathcal{L}_{\mathbf{sp}}(s_1) = \{\neg\mathsf{WinnerTeam}_{\mathbf{wc10}}\}$ by $\exists$-rule;

(3) $\mathcal{L}_{\mathbf{sp}}(s_0) := \mathcal{L}_{\mathbf{sp}}(s_0) \cup \{\top_{\mathbf{wc10}}\}$, $\mathcal{L}_{\mathbf{sp}}(s_1) := \mathcal{L}_{\mathbf{sp}}(s_1) \cup \{\top_{\mathbf{wc10}}\}$ by $\top_R$-rule;

(4) $\mathcal{L}_{\mathbf{sp}}(s_0) := \mathcal{L}_{\mathbf{sp}}(s_0) \cup \{\neg\top_{\mathbf{wc10}} \sqcup \top_{\mathbf{fb}}, \neg\top_{\mathbf{nfl10}} \sqcup \top_{\mathbf{fb}}, \neg\top_{\mathbf{wc10}} \sqcup \top_{\mathbf{sp}}, \neg\top_{\mathbf{nfl10}} \sqcup \top_{\mathbf{sp}}, \neg\top_{\mathbf{fb}} \sqcup \top_{\mathbf{sp}}\}$, $\mathcal{L}_{\mathbf{sp}}(s_1) := \mathcal{L}_{\mathbf{sp}}(s_1) \cup \{\neg\top_{\mathbf{wc10}} \sqcup \top_{\mathbf{fb}}, \neg\top_{\mathbf{nfl10}} \sqcup \top_{\mathbf{fb}}, \neg\top_{\mathbf{wc10}} \sqcup \top_{\mathbf{sp}}, \neg\top_{\mathbf{nfl10}} \sqcup \top_{\mathbf{sp}}, \neg\top_{\mathbf{fb}} \sqcup \top_{\mathbf{sp}}\}$ by multiple applications of the $\top_{\sqsubseteq}$-rule;

(5) $\mathcal{L}_{\mathbf{sp}}(s_0) := \mathcal{L}_{\mathbf{sp}}(s_0) \cup \{\top_{\mathbf{fb}}, \top_{\mathbf{sp}}\}$, $\mathcal{L}_{\mathbf{sp}}(s_1) := \mathcal{L}_{\mathbf{sp}}(s_1) \cup \{\top_{\mathbf{fb}}, \top_{\mathbf{sp}}\}$ by $\sqcup$-rule;

(6) $V_{\mathbf{fb}} = \{s_0, s_1\}$, $\mathcal{L}_{\mathbf{fb}}(s_0) = \{\mathsf{TopSportsman}_{\mathbf{sp}}\}$ by $\Delta{\downarrow}$- and $A$-rules;

(7) $\mathcal{L}_{\mathbf{fb}}(s_0) := \mathcal{L}_{\mathbf{fb}}(s_0) \cup \{\mathsf{WorldChampionPlayer}\}$ by $\mathcal{T}$-rule[1] and $\sqcup$-rule;

(8) $\mathcal{L}_{\mathbf{fb}}(s_0) := \mathcal{L}_{\mathbf{fb}}(s_0) \cup \{\mathsf{ChampionPlayer}_{\mathbf{wc10}}\}$ by $\mathcal{T}$- and $\sqcup$-rules;

(9) $V_{\mathbf{wc10}} = \{s_0, s_1\}$, $\mathcal{L}_{\mathbf{wc10}}(s_0) = \{\mathsf{ChampionPlayer}\}$, $\mathcal{L}_{\mathbf{wc10}}(\langle s_0, s_1\rangle) = \{\mathsf{playsFor}\}$ by $\Delta{\downarrow}$- $A$ and $R$-rules;

(10) $\mathcal{L}_{\mathbf{wc10}}(s_0) := \mathcal{L}_{\mathbf{wc10}}(s_0) \cup \{\forall\mathsf{playsFor}.\mathsf{WinnerTeam}\}$ by $\mathcal{T}$- and $\sqcup$-rules;

(11) $\mathcal{L}_{\mathbf{wc10}}(s_1) := \mathcal{L}_{\mathbf{wc10}}(s_1) \cup \{\mathsf{WinnerTeam}\}$ by $\forall$-rule;

(12) $\mathcal{L}_{\mathbf{sp}}(s_1) := \mathcal{L}_{\mathbf{sp}}(s_1) \cup \{\mathsf{WinnerTeam}_{\mathbf{wc10}}\}$ by $A$-rule;

Similarly to the previous example, the last rule application causes a clash since we obtain $\mathcal{L}_{\mathbf{sp}}(s_1) = \{\neg\mathsf{WinnerTeam}_{\mathbf{wc10}}, \mathsf{WinnerTeam}_{\mathbf{wc10}}\}$. Notice that out of the ten applications of the $\top_{\sqsubseteq}$-rule in step (4), only the one resulting into adding $\neg\top_{\mathbf{wc10}} \sqcup \top_{\mathbf{fb}}$ into $\mathcal{L}_{\mathbf{fb}}(s_0)$ is actually needed propagation of the concept TopSportsman into $\mathcal{L}_{\mathbf{fb}}(s_0)$ eventually leading into recognition of a clash.

However, note that the application of $\top_{\sqsubseteq}$-rule to $s_1$ and the result $\neg\top_{\mathbf{nfl10}} \sqcup \top_{\mathbf{fb}}$ in the labels of both nodes do not contribute to the recognition of the clash: moreover, in particular in the second case, they require additional branching in the execution of the algorithm.

If instead the optimized algorithm $C_{\mathcal{T}}{}^*$ is used, a similar derivation is obtained in which steps (3)–(5) are replaced with:

(3') $\mathcal{L}_{\mathbf{sp}}(s_0) := \mathcal{L}_{\mathbf{sp}}(s_0) \cup \{\top_{\mathbf{wc10}}, \top_{\mathbf{fb}}, \top_{\mathbf{sp}}\}$, $\mathcal{L}_{\mathbf{sp}}(s_1) := \mathcal{L}_{\mathbf{sp}}(s_1) \cup \{\top_{\mathbf{wc10}}, \top_{\mathbf{fb}}, \top_{\mathbf{sp}}\}$ by $\top^*_R$-rule;

and the remainder of the derivation is just the same. ◇

Thus we see that the optimized rules constrain the propagation of $\top_{\mathbf{e}}$ concepts, which are needed to reflect the context hierarchy in the reasoning, to necessary propagations only, and we avoid the introduction of a number of disjunctive concepts which may cause unnecessary branching.

Observe that the derivations in Examples 5 and 6 are in fact very short because we chose to apply most of the rules

[1] Note that the two disjunctive concepts $\neg\mathsf{TopSportsman}_{\mathbf{sp}} \sqcup \mathsf{WorldChampionPlayer}$ and $\neg\mathsf{WorldChampionPlayer} \sqcup \mathsf{ChampionPlayer}_{\mathbf{wc10}}$ which are added to $\mathcal{L}_{\mathbf{fb}}$ in steps (7) and (8) respectively by the $\mathcal{T}$-rule are not listed here to improve readability.

in the right order. However, the algorithm does not know the correct order beforehand and hence additional rules may be applied before a clash is reached. For instance, due to the axiom $\mathsf{WinnerTeam} \equiv \mathsf{WinnerFinal}$ in $\mathrm{K}(\mathcal{C}_{\mathbf{wc10}})$ the algorithm may add $\mathsf{WinnerFinal}$ into $\mathcal{L}_{\mathbf{wc10}}(s_1)$ after step (11) in Example 6 (by $\mathcal{T}$- and $\sqcup$-rules) and consequently propagate $\mathsf{WinnerFinal}_{\mathbf{wc10}}$ into $\mathcal{L}_{\mathbf{fb}}(s_1)$ and $\mathcal{L}_{\mathbf{sp}}(s_1)$ by $A$-rule. Therefore in the future we would like to investigate also when it is really necessary to propagate qualified concept and role symbols (apart from $\top_{\mathbf{e}}$ symbols which we investigated above).

## 5 Related Works

The most directly related approaches to our work are the distributed tableaux algorithm for DDL and P-DL. Homola and Serafini (Homola and Serafini 2010) proposed an extension of the $\mathcal{ALC}$ algorithm that handles bridge rules. Bridge rules are a more general mechanism than qualified symbols used in CKR, however the effect of bridge rules on the tableau is similar: if a particular concept $C$ is found in the label of some node $x$ in the local tableau $T_i$, then due to some onto-bridge rule of the form $j : D \xrightarrow{\sqsupseteq} i : C$, we must add $D$ in the corresponding node of $T_j$; similarly for into-bridge rules of the form $j : D \xrightarrow{\sqsubseteq} i : C$. Hence the tableaux expansion rules that handle bridge rules are similar to our $A$-rule.

Bao et al. (Bao, Caragea, and Honavar 2006) proposed a tableaux algorithm for P-DL that handles semantic imports of concepts in $\mathcal{ALC}$. Instead of introducing new tableaux rules the algorithm is extended by messages passed between the local tableaux. The effect of these messages is again very similar to our $A$-rule – imported concept labels are propagated between the corresponding nodes of local tableaux.

On one hand, these two tableaux algorithms are fully distributed in the sense that each local tableau is handled independently by separate agents communicating by exchanging messages. The CKR algorithm works on a global tableau, which is however still broken into local tableaux which can be handled by parallel processors. On the other hand, we were able to develop a suitable blocking policy that permits arbitrary dependencies between qualified symbols, whereas the DDL and P-DL algorithms allow only acyclic bridge rules/imports. Also, the CKR algorithm handles qualified roles, while bridge rules between roles/imports between roles are not handled by the DDL and P-DL algorithms.

More importantly, the semantics of CKR is similar to that of semantic imports (and hence bridge rules), but these basic operations of knowledge reuse are employed in different specific ways in CKR with respect to the coverage relation of the contexts between which we are importing (Serafini and Homola 2012). Hence knowledge reuse in CKR is more complex than in P-DL or DDL thus a number of additional tableaux rules were needed.

Another work considering a parallelization of a DL reasoning procedure is presented in (Kazakov, Krötzsch, and Simancik 2011). Basically, this paper propose a saturation procedure for the classification of the polynomial fragment $\mathcal{ELH}_{\mathcal{R}+}$ of OWL 2 EL profile, with the distinguishing feature of being distributable among multiple processors in the form of a concurrent algorithm. In particular, the paper first formulates a "consequence-based" classification procedure for the $\mathcal{ELH}_{\mathcal{R}+}$ fragment and then describes a concurrent strategy for its application: notably, the concurrent strategy works by distributing input axioms into different "contexts" in which inferences can be carried out independently. The work also presents an implementation of such concurrent procedure in the Java based reasoner ELK together with a promising evaluation with respect to existing reasoners over known $\mathcal{EL}$ ontologies. Even if the scope of (Kazakov, Krötzsch, and Simancik 2011) is different from our work, it highlights some aspects that support our approach. In particular, it shows that there is interest in a parallelized and distributed vision of DL reasoning algorithms. Moreover, it suggests that the sort of knowledge distribution and independence between contexts which we point to can effectively result in promising performance improvements.

## 6 Conclusions

Contextualized Knowledge Repository (CKR) is a novel knowledge representation framework that provides a contextual layer for SW and LOD knowledge resources. CKR knowledge bases can be built on top of OWL 2 or any of its fragments. Knowledge can be sorted into multiple contexts with contextual meta data explicitly assigned. Thanks to CKR semantics rooted in the well studied principles of contextual knowledge representation, knowledge can be accessed and reused between contexts, depending on the relation between them.

The only practical reasoning algorithm for CKR that is currently known is a forward chaining algorithm built on top OWL Horst (Joseph and Serafini 2011). In this paper we have made the first step towards practical reasoning with CKR based on description logics – we introduced the first tableaux reasoning algorithm for CKR. This algorithm is for CKR built on top of $\mathcal{ALC}$. It is an extension of a well known algorithm for $\mathcal{ALC}$ (Buchheit, Donini, and Schaerf 1993). $\mathcal{ALC}$ tableaux rules are used inside contexts, and a number of new tableaux rules were introduced in order to deal with relations between contexts. We have shown soundness and completeness of the algorithm, and also that it belongs to the complexity class NEXPTIME as much as the original $\mathcal{ALC}$ algorithm that we extended.

As reasoning in $\mathcal{ALC}$ is known to be EXPTIME-complete it is an interesting open issue to verify this for $\mathcal{ALC}$-based CKR as well. In the future we would also like to extend this tableaux algorithm for more expressive DL such as $\mathcal{SHIQ}$ and eventually $\mathcal{SROIQ}$ in order to reach the full expressivity of OWL 2. Another interesting line of research is optimization where the fact that contexts are compatible but, to a certain extent, also independent, and hence part of the reasoning process can be parallelized, may be exploited.

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# A Appendix: Result Proofs

**Lemma 1.** *Given a CKR $\mathfrak{K}$ with no ABoxes and some $\mathbf{d} \in \mathfrak{D}_\Gamma$, a concept $C$ is $\mathbf{d}$-satisfiable w.r.t. $\mathfrak{K}$ iff there exists a $\mathbf{d}$-tableau for $C$.*

*Proof.* We prove the lemma by showing the two inclusions directions of the assertion: in other words, if there exists a CKR model $\mathfrak{I}$ for $\mathfrak{K}$ that supports the $\mathbf{d}$-satisfiability of the concept $C$, then we are able to build a $\mathbf{d}$-tableau $\mathbf{T}$ for $C$ (if direction); Vice versa (only-if direction), whenever there exists a $\mathbf{d}$-tableau $\mathbf{T}$ for $C$, then we can construct a CKR model $\overline{\mathfrak{I}}$ for $\mathfrak{K}$ such that $C^{\overline{\mathcal{I}}_{\mathbf{d}}} \neq \emptyset$.

**If direction.** As $C$ is $\mathbf{d}$-satisfiable, there is a CKR-interpretation $\mathfrak{I} = \{\mathcal{I}_{\mathbf{f}}\}_{\mathbf{f} \in \mathfrak{D}_\Gamma}$ such that $C^{\mathcal{I}_{\mathbf{d}}} \neq \emptyset$. Let us construct $\mathbf{T} = \{\mathbf{T}_{\mathbf{f}}\}_{\mathbf{f} \in \mathfrak{D}_\Gamma}$ where for each $\mathbf{f} \in \mathfrak{D}_\Gamma$, $\mathbf{T}_{\mathbf{f}} = \langle \mathcal{S}_{\mathbf{f}}, \mathcal{E}_{\mathbf{f}}, \mathcal{L}_{\mathbf{f}} \rangle$ is defined as follows:

- $\mathcal{S}_{\mathbf{f}} = \Delta_{\mathbf{f}}$;
- $\mathcal{E}_{\mathbf{f}}(R) = R^{\mathcal{I}_{\mathbf{f}}}$ for every $R \in \mathcal{R}_{\mathfrak{K},C}$;
- $\mathcal{L}_{\mathbf{f}}(t) = \{D \in \mathrm{clos}_{\mathfrak{K}}(C) \mid t \in D^{\mathcal{I}_{\mathbf{f}}}\}$ for every $t \in \mathcal{S}_{\mathbf{f}}$.

We will now show that $\mathbf{T}$ is in fact a $\mathbf{d}$-tableau for $C$. First of all, as $C^{\mathcal{I}_{\mathbf{d}}} \neq \emptyset$ then there must be some $s_0 \in C^{\mathcal{I}_{\mathbf{d}}}$ and directly from the construction we have $s_0 \in \mathcal{S}_{\mathbf{d}}$ and $C \in \mathcal{L}_{\mathbf{d}}(s_0)$.

We have now to show that the conditions asserted on CKR-tableaux are satisfied. In the following assume $\mathbf{e}, \mathbf{f}, \mathbf{g}$ are arbitrary dimensional vectors of $\mathfrak{D}_\Gamma$; $s, t$ are any two elements of $\mathcal{S}_{\mathbf{e}}$; $A, C_1, C_2$ are any concepts from $\mathrm{clos}_{\mathfrak{K}}(C)$ such that $A$ is atomic; and $R$ is arbitrary role of $\mathcal{R}_{\mathfrak{K},C}$. Let us prove the conditions step by step:

(1) suppose that $A \in \mathcal{L}_{\mathbf{e}}(s)$ from the construction this implies $s \in A^{\mathcal{I}_{\mathbf{e}}}$. If also $\neg A \in \mathcal{L}_{\mathbf{e}}(s)$, then by the construction we will get $s \in \neg A^{\mathcal{I}_{\mathbf{e}}}$ which is in contradiction with the fact that $\mathcal{I}_{\mathbf{e}}$ is DL-interpretation. Hence $\neg A \notin \mathcal{L}_{\mathbf{e}}(s)$;

(2) if $C_1 \sqcap C_2 \in \mathcal{L}_{\mathbf{e}}(s)$, then from the construction $s \in C_1 \sqcap C_2^{\mathcal{I}_{\mathbf{e}}}$ and hence both $s \in C_1^{\mathcal{I}_{\mathbf{e}}}$ and $s \in C_2^{\mathcal{I}_{\mathbf{e}}}$ as $\mathcal{I}_{\mathbf{e}}$ is a DL-interpretation. Now from the construction this implies $C_1 \in \mathcal{L}_{\mathbf{e}}(s)$ and $C_2 \in \mathcal{L}_{\mathbf{e}}(s)$;

(3) if $C_1 \sqcup C_2 \in \mathcal{L}_{\mathbf{e}}(s)$, then similarly to the previous case $s \in C_1 \sqcup C_2^{\mathcal{I}_{\mathbf{e}}}$ due to the construction, which implies that either $s \in C_1^{\mathcal{I}_{\mathbf{e}}}$ or $s \in C_2^{\mathcal{I}_{\mathbf{e}}}$, and then from the construction either $C_1 \in \mathcal{L}_{\mathbf{e}}(s)$ or $C_2 \in \mathcal{L}_{\mathbf{e}}(s)$;

(4) if $\exists R.C_1 \in \mathcal{L}_{\mathbf{e}}(s)$ then from the construction $s \in \exists R.C_1^{\mathcal{I}_{\mathbf{e}}}$. Then there must me some $t \in \Delta_{\mathbf{e}}$ such that $\langle s, t\rangle \in R^{\mathcal{I}_{\mathbf{e}}}$ and $t \in C_1^{\mathcal{I}_{\mathbf{e}}}$. Finally, from the construction we have $t \in \mathcal{S}_{\mathbf{e}}$, $\langle s, t\rangle \in \mathcal{E}_{\mathbf{e}}(R)$ and $C_1 \in \mathcal{L}_{\mathbf{e}}(t)$;

(5) let $\forall R.C_1 \in \mathcal{L}_{\mathbf{e}}(s)$ and let $\langle s, t\rangle \in \mathcal{E}_{\mathbf{e}}(R)$. From the construction $s \in \forall R.C_1^{\mathcal{I}_{\mathbf{e}}}$ and $\langle s, t\rangle \in R^{\mathcal{I}_{\mathbf{e}}}$. This however implies $t \in C_1^{\mathcal{I}_{\mathbf{e}}}$ which due to the construction implies $C_1 \in \mathcal{L}_{\mathbf{e}}(t)$;

(6) let $C_1 \sqsubseteq C_2 \in \mathrm{K}(\mathcal{C}_{\mathbf{e}})$. As $\mathfrak{I}$ is a model of $\mathfrak{K}$, $\mathcal{I}_{\mathbf{e}} \models C_1 \sqsubseteq C_2$. Since $C_1 \sqsubseteq C_2$ is equivalent to $\top \sqsubseteq \mathrm{nnf}(\neg C_1 \sqcup C_2)$, we have $s \in \mathrm{nnf}(\neg C_1 \sqcup C_2)^{\mathcal{I}_{\mathbf{e}}}$ because it must hold for every element of $\top^{\mathcal{I}_{\mathbf{e}}} = \Delta_{\mathbf{e}}$. From the construction this implies $\mathrm{nnf}(\neg C_1 \sqcup C_2) \in \mathcal{L}_{\mathbf{e}}(s)$;

(7) let $s \in \mathcal{S}_{\mathbf{e}}$ and $\mathbf{e} \prec \mathbf{f}$. By the construction we have $s \in \Delta_{\mathbf{e}}$. If $\mathbf{e} \prec \mathbf{f}$, then in every CKR-interpretation the following holds: $\Delta_{\mathbf{e}} = \top^{\mathcal{I}_{\mathbf{e}}} = \top_{\mathbf{e}}^{\mathcal{I}_{\mathbf{e}}} = \top_{\mathbf{e}}^{\mathcal{I}_{\mathbf{f}}} \subseteq \top_{\mathbf{f}}^{\mathcal{I}_{\mathbf{f}}} = \top^{\mathcal{I}_{\mathbf{f}}} = \Delta_{\mathbf{f}}$. Hence $s \in \Delta_{\mathbf{f}}$ and therefore due to the construction $s \in \mathcal{S}_{\mathbf{f}}$;

(8) let $s \in \mathcal{S}_{\mathbf{e}}$, $\mathbf{e} \succ \mathbf{f}$ and $\top_{\mathbf{f}} \in \mathcal{L}_{\mathbf{e}}(s)$. From the construction, we have $s \in \top_{\mathbf{f}}^{\mathcal{I}_{\mathbf{e}}}$. If $\mathbf{e} \succ \mathbf{f}$, then in every CKR-interpretation we have: $\top_{\mathbf{f}}^{\mathcal{I}_{\mathbf{e}}} = \top_{\mathbf{f}}^{\mathcal{I}_{\mathbf{f}}} = \top_{\mathbf{f}}^{\mathcal{I}_{\mathbf{f}}} = \top^{\mathcal{I}_{\mathbf{f}}} = \Delta_{\mathbf{f}}$. Hence $s \in \Delta_{\mathbf{f}}$ and due to the construction $s \in \mathcal{S}_{\mathbf{f}}$;

(9) let $\mathbf{f} \prec \mathbf{g}$. In this case, in every CKR-interpretation we have $\top_{\mathbf{f}}^{\mathcal{I}_{\mathbf{e}}} \subseteq \top_{\mathbf{g}}^{\mathcal{I}_{\mathbf{e}}}$ which in fact implies that $\mathcal{I}_{\mathbf{e}} \models \top_{\mathbf{f}} \sqsubseteq \top_{\mathbf{g}}$ and hence also $\mathcal{I}_{\mathbf{e}} \models \top \sqsubseteq (\neg\top_{\mathbf{f}} \sqcup \top_{\mathbf{g}})$. This implies $s \in (\neg\top_{\mathbf{f}} \sqcup \top_{\mathbf{g}})^{\mathcal{I}_{\mathbf{e}}}$ and consequently from the construction we have $(\neg\top_{\mathbf{f}} \sqcup \top_{\mathbf{g}}) \in \mathcal{L}_{\mathbf{e}}(s)$;

(10) if $A_{\mathbf{e}} \in \mathcal{L}_{\mathbf{f}}(s)$, then we have $s \in A_{\mathbf{e}}^{\mathcal{I}_{\mathbf{f}}}$ from the construction. As $\mathfrak{I}$ is a CKR-interpretation we know that $A_{\mathbf{e}}^{\mathcal{I}_{\mathbf{f}}} \subseteq \top_{\mathbf{e}}^{\mathcal{I}_{\mathbf{f}}}$ (Definition 6, condition 2). Then however $s \in \top_{\mathbf{e}}^{\mathcal{I}_{\mathbf{f}}}$ and from the construction $\top_{\mathbf{e}} \in \mathcal{L}_{\mathbf{f}}(s)$;

(11) if $\langle s, t\rangle \in \mathcal{E}_{\mathbf{f}}(R_{\mathbf{e}})$, then from the construction $\langle s, t\rangle \in R_{\mathbf{e}}^{\mathcal{I}_{\mathbf{f}}}$. From $\mathfrak{I}$ being a CKR-interpretation we know that $R_{\mathbf{e}}^{\mathcal{I}_{\mathbf{f}}} \subseteq \top_{\mathbf{e}}^{\mathcal{I}_{\mathbf{f}}} \times \top_{\mathbf{e}}^{\mathcal{I}_{\mathbf{f}}}$ (Definition 6, condition 3). Thus $s, t \in \top_{\mathbf{e}}^{\mathcal{I}_{\mathbf{f}}}$ and therefore $\top_{\mathbf{e}} \in \mathcal{L}_{\mathbf{f}}(s)$ and $\top_{\mathbf{e}} \in \mathcal{L}_{\mathbf{f}}(t)$;

(12) let $s \in \mathcal{S}_{\mathbf{e}} \cap \mathcal{S}_{\mathbf{f}}$ with $A_{\mathbf{g}} \in \mathcal{L}_{\mathbf{e}}(s)$ and let us first suppose that $\mathbf{e} \prec \mathbf{f}$. From the construction we have $s \in A_{\mathbf{g}}^{\mathcal{I}_{\mathbf{e}}}$. As $\mathfrak{I}$ is a CKR-interpretation, $A_{\mathbf{g}}^{\mathcal{I}_{\mathbf{e}}} = A_{\mathbf{g}}^{\mathcal{I}_{\mathbf{f}}} \cap \Delta_{\mathbf{e}}$ (Definition 6, condition 7). Therefore $s \in A_{\mathbf{g}}^{\mathcal{I}_{\mathbf{f}}} \cap \Delta_{\mathbf{e}}$, that is, surely $s \in A_{\mathbf{g}}^{\mathcal{I}_{\mathbf{f}}}$. Finally, due to the construction that $A_{\mathbf{g}} \in \mathcal{L}_{\mathbf{f}}(s)$.
Suppose now that $\mathbf{f} \prec \mathbf{e}$. As $A_{\mathbf{g}} \in \mathcal{L}_{\mathbf{e}}(s)$ and $s \in \mathcal{S}_{\mathbf{f}}$, from the construction we have $s \in A_{\mathbf{g}}^{\mathcal{I}_{\mathbf{e}}} \cap \Delta_{\mathbf{f}}$. Similarly to the previous case, from Definition 6, condition 7 we get $A_{\mathbf{g}}^{\mathcal{I}_{\mathbf{e}}} \cap \Delta_{\mathbf{f}} = A_{\mathbf{g}}^{\mathcal{I}_{\mathbf{f}}}$. Hence $s \in A_{\mathbf{g}}^{\mathcal{I}_{\mathbf{f}}}$ and from the construction $A_{\mathbf{g}} \in \mathcal{L}_{\mathbf{f}}(s)$;

(13) let $s, t \in \mathcal{S}_{\mathbf{e}} \cap \mathcal{S}_{\mathbf{f}}$ with $\langle s, t\rangle \in \mathcal{E}_{\mathbf{e}}(R_{\mathbf{g}})$. Let us first suppose that $\mathbf{e} \prec \mathbf{f}$. From the construction we have $\langle s, t\rangle \in R_{\mathbf{g}}^{\mathcal{I}_{\mathbf{e}}}$. As $\mathfrak{I}$ is a CKR-interpretation, it follows that $R_{\mathbf{g}}^{\mathcal{I}_{\mathbf{e}}} = R_{\mathbf{g}}^{\mathcal{I}_{\mathbf{f}}} \cap \Delta_{\mathbf{e}} \times \Delta_{\mathbf{e}}$ (Definition 6, condition 8). Therefore in fact $\langle s, t\rangle \in R_{\mathbf{g}}^{\mathcal{I}_{\mathbf{f}}}$ and then from the construction $\langle s, t\rangle \in \mathcal{E}_{\mathbf{d}}(R_{\mathbf{g}})$.
Suppose now that $\mathbf{f} \prec \mathbf{e}$. As $\langle s, t\rangle \in \mathcal{E}_{\mathbf{e}}(R_{\mathbf{g}})$ and $s, t \in \mathcal{S}_{\mathbf{f}}$, from the construction we have $\langle s, t\rangle \in A_{\mathbf{g}}^{\mathcal{I}_{\mathbf{e}}} \cap \Delta_{\mathbf{f}} \times \Delta_{\mathbf{f}}$. From Definition 6, condition 8 we have $R_{\mathbf{g}}^{\mathcal{I}_{\mathbf{e}}} \cap \Delta_{\mathbf{f}} \times \Delta_{\mathbf{f}} = R_{\mathbf{g}}^{\mathcal{I}_{\mathbf{f}}}$ and hence $\langle s, t\rangle \in R_{\mathbf{g}}^{\mathcal{I}_{\mathbf{f}}}$. Finally from the construction $\langle s, t\rangle \in \mathcal{E}_{\mathbf{d}}(R_{\mathbf{g}})$.

**Only-if direction.** Given $\mathfrak{K}$, $\mathbf{d}$, and $C$ as assumed by the theorem, let us assume that a $\mathbf{d}$-tableau $\mathbf{T} = \{\mathbf{T}_{\mathbf{d}}\}_{\mathbf{d} \in \mathfrak{D}_\Gamma}$ exists for $C$. Let us construct a CKR-interpretation $\overline{\mathfrak{I}} = \{\overline{\mathcal{I}}_{\mathbf{f}}\}_{\mathbf{f} \in \mathfrak{D}_\Gamma}$ with $\overline{\mathcal{I}}_{\mathbf{f}} = \left\langle \Delta_{\mathbf{f}}, \cdot^{\overline{\mathcal{I}}_{\mathbf{f}}} \right\rangle$ as follows:

- $\Delta_{\mathbf{f}} = \mathcal{S}_{\mathbf{f}}$;

– $A_{\mathbf{g}}^{\overline{\mathcal{I}}_{\mathbf{f}}} = \{t \in \mathcal{S}_{\mathbf{f}} \mid A_{\mathbf{g}} \in \mathcal{L}_{\mathbf{f}}(s)\}$ for every atomic concept $A_{\mathbf{g}} \in \mathrm{clos}_{\mathfrak{K}}(C)$;

– $R_{\mathbf{g}}^{\overline{\mathcal{I}}_{\mathbf{f}}} = \mathcal{E}_{\mathbf{f}}(R)$ for every role $R_{\mathbf{g}} \in \mathcal{R}_{\mathfrak{K},C}$;

where the interpretation of complex concepts is defined recursively as required by DL-interpretations. Let us first show the property (†):

> For any complex concept $D \in \mathrm{clos}_{\mathfrak{K}}(C)$, for any $\mathbf{f} \in \mathfrak{D}_\Gamma$, and for any $t \in \mathcal{S}_{\mathbf{f}}$: if $D \in \mathcal{L}_{\mathbf{f}}(t)$ then $t \in D^{\overline{\mathcal{I}}_{\mathbf{f}}}$.

We prove the proposition by induction on the structure of concepts. We distinguish between the following cases

**$D = A_{\mathbf{g}}$:** where $A_{\mathbf{g}}$ is an atomic concept. If $A_{\mathbf{g}} \in \mathcal{L}_{\mathbf{f}}(t)$ then from the construction $t \in A_{\mathbf{g}}^{\overline{\mathcal{I}}_{\mathbf{d}}}$;

**$D = \neg C_1$:** where $C_1 = A_{\mathbf{g}}$ is some atomic concept (no other case is possible as we assumed $\mathfrak{K}$ and $C$ in NNF). Suppose $D = \neg A_{\mathbf{g}} \in \mathcal{L}_{\mathbf{f}}(t)$. As $\mathbf{T}$ is a tableau we have $A_{\mathbf{g}} \notin \mathcal{L}_{\mathbf{f}}(t)$. From the construction this implies $t \notin A_{\mathbf{g}}^{\overline{\mathcal{I}}_{\mathbf{f}}}$. As $t \in \Delta_{\mathbf{f}}$ (from the construction) and $t \notin A_{\mathbf{g}}^{\overline{\mathcal{I}}_{\mathbf{f}}}$ then we have $t \in \Delta_{\mathbf{f}} \setminus A_{\mathbf{g}}^{\overline{\mathcal{I}}_{\mathbf{f}}} = \neg A_{\mathbf{g}}^{\overline{\mathcal{I}}_{\mathbf{f}}} = D^{\overline{\mathcal{I}}_{\mathbf{f}}}$;

**$D = C_1 \sqcap C_2$:** since $\mathbf{T}$ is a tableau, in this case both $C_1 \in \mathcal{L}_{\mathbf{f}}(t)$ and $C_2 \in \mathcal{L}_{\mathbf{f}}(t)$. From the induction hypothesis, $t \in C_1^{\overline{\mathcal{I}}_{\mathbf{f}}}$ and $t \in C_2^{\overline{\mathcal{I}}_{\mathbf{f}}}$ which in turn implies $t \in (C_1 \sqcap C_2)^{\overline{\mathcal{I}}_{\mathbf{f}}}$;

**$D = C_1 \sqcup C_2$:** as $\mathbf{T}$ is a tableau, either $C_1 \in \mathcal{L}_{\mathbf{f}}(t)$ or $C_2 \in \mathcal{L}_{\mathbf{f}}(t)$. By the induction hypothesis we have either $t \in C_1^{\overline{\mathcal{I}}_{\mathbf{f}}}$ or $t \in C_2^{\overline{\mathcal{I}}_{\mathbf{f}}}$ which in implies $t \in (C_1 \sqcup C_2)^{\overline{\mathcal{I}}_{\mathbf{f}}}$;

**$D = \exists R.C_1$:** due to $\mathbf{T}$ being a tableau there must be some $z \in \mathcal{S}_{\mathbf{f}}$ such that $\langle t, z\rangle \in \mathcal{E}_{\mathbf{f}}(R)$ and $C_1 \in \mathcal{L}_{\mathbf{f}}(z)$. From the construction we have that $\langle t, z\rangle \in R^{\overline{\mathcal{I}}_{\mathbf{f}}}$. Moreover, by the induction hypothesis $z \in C_1^{\overline{\mathcal{I}}_{\mathbf{f}}}$, which implies $t \in \exists R.C_1^{\overline{\mathcal{I}}_{\mathbf{f}}}$;

**$D = \forall R.C_1$:** in order to show that $t \in \forall R.C_1^{\overline{\mathcal{I}}_{\mathbf{f}}}$ we must show that for every $s \in \Delta_{\mathbf{f}}$ such that $\langle t, s\rangle \in R^{\overline{\mathcal{I}}_{\mathbf{f}}}$ we have $s \in C_1^{\overline{\mathcal{I}}_{\mathbf{f}}}$. Let $s$ be an arbitrary element of $\Delta_{\mathbf{f}}$ with $\langle t, s\rangle \in R^{\overline{\mathcal{I}}_{\mathbf{f}}}$. From the construction, $\langle t, s\rangle \in \mathcal{E}_{\mathbf{f}}$ and hence, since $\mathbf{T}$ is a tableau, $C_1 \in \mathcal{L}_{\mathbf{f}}(s)$. By the induction hypothesis this gives us $s \in C_1^{\overline{\mathcal{I}}_{\mathbf{f}}}$ and hence $t \in \forall R.C_1^{\overline{\mathcal{I}}_{\mathbf{f}}}$.

We will now show that $\overline{\mathfrak{I}}$ is indeed a model of $\mathfrak{K}$, i.e., that it satisfies the conditions 1–9 on the CKR-model definition:

1. we must prove that $\top_{\mathbf{e}}^{\overline{\mathcal{I}}_{\mathbf{g}}} \subseteq \top_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}}$ if $\mathbf{e} \prec \mathbf{f}$ for any $\mathbf{g} \in \mathfrak{D}_\Gamma$. Assume $x \in \top_{\mathbf{e}}^{\overline{\mathcal{I}}_{\mathbf{g}}}$. From the construction we have $x \in \mathcal{S}_{\mathbf{g}}$. As $\mathbf{e} \prec \mathbf{f}$, from condition (9) of CKR tableaux we have $(\neg\top_{\mathbf{e}} \sqcup \top_{\mathbf{f}}) \in \mathcal{L}_{\mathbf{f}}(x)$. Due to (†) this implies that $x \in (\neg\top_{\mathbf{e}} \sqcup \top_{\mathbf{f}})^{\overline{\mathcal{I}}_{\mathbf{g}}}$. Finally, as $x \in (\neg\top_{\mathbf{e}} \sqcup \top_{\mathbf{f}})^{\overline{\mathcal{I}}_{\mathbf{g}}}$ then either $x \in \neg\top_{\mathbf{e}}^{\overline{\mathcal{I}}_{\mathbf{g}}}$ or $x \in \top_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}}$, but since we have assumed $x \in \top_{\mathbf{e}}^{\overline{\mathcal{I}}_{\mathbf{g}}}$, it must be the case that $x \in \top_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}}$;
2. we must prove that $A_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}} \subseteq \top_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}}$ for any $\mathbf{g}, \mathbf{f} \in \mathfrak{D}_\Gamma$ and for any atomic concept $A_{\mathbf{f}}$. Let $x \in A_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}}$ by arbitrary. From the construction this implies $x \in \mathcal{S}_{\mathbf{g}}$ and $A_{\mathbf{f}} \in \mathcal{L}_{\mathbf{g}}(x)$. From condition (10) of CKR-tableau this implies $\top_{\mathbf{f}} \in \mathcal{L}_{\mathbf{g}}(x)$; and finally due to the construction $x \in \top_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}}$. Thus it holds that $A_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}} \subseteq \top_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}}$;
3. we must prove that $R_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}} \subseteq \top_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}} \times \top_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}}$ for any $\mathbf{g}, \mathbf{f} \in \mathfrak{D}_\Gamma$ and for any role $R_{\mathbf{f}}$. Let $\langle x, y\rangle \in R_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}}$ by arbitrary. From the construction this implies $x, y \in \mathcal{S}_{\mathbf{g}}$ and $R_{\mathbf{f}} \in \mathcal{L}_{\mathbf{g}}(\langle x, y\rangle)$. From condition (11) of CKR-tableau this implies $\top_{\mathbf{f}} \in \mathcal{L}_{\mathbf{g}}(x)$ and $\top_{\mathbf{f}} \in \mathcal{L}_{\mathbf{g}}(x)$; which implies $x, y \in \top_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}}$ due to the construction. And hence $R_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}} \subseteq \top_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}} \times \top_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}}$;
4. there are no individuals interpreted by $\overline{\mathfrak{I}}$ because we do not deal with ABoxes so far. Hence this point is trivially satisfied;
5. we have assumed that partially qualified symbols do not occur in $\mathfrak{K}$ or $C$, hence also this point is trivially satisfied;
6. we must prove that $X_{\mathbf{e}}^{\overline{\mathcal{I}}_{\mathbf{f}}} = X_{\mathbf{e}}^{\overline{\mathcal{I}}_{\mathbf{e}}}$ for any $\mathbf{e}, \mathbf{f} \in \mathfrak{D}_\Gamma$ with $\mathbf{e} \prec \mathbf{f}$. And for any atomic concept or role $X_{\mathbf{e}}$.

   Let $X_{\mathbf{e}} = A_{\mathbf{e}}$ be an atomic concept and let $x \in A_{\mathbf{e}}^{\overline{\mathcal{I}}_{\mathbf{e}}}$. From the construction $x \in \mathcal{S}_{\mathbf{e}}$, and as $\mathbf{e} \prec \mathbf{f}$ then from condition (7) of the $\mathbf{d}$-tableaux also $x \in \mathcal{S}_{\mathbf{f}}$, hence $x \in \mathcal{S}_{\mathbf{e}} \cap \mathcal{S}_{\mathbf{f}}$. This together with the fact that $A_{\mathbf{e}} \in \mathcal{L}_{\mathbf{e}}(x)$ (which follows from the construction) implies that $A_{\mathbf{e}} \in \mathcal{L}_{\mathbf{f}}(x)$ by condition (12) on $\mathbf{d}$-tableaux. Finally, from the construction we have $x \in A_{\mathbf{e}}^{\overline{\mathcal{I}}_{\mathbf{f}}}$. Vice versa, assume $x \in A_{\mathbf{e}}^{\overline{\mathcal{I}}_{\mathbf{f}}}$. From the construction $x \in \mathcal{S}_{\mathbf{f}}$ and $A_{\mathbf{e}} \in \mathcal{L}_{\mathbf{f}}(x)$. By condition (10) of $\mathbf{d}$-tableaux we get that $\top_{\mathbf{e}} \in \mathcal{L}_{\mathbf{f}}(x)$ and consequently by condition (8) we have $x \in \mathcal{S}_{\mathbf{e}}$. Hence in fact $x \in \mathcal{S}_{\mathbf{e}} \cap \mathcal{S}_{\mathbf{f}}$ and hence by condition (12) we derive $A_{\mathbf{e}} \in \mathcal{L}_{\mathbf{e}}(x)$ which by construction implies $x \in A_{\mathbf{e}}^{\overline{\mathcal{I}}_{\mathbf{e}}}$.

   Let $X_{\mathbf{e}} = R_{\mathbf{e}}$ be a role and let $\langle x, y\rangle \in R_{\mathbf{e}}^{\overline{\mathcal{I}}_{\mathbf{e}}}$. From the construction $x, y \in \mathcal{S}_{\mathbf{e}}$, and as $\mathbf{e} \prec \mathbf{f}$ then from condition (7) of the $\mathbf{d}$-tableaux $x, y \in \mathcal{S}_{\mathbf{e}} \cap \mathcal{S}_{\mathbf{f}}$. This together with the fact that $\langle x, y\rangle \in \mathcal{E}_{\mathbf{e}}(R_{\mathbf{e}})$ (which follows from the construction) implies that $\langle x, y\rangle \in \mathcal{E}_{\mathbf{f}}(R_{\mathbf{e}})$ by condition (13) of $\mathbf{d}$-tableaux. Finally, from the construction we have $\langle x, y\rangle \in R_{\mathbf{e}}^{\overline{\mathcal{I}}_{\mathbf{f}}}$. Vice versa, assuming $\langle x, y\rangle \in R_{\mathbf{e}}^{\overline{\mathcal{I}}_{\mathbf{f}}}$, we know from the construction that $x, y \in \mathcal{S}_{\mathbf{f}}$ and $\langle x, y\rangle \in \mathcal{E}_{\mathbf{f}}(R_{\mathbf{e}})$. By condition (11) of $\mathbf{d}$-tableaux we get both $\top_{\mathbf{e}} \in \mathcal{L}_{\mathbf{f}}(x)$ and $\top_{\mathbf{e}} \in \mathcal{L}_{\mathbf{f}}(y)$; and then by condition (8) we have $x, y \in \mathcal{S}_{\mathbf{e}}$. Hence $x, y \in \mathcal{S}_{\mathbf{e}} \cap \mathcal{S}_{\mathbf{f}}$ and hence by condition (13) we derive $\langle x, y\rangle \in \mathcal{E}_{\mathbf{e}}(R_{\mathbf{e}})$ which by construction implies $\langle x, y\rangle \in R_{\mathbf{e}}^{\overline{\mathcal{I}}_{\mathbf{e}}}$;
7. we must prove that $A_{\mathbf{g}}^{\overline{\mathcal{I}}_{\mathbf{e}}} = A_{\mathbf{g}}^{\overline{\mathcal{I}}_{\mathbf{f}}} \cap \Delta_{\mathbf{e}}$ for $\mathbf{e} \prec \mathbf{f}$. Let $x \in A_{\mathbf{g}}^{\overline{\mathcal{I}}_{\mathbf{e}}}$. From the construction we have $x \in \mathcal{S}_{\mathbf{e}}$ and $A_{\mathbf{g}} \in \mathcal{L}_{\mathbf{e}}(x)$. By condition (7) of $\mathbf{d}$-tableaux $x \in \mathcal{S}_{\mathbf{f}}$. Hence $x \in \mathcal{S}_{\mathbf{e}} \cap \mathcal{S}_{\mathbf{f}}$ and from condition (12) $A_{\mathbf{g}} \in \mathcal{L}_{\mathbf{f}}(x)$. Therefore due to the construction $x \in A_{\mathbf{g}}^{\overline{\mathcal{I}}_{\mathbf{f}}}$. But since

$x \in \mathcal{S}_{\mathbf{e}}$ and from the construction $\mathcal{S}_{\mathbf{e}} = \Delta_{\mathbf{e}}$ then also $x \in A_{\mathbf{g}}^{\overline{\mathcal{I}}_{\mathbf{f}}} \cap \Delta_{\mathbf{e}}$. On the other hand, suppose that $x \in A_{\mathbf{g}}^{\overline{\mathcal{I}}_{\mathbf{f}}} \cap \Delta_{\mathbf{e}}$. Directly from the construction this implies $x \in \mathcal{S}_{\mathbf{e}} \cap \mathcal{S}_{\mathbf{f}}$, $A_{\mathbf{g}} \in \mathcal{L}_{\mathbf{f}}(x)$. Consequently, by condition (12) we have that $A_{\mathbf{g}} \in \mathcal{L}_{\mathbf{e}}(x)$, and by the construction $x \in A_{\mathbf{g}}^{\overline{\mathcal{I}}_{\mathbf{e}}}$;

8. we have to prove that $(R_{\mathbf{g}})^{\overline{\mathcal{I}}_{\mathbf{e}}} = (R_{\mathbf{g}})^{\overline{\mathcal{I}}_{\mathbf{f}}} \cap (\Delta_{\mathbf{e}} \times \Delta_{\mathbf{e}})$ for $\mathbf{e} \prec \mathbf{f}$. The proof is similar to the previous case, but we have to use condition (13) of $\mathbf{d}$-tableaux instead of condition (12). Let $\langle x, y\rangle \in R_{\mathbf{g}}^{\overline{\mathcal{I}}_{\mathbf{e}}}$. From the construction we have $x, y \in \mathcal{S}_{\mathbf{e}}$ and $\langle x, y\rangle \in \mathcal{E}_{\mathbf{e}}(R_{\mathbf{g}})$. By condition (7) of $\mathbf{d}$-tableaux $x, y \in \mathcal{S}_{\mathbf{f}}$. Hence $x, y \in \mathcal{S}_{\mathbf{e}} \cap \mathcal{S}_{\mathbf{f}}$ and from condition (13) we get $\langle x, y\rangle \in \mathcal{E}_{\mathbf{f}}(R_{\mathbf{g}})$. Therefore due to the construction $\langle x, y\rangle \in R_{\mathbf{g}}^{\overline{\mathcal{I}}_{\mathbf{f}}}$. But since $x, y \in \mathcal{S}_{\mathbf{e}} = \Delta_{\mathbf{e}}$ then also $\langle x, y\rangle \in R_{\mathbf{g}}^{\overline{\mathcal{I}}_{\mathbf{f}}} \cap \Delta_{\mathbf{e}}$. On the other hand, suppose that $\langle x, y\rangle \in R_{\mathbf{g}}^{\overline{\mathcal{I}}_{\mathbf{f}}} \cap (\Delta_{\mathbf{e}} \times \Delta_{\mathbf{e}})$. Directly from the construction this implies $x, y \in \mathcal{S}_{\mathbf{e}} \cap \mathcal{S}_{\mathbf{f}}$, $\langle x, y\rangle \in \mathcal{E}_{\mathbf{f}}(R_{\mathbf{g}})$. Consequently, by condition (13) we have that $\langle x, y\rangle \in \mathcal{E}_{\mathbf{e}}(R_{\mathbf{g}})$, and by the construction $\langle x, y\rangle \in R_{\mathbf{g}}^{\overline{\mathcal{I}}_{\mathbf{e}}}$;
9. we have to prove that for every $\mathbf{f} \in \mathfrak{D}_\Gamma$ that every axiom of $\mathrm{K}(\mathcal{C}_{\mathbf{f}})$ is satisfied by $\overline{\mathcal{I}}_{\mathbf{f}} \models_{DL}$. Since we assumed that the ABox of $\mathcal{C}_{\mathbf{f}}$ is empty, the only form of axioms that possibly occur in $\mathrm{K}(\mathcal{C}_{\mathbf{f}})$ are the GCI axioms. Let $C_1 \sqsubseteq C_2$ be any GCI in $\mathrm{K}(\mathcal{C}_{\mathbf{f}})$. We must show that $C_1^{\overline{\mathcal{I}}_{\mathbf{f}}} \subseteq C_2^{\overline{\mathcal{I}}_{\mathbf{f}}}$. Let $x$ be any element of $C_1^{\overline{\mathcal{I}}_{\mathbf{f}}}$. From the construction we have $x \in \mathcal{S}_{\mathbf{f}}$. As $C_1 \sqsubseteq C_2 \in \mathrm{K}(\mathcal{C}_{\mathbf{f}})$, from condition (6) we have $\mathrm{nnf}(\neg C_1 \sqcup C_2) \in \mathcal{L}_{\mathbf{f}}(x)$. By (†) this implies $x \in \mathrm{nnf}(\neg C_1 \sqcup C_2)^{\overline{\mathcal{I}}_{\mathbf{f}}}$, and therefore $x \in (\neg C_1 \sqcup C_2)^{\overline{\mathcal{I}}_{\mathbf{f}}}$ by NNF. Then either $x \in \neg C_1^{\overline{\mathcal{I}}_{\mathbf{f}}}$ or $x \in C_2^{\overline{\mathcal{I}}_{\mathbf{f}}}$, but we have assumed $x \in C_1^{\overline{\mathcal{I}}_{\mathbf{f}}}$ therefore it must be the case that $x \in C_2^{\overline{\mathcal{I}}_{\mathbf{f}}}$.

Finally, since $\mathbf{T}$ is a $\mathbf{d}$-tableau for $C$ there is $s_0 \in \mathcal{S}_{\mathbf{d}}$ such that $C \in \mathcal{L}_{\mathbf{d}}(s_0)$. Hence by (†) $s_0 \in C^{\overline{\mathcal{I}}_{\mathbf{d}}}$, which implies that $C$ is $\mathbf{d}$-satisfiable w.r.t. $\mathfrak{K}$. □

**Lemma 2** (Tree model property)**.** *Given a CKR $\mathfrak{K}$ in without ABoxes, a concept $C$ and $\mathbf{d} \in \mathfrak{D}_\Gamma$. If there is a CKR model $\mathfrak{I}$ of $\mathfrak{K}$ with $C^{\mathcal{I}_{\mathbf{d}}} \neq \emptyset$, then there also exists a CKR model $\mathfrak{I}'$ of $\mathfrak{K}$ such that $C^{\mathcal{I}'_{\mathbf{d}}} \neq \emptyset$ and $\mathfrak{I}'$ is tree-shaped.*

*Proof.* First of all, a tree-shaped structure is usually defined as follows (Chagrov and Zakharyaschev 1997; Blackburn, Rijke, and Venema 2002): a tree is a relational structure $(T, S)$ where:

- the set of nodes $T$ contains a unique $r \in T$ (the root of the tree) such that for every $t \in T$, $r$ is an $S$-ancestor for every node in $T$.
- Every element $t \in T$ different from $r$ has a unique $S$-predecessor. That is, for every $t \neq r$, there exists a unique $t' \in T$ such that $S(t', t)$.
- $S$ is acyclic. That is, for every $t \in T$, there does not exists a $S$-path from $t$ to itself.

We say that a CKR-interpretation or tableaux is *tree-shaped* if it is a tree-shaped structure with respect to the $R$-successor relation. This definition allows us to prove the tree model property for $\mathcal{ALC}$-based CKR.

Since the concept $C$ is $\mathbf{d}$-satisfiable w.r.t. $\mathfrak{K}$ by the model $\mathfrak{I}$, we can then repeat the tableau construction that we proposed in Lemma 1 and obtain a $\mathbf{d}$-tableaux $\mathbf{T}$ for $\mathfrak{K}$ and $C$. We suppose that $s_0 \in \mathcal{S}_{\mathbf{d}}$ is the node in $\mathbf{T}$ such that it satisfies the condition $C \in \mathcal{L}_{\mathbf{d}}(s_0)$. Note that by the definition of $\mathcal{E}_{\mathbf{e}}$, $\mathbf{T}$ is not assured to have a tree form: however, we show that we can define a tree shaped tableau $\mathbf{T}^*$ for $\mathfrak{K}$ and $C$. $\mathbf{T}^*$ is defined by *unravelling* (Chagrov and Zakharyaschev 1997; Blackburn, Rijke, and Venema 2002) $\mathbf{T}$ around its root $s_0$ as follows: given $\mathbf{T} = \{\mathbf{T}_{\mathbf{e}}\}_{\mathbf{e} \in \mathfrak{D}_\Gamma}$ with $\mathbf{T}_{\mathbf{e}} = \langle \mathcal{S}_{\mathbf{e}}, \mathcal{E}_{\mathbf{e}}, \mathcal{L}_{\mathbf{e}}\rangle$ for $\mathbf{e} \in \mathfrak{D}_\Gamma$, then $\mathbf{T}^* = \{\mathbf{T}^*_{\mathbf{e}}\}_{\mathbf{e} \in \mathfrak{D}_\Gamma}$ with $\mathbf{T}^*_{\mathbf{e}} = \langle \mathcal{S}^*_{\mathbf{e}}, \mathcal{E}^*_{\mathbf{e}}, \mathcal{L}^*_{\mathbf{e}}\rangle$ where:

- $\mathcal{S}^*_{\mathbf{e}} = \{\langle x_0, \ldots, x_n\rangle \mid x_n \in \mathcal{S}_{\mathbf{e}},\ x_0 = s_0,\ (x_i, x_{i+1}) \in \mathcal{E}_{\mathbf{g}}(S_i)$ for $S_i \in \mathcal{R}_{\mathfrak{K},C},\ \mathbf{g} \in \mathfrak{D}_\Gamma$ and $i \in \{0, \ldots, n-1\}\}$;
- $\mathcal{L}^*_{\mathbf{e}}(\langle x_0, \ldots, x_n\rangle) = \mathcal{L}_{\mathbf{e}}(x_n)$ for every $\langle x_0, \ldots, x_n\rangle \in \mathcal{S}^*_{\mathbf{e}}$;
- $\mathcal{E}^*_{\mathbf{e}}(R) = \{\ (\langle x_0, \ldots, x_n\rangle, \langle y_0, \ldots, y_m\rangle) \in \mathcal{S}^*_{\mathbf{e}} \times \mathcal{S}^*_{\mathbf{e}} \mid m = n + 1,\ x_i = y_i,\ (x_i, x_{i+1}) \in \mathcal{E}_{\mathbf{g}}(S_i),$ $(x_n, y_m) \in \mathcal{E}_{\mathbf{e}}(R)$ for $S_i \in \mathcal{R}_{\mathfrak{K},C},\ \mathbf{g} \in \mathfrak{D}_\Gamma$ and $i \in \{0, \ldots, n\}\ \}$

As noted for the similar construction in (Blackburn, Rijke, and Venema 2002), it is easy to verify that the obtained construction is in fact a tree:

- $\mathbf{T}^*$ has a root in the node $\langle s_0\rangle$ around which the unravelling process takes place. This can also be verified by the definition of $\mathcal{S}^*_{\mathbf{e}}$, in which the condition $x_0 = s_0$ implies the existence of a path connecting each node to $\langle s_0\rangle$.
- Every element $\langle x_0, \ldots, x_n\rangle$ different from $\langle s_0\rangle$ has a unique predecessor: that is, there exists a single node $\langle y_0, \ldots, y_m\rangle$ such that $(\langle x_0, \ldots, x_n\rangle, \langle y_0, \ldots, y_m\rangle) \in \mathcal{E}^*_{\mathbf{e}}(R)$ for some $\mathbf{e} \in \mathfrak{D}_\Gamma$ and $R \in \mathcal{R}_{\mathfrak{K},C}$. By definition, assuming $\langle x_0, \ldots, x_n\rangle \in \mathcal{S}^*_{\mathbf{g}}$, there exists an $R \in \mathcal{R}_{\mathfrak{K},C}$ and $\mathbf{e} \in \mathfrak{D}_\Gamma$ such that $(x_{n-1}, x_n) \in \mathcal{E}_{\mathbf{e}}(R)$. By definition of $\mathcal{E}^*_{\mathbf{e}}(R)$, hence there exists a $R$-predecessor $\langle y_0, \ldots, y_m\rangle \in \mathcal{S}^*_{\mathbf{e}}$ with $m = n - 1$ such that $(\langle y_0, \ldots, y_m\rangle, \langle x_0, \ldots, x_n\rangle) \in \mathcal{E}^*_{\mathbf{e}}(R)$. By definition it also follows that it is unique, since it is required that for $i \in \{0, \ldots, n-1\}$ it holds that $x_i = y_i$ (and thus it is univocally determined).
- Edges in $\mathcal{E}^*_{\mathbf{e}}$ have no cycles, for every $\mathbf{e} \in \mathfrak{D}_\Gamma$. This can be simply verified by definition of $\mathcal{E}^*_{\mathbf{e}}$: for every $R \in \mathcal{R}_{\mathfrak{K},C}$ we have that $\mathcal{E}^*_{\mathbf{e}}(R)$ is composed by pairs $(\langle x_0, \ldots, x_n\rangle, \langle y_0, \ldots, y_m\rangle)$ in which $m > n$. In other words, roles are defined only on distinct node sequences from $\mathbf{T}$ with increasing length. By this fact, it is clear that there does not exist a path in $\mathbf{T}^*$ such that a node $\langle x_0, \ldots, x_n\rangle$ is reachable from itself.

We can easily show, by verifying the conditions on the $\mathbf{d}$-tableau definition, that also $\mathbf{T}^*$ is a $\mathbf{d}$-tableaux for $\mathfrak{K}$ and $C$. First of all, we note that there exists the node $\langle s_0\rangle \in \mathcal{S}^*_{\mathbf{d}}$ such that $C \in \mathcal{L}^*_{\mathbf{d}}(\langle s_0\rangle)$: this is directly implied by the fact that, by the definition of $\mathbf{T}$, there exists $s_0 \in \mathcal{S}_{\mathbf{d}}$ s.t.

$C \in \mathcal{L}_{\mathbf{d}}(s_0)$ and, by construction of $\mathbf{T}^*$, $\langle s_0\rangle \in \mathcal{S}^*_{\mathbf{d}}$ and $\mathcal{L}^*_{\mathbf{d}}(\langle s_0\rangle) = \mathcal{L}_{\mathbf{d}}(s_0)$.

To conclude the proof, we show that the 13 conditions on CKR-tableaux are preserved by the construction: similarly to the proof of Lemma 1, in the following we assume $\mathbf{e}, \mathbf{f}, \mathbf{g}$ to be any dimensional vectors of $\mathfrak{D}_\Gamma$, $\langle x_0, \ldots, x_n\rangle$ and $\langle y_0, \ldots, y_m\rangle$ any elements of $\mathcal{S}^*_{\mathbf{e}}$, $A \in \mathrm{clos}_{\mathfrak{K}}(C)$ any atomic concept, $C_1, C_2 \in \mathrm{clos}_{\mathfrak{K}}(C)$ any possibly complex concepts and $R \in \mathcal{R}_{\mathfrak{K},C}$ any role. We only show some of the relevant cases: the other cases mostly follow immediately from the definition of $\mathcal{L}^*_{\mathbf{e}}$.

(1). If $A \in \mathcal{L}^*_{\mathbf{e}}(\langle x_0, \ldots, x_n\rangle)$, by construction we have that $A \in \mathcal{L}_{\mathbf{e}}(x_n)$. Thus, by condition (1) on $\mathbf{T}$ definition, $\neg A \notin \mathcal{L}_{\mathbf{e}}(x_n)$. By the construction of $\mathcal{L}^*_{\mathbf{e}}$, this implies that $\neg A \notin \mathcal{L}^*_{\mathbf{e}}(\langle x_0, \ldots, x_n\rangle)$.

(4). If $\exists R.C_1 \in \mathcal{L}^*_{\mathbf{e}}(\langle x_0, \ldots, x_n\rangle)$, by construction we have that $\exists R.C_1 \in \mathcal{L}_{\mathbf{e}}(x_n)$. By condition (4) on $\mathbf{T}$ definition, there exists $y \in \mathcal{S}_{\mathbf{e}}$ such that $\langle x_n, y\rangle \in \mathcal{E}_{\mathbf{e}}(R)$ and $C_1 \in \mathcal{L}_{\mathbf{e}}(y)$. By the construction of $\mathcal{S}^*_{\mathbf{e}}$, it holds that $\langle x_0, \ldots, x_n, y\rangle \in \mathcal{S}^*_{\mathbf{e}}$. Moreover, by construction of $\mathcal{L}^*_{\mathbf{e}}$ and $\mathcal{E}^*_{\mathbf{e}}$, we have $(\langle x_0, \ldots, x_n\rangle, \langle x_0, \ldots, x_n, y\rangle) \in \mathcal{E}^*_{\mathbf{e}}(R)$ and $C_1 \in \mathcal{L}^*_{\mathbf{e}}(\langle x_0, \ldots, x_n, y\rangle) = \mathcal{L}_{\mathbf{e}}(y)$.

(7). Let $\mathbf{e} \prec \mathbf{f}$; since $\langle x_0, \ldots, x_n\rangle \in \mathcal{S}^*_{\mathbf{e}}$, it holds that $x_n \in \mathcal{S}_{\mathbf{e}}$. By condition (7) on $\mathbf{T}$ definition, we have that $x_n \in \mathcal{S}_{\mathbf{f}}$. By construction, we directly obtain that $\langle x_0, \ldots, x_n\rangle \in \mathcal{S}^*_{\mathbf{f}}$.

(8). Let $\mathbf{e} \succ \mathbf{f}$ and $\top_{\mathbf{f}} \in \mathcal{L}^*_{\mathbf{e}}(\langle x_0, \ldots, x_n\rangle)$; by construction of $\mathcal{L}^*_{\mathbf{e}}$, we have that $\top_{\mathbf{f}} \in \mathcal{L}_{\mathbf{e}}(x_n)$. By condition (8) on the definition of $\mathbf{T}$, $x_n \in \mathcal{S}_{\mathbf{f}}$. Hence, by construction, it holds that $\langle x_0, \ldots, x_n\rangle \in \mathcal{S}^*_{\mathbf{f}}$.

(11). If $(\langle x_0, \ldots, x_n\rangle, \langle y_0, \ldots, y_m\rangle) \in \mathcal{E}^*_{\mathbf{e}}(R_{\mathbf{f}})$, then, by construction of $\mathcal{S}^*_{\mathbf{e}}$, $x_n \in \mathcal{S}_{\mathbf{e}}$ and $y_m \in \mathcal{S}_{\mathbf{e}}$ with $(x_n, y_m) \in \mathcal{E}_{\mathbf{e}}(R_{\mathbf{f}})$. By condition (11) on $\mathbf{T}$, we have that $\top_{\mathbf{f}} \in \mathcal{L}_{\mathbf{e}}(x_n)$ and $\top_{\mathbf{f}} \in \mathcal{L}_{\mathbf{e}}(y_m)$. Thus, by construction of $\mathcal{L}^*_{\mathbf{e}}$, $\top_{\mathbf{f}} \in \mathcal{L}_{\mathbf{e}}(\langle x_0, \ldots, x_n\rangle)$ and $\top_{\mathbf{f}} \in \mathcal{L}_{\mathbf{e}}(\langle y_0, \ldots, y_m\rangle)$.

(13). Let $\langle x_0, \ldots, x_n\rangle, \langle y_0, \ldots, y_m\rangle \in \mathcal{S}^*_{\mathbf{e}} \cap \mathcal{S}^*_{\mathbf{f}}$. with $\mathbf{e} \prec \mathbf{f}$ or $\mathbf{f} \prec \mathbf{e}$ and $(\langle x_0, \ldots, x_n\rangle, \langle y_0, \ldots, y_m\rangle) \in \mathcal{E}^*_{\mathbf{e}}(R_{\mathbf{g}})$. Then we have that $(x_n, y_m) \in \mathcal{E}_{\mathbf{e}}(R_{\mathbf{g}})$ with $x_n, y_m \in \mathcal{S}_{\mathbf{e}} \cap \mathcal{S}_{\mathbf{f}}$. From the condition (13) of $\mathbf{T}$, it is implied that $(x_n, y_m) \in \mathcal{E}_{\mathbf{f}}(R_{\mathbf{g}})$. Hence, by construction, we have that $(\langle x_0, \ldots, x_n\rangle, \langle y_0, \ldots, y_m\rangle) \in \mathcal{E}^*_{\mathbf{f}}(R_{\mathbf{g}})$.

From the tree tableaux $\mathbf{T}^*$, by repeating the construction of Lemma 1 (only-if direction), we can then build a CKR-model $\mathfrak{I}'$ of $\mathfrak{K}$ where $C$ is $\mathbf{d}$-satisfiable. It is immediate to verify that the construction does not modify the (tree-shaped) structure of $\mathbf{T}^*$, since for every $\mathbf{f}, \mathbf{g} \in \mathfrak{D}_\Gamma$, we have $\Delta_{\mathbf{f}} = \mathcal{S}^*_{\mathbf{f}}$ and $R_{\mathbf{g}}^{\mathcal{I}'_{\mathbf{f}}} = \mathcal{E}^*_{\mathbf{f}}(R_{\mathbf{g}})$. The construction of $\mathfrak{I}'$ proves our assertion: we have shown that given a $\mathbf{d}$-satisfiable concept $C$ there always exists a tree-shaped CKR model in which $C$ is $\mathbf{d}$-satisfiable. □

**Theorem 1** (Correctness). *Given a CKR $\mathfrak{K}$ without ABoxes, $\mathbf{d} \in \mathfrak{D}_\Gamma$, and a concept $C$ in NNF on the input, the tableaux algorithm $C_{\mathcal{T}}$ always terminates and it is sound and complete.*

*Proof.* We show the result by first providing a proof for termination, then soundness and finally completeness.

**Termination.** We must show that the tableaux algorithm $C_{\mathcal{T}}$ terminates after a finite number of steps. Termination is implied by the blocking policy adopted by the algorithm. Intuitively, we will show that there is a finite bound on the size of the labels of each of the nodes. It follows that the generation of the completion tree cannot continue indefinitely because eventually the labels of some node must necessarily be subsets of the labels of some of the node's ancestors. The node then becomes blocked and the completion tree is not expanded further below. Therefore the computation eventually terminates.

Let us denote the number of context by $c = |\mathfrak{D}_\Gamma|$. This number is a constant and it is exponentially bounded by the number of dimensions $k$, that is, $c \leq 2^k$. Let us denote by $n$ the number of concepts and syntactic subconcepts that appear in the input, that is, $n = |\mathrm{clos}_{\mathfrak{K}}(C)|$.

Let $T = \langle V, E, \mathcal{L}\rangle$ be the completion tree constructed by the algorithm for input the concept $C$, $\mathfrak{K}$ and $\mathbf{d} \in \mathfrak{D}_\Gamma$. Observe that for any $D \in \mathcal{L}_{\mathbf{e}}(x)$, $x \in V_{\mathbf{e}}$, $\mathbf{e} \in \mathfrak{D}_\Gamma$, we have $D \in \mathrm{clos}_{\mathfrak{K}}(C)$ and therefore the size of $\mathcal{L}_{\mathbf{e}}(x)$ is bounded by $n$. Hence each node $x \in V$ has $c$ labels, each of which has at most $2^n$ possible values. There are therefore $2^{c\times n}$ possible unique combinations of all $c$ labels for each node. This however implies, that there is at most $2^{c\times n}$ nodes in $V$ which are not blocked, which is proved as follows: assume that there are $2^{c\times n} + 1$ such nodes, then there must be two nodes $x, y \in V$ which are not blocked and such that for all $\mathbf{e} \in \mathfrak{D}_\Gamma$ we have $\mathcal{L}_{\mathbf{e}}(x) = \mathcal{L}_{\mathbf{e}}(y)$. We have either $x <_V y$ or $y <_V x$, as the other case is symmetric, without loss of generality we may assume that $x <_V y$. In this case either $x$ is the witness for $y$, or there is some other node $w <_V x$ which is the witness for $y$, therefore $y$ is blocked, which contradicts the assumption.

We also observe that the $\exists$-rule is only applicable finitely many times on each node $x \in V$ as for each existential restriction $\exists R.E$ in each of the labels of $x$ it is only applicable once. Therefore each of the non-blocked nodes of $V$ has at most $c \times n$ successors, and the total number of nodes in $V$ is therefore smaller than $c \times n \times 2^{c\times n}$.

Finally, none of the rules can be applied infinitely many times on a finite tree. The $\Delta\!\uparrow$-rule and the $\Delta\!\downarrow$-rule add nodes into $V_{\mathbf{f}}$, for some $\mathbf{f} \in \mathfrak{D}_\Gamma$ during each application, hence for one $x \in V$ they can together only be applied $c$ times. All the remaining rules add concepts into node labels and are only applicable if the concept to be added is not already contained in the target label. Each node has $c$ labels and each label my contain at most $n$ concepts. Hence all these rules together can only be applied $c \times n$ times on each node.

Summing up, the $\exists$-rule may be applied $c \times n \times 2^{c\times n}$ times, resulting into $c \times n \times 2^{c\times n}$ nodes, the $\Delta\!\uparrow$-rule and the $\Delta\!\downarrow$-rule may be applied together at most $c^2 \times n \times 2^{c\times n}$ on these nodes, and the remaining rules may be applied together at most $c^2 \times n^2 \times 2^{c\times n}$ times. This results into at most $(c \times n + c^2 \times n + c^2 \times n^2) \times 2^{c\times n}$ rule application before the algorithm terminates, which is smaller than $O(2^{2\times c\times n})$.

**Soundness.** We have to show that tf the $C_\mathcal{T}$ algorithm generates a complete and clash-free completion tree for a given input CKR $\mathfrak{K}$ without ABoxes, $\mathbf{d} \in \mathfrak{D}_\Gamma$, and a concept $C$, then $C$ is $\mathbf{d}$-satisfiable w.r.t. $\mathfrak{K}$.

It suffices to show that there exists a $\mathbf{d}$-tableau for $C$ and the given $\mathfrak{K}$: the existence of the model with $C^{\mathcal{I}_\mathbf{d}} \neq \emptyset$ then follows from Lemma 1 and tableaux definition.

Let $T = \langle V, E, \mathcal{L}\rangle$ be a clash-free and complete completion tree generated by the $C_\mathcal{T}$ algorithm on the input $\mathfrak{K}$, $\mathbf{d}$, and $C$. Let us construct a CKR-tableau $\mathbf{T}' = \{\mathbf{T}'_\mathbf{e}\}_{\mathbf{e}\in\mathfrak{D}_\Gamma}$: for every $\mathbf{e} \in \mathfrak{D}_\Gamma$ let $\mathbf{T}'_\mathbf{e} = \langle \mathcal{S}'_\mathbf{e}, \mathcal{E}'_\mathbf{e}, \mathcal{L}'_\mathbf{e}\rangle$ be constructed as follows:

- $\mathcal{S}'_\mathbf{e} = \{x \mid x \in V_\mathbf{e}$ and $x$ is not blocked$\}$;
- $\mathcal{L}'_\mathbf{e}(x) = \mathcal{L}_\mathbf{e}(x)$ for every $x \in \mathcal{S}'_\mathbf{e}$;
- $\mathcal{E}'_\mathbf{e}(R) = \{\, (x, y) \in \mathcal{S}'_\mathbf{e} \times \mathcal{S}'_\mathbf{e} \mid R \in \mathcal{L}_\mathbf{e}(\langle x, y\rangle)$
  and $y$ is not blocked or there exists $z$ s.t.
  $R \in \mathcal{L}_\mathbf{e}(\langle x, z\rangle)$ and $z$ is blocked by $y\,\}$

We show that $\mathbf{T}'$ is indeed a $\mathbf{d}$-tableaux for $\mathfrak{K}$ and $C$. First we have to show that there is at least one node of $\mathcal{S}'_\mathbf{d}$ with $C$ in its label. From the definition of the tableaux algorithm (initialization) we know that surely $s_0 \in V_\mathbf{d}$ with $C \in \mathcal{L}_\mathbf{d}(s_0)$. Moreover, $s_0$ is not blocked as it is the root node of the tree $\langle V_\mathbf{d}, E_\mathbf{d}\rangle$ and only nodes with ancestors may possibly be blocked. Therefore from the construction of $\mathbf{T}'$, $s_0 \in \mathcal{S}'_\mathbf{d}$ and $C \in \mathcal{L}'_\mathbf{d}(s_0)$.

We now need to verify by cases that the 13 conditions placed on $\mathbf{d}$-tableaux by of Definition 7 are satisfied. Let $\mathbf{e}, \mathbf{f}$ be any two dimensional vectors of $\mathfrak{D}_\Gamma$ and let $s, t$ be any elements of $\mathcal{S}'_\mathbf{e}$, let $A$ be any atomic concept, $C_1$ and $C_2$ any possibly complex concepts and $R$ any role.

(1). if $A \in \mathcal{L}'_\mathbf{e}(s)$, then due to the construction $A \in \mathcal{L}_\mathbf{e}(s)$. Since $T$ is clash-free, we know that $\neg A \notin \mathcal{L}_\mathbf{e}(s)$ which implies $\neg A \notin \mathcal{L}'_\mathbf{e}(s)$;

(2). if $C_1 \sqcap C_2 \in \mathcal{L}'_\mathbf{e}(s)$, then again from the construction $C_1 \sqcap C_2 \in \mathcal{L}_\mathbf{e}(s)$ and since $T$ is complete this implies $C_1, C_2 \in \mathcal{L}_\mathbf{e}(s)$ (otherwise the $\sqcap$-rule is applicable on $s$). Hence due to the construction $C_1, C_2 \in \mathcal{L}'_\mathbf{e}(s)$;

(3). if $C_1 \sqcup C_2 \in \mathcal{L}'_\mathbf{e}(s)$, then from the construction $C_1 \sqcup C_2 \in \mathcal{L}_\mathbf{e}(s)$. Since $T$ is complete, the $\sqcup$-rule is not applicable on $s$ and hence either $C_1 \in \mathcal{L}_\mathbf{e}(s)$ or $C_2 \in \mathcal{L}_\mathbf{e}(s)$. From the construction this implies that either $C_1 \in \mathcal{L}'_\mathbf{e}(s)$ or $C_2 \in \mathcal{L}'_\mathbf{e}(s)$;

(4). if $\exists R.C_1 \in \mathcal{L}'_\mathbf{e}(s)$, then from the construction $\exists R.C_1 \in \mathcal{L}_\mathbf{e}(s)$. Since $T$ is complete, there is $t \in V_\mathbf{e}$ such that $R \in \mathcal{L}_\mathbf{e}(\langle s, t\rangle)$ and $C_1 \in \mathcal{L}_\mathbf{e}(t)$ (otherwise the $\exists$-rule is applicable on $s$). Now we have to distinguish between two cases.
If $t$ is not blocked, then directly from the construction $t \in \mathcal{S}'_\mathbf{e}$, $\langle s, t\rangle \in \mathcal{E}'_\mathbf{e}(R)$, and $C_1 \in \mathcal{L}'_\mathbf{e}(t)$ and hence the condition is verified.
If $t$ is blocked, then there is $t' \in V_\mathbf{e}$ which blocks $t$. Moreover, from the construction $t' \in \mathcal{S}'_\mathbf{e}$ (because it is not blocked), $\langle s, t'\rangle \in \mathcal{E}'_\mathbf{e}(R)$ (because it was added instead of $\langle s, t\rangle$), and $C_1 \in \mathcal{L}'_\mathbf{e}(t')$ (because $\mathcal{L}_\mathbf{e}(t) = \mathcal{L}_\mathbf{e}(t')$ as $t'$ blocks $t$). Hence also in this case the condition holds;

(5). if $\forall R.C_1 \in \mathcal{L}'_\mathbf{e}(s)$ and therefore from the construction also $\forall R.C_1 \in \mathcal{L}_\mathbf{e}(s)$ and for some $t \in \mathcal{S}'_\mathbf{e}$, we have $\langle s, t\rangle \in \mathcal{E}'_\mathbf{e}(R)$. Again, we have two possible cases:
if $\langle s, t\rangle \in E$, then due to the construction it must be the case that $R \in \mathcal{L}_\mathbf{e}(\langle s, t\rangle)$. It follows that $C_1 \in \mathcal{L}_\mathbf{e}(t)$ (otherwise the $\forall$-rule is applicable on $s$). From the construction then also $C_1 \in \mathcal{L}'_\mathbf{e}(t)$ which verifies the condition.
If $\langle s, t\rangle \notin E$, then from the construction $s$ must have some other $R$-successor $u \in V_\mathbf{e}$ which is blocked by $t$. Since $T$ is complete and we have $s, u \in V_\mathbf{e}$, $\forall R.C_1 \in \mathcal{L}_\mathbf{e}(s)$, and $R \in \mathcal{L}_\mathbf{e}(\langle s, u\rangle)$, then $C_1 \in \mathcal{L}_\mathbf{e}(u)$. As $t$ blocks $u$, we have $\mathcal{L}_\mathbf{e}(u) = \mathcal{L}_\mathbf{e}(t)$ and hence $C_1 \in \mathcal{L}_\mathbf{e}(t)$. Finally from the construction $C_1 \in \mathcal{L}'_\mathbf{e}(t)$ which again verifies the condition;

(6). if $C_1 \sqsubseteq C_2 \in \mathrm{K}(\mathcal{C}_\mathbf{e})$, then since $T$ is complete, then $\mathrm{nnf}(\neg C_1 \sqcup C_2) \in \mathcal{L}_\mathbf{e}(s)$ (otherwise $\mathcal{T}$-rule is applicable and hence $T$ is not complete). Hence from the construction $\mathrm{nnf}(\neg C_1 \sqcup C_2) \in \mathcal{L}'_\mathbf{e}(s)$;

(7). assume $\mathbf{e} \prec \mathbf{f}$, As $s \in \mathcal{S}'_\mathbf{e}$. From the construction $s \in V_\mathbf{e}$ and it is not blocked. Due to $\Delta\!\uparrow$-rule, we have $s \in V_\mathbf{f}$, because $T$ is complete and otherwise the rule would be applicable, and hence due to the construction $s \in \mathcal{S}'_\mathbf{f}$;

(8). assume $\mathbf{f} \prec \mathbf{e}$ and $\top_\mathbf{f} \in \mathcal{L}'_\mathbf{e}(s)$. From the construction we get: $s \in V_\mathbf{e}$, $\top_\mathbf{f} \in \mathcal{L}_\mathbf{e}(s)$ and $s$ not blocked. As $T$ is complete it must be the case that $s \in V_\mathbf{f}$ (otherwise $\Delta\!\downarrow$-rule is applicable). From the construction $s \in \mathcal{S}'_\mathbf{f}$;

(9). assume $\mathbf{f} \prec \mathbf{g}$. As $s \in \mathcal{S}'_\mathbf{e}$. From the construction $s \in V_\mathbf{e}$ and it is not blocked. Due to $\top_\sqsubseteq$-rule, we have $(\neg\top_\mathbf{f} \sqcup \top_\mathbf{g}) \in \mathcal{L}_\mathbf{e}(s)$, otherwise the rule would be applicable which is not the case as $T$ is complete. From the construction $(\neg\top_\mathbf{f} \sqcup \top_\mathbf{g}) \in \mathcal{L}'_\mathbf{e}(s)$;

(10). assume $A_\mathbf{f} \in \mathcal{L}'_\mathbf{e}(s)$. From the construction $s \in V_\mathbf{e}$ and it is not blocked. As $T$ is complete we have $\top_\mathbf{f} \in \mathcal{L}_\mathbf{e}(s)$, otherwise $\top_A$-rule is applicable. From the construction $\top_\mathbf{f} \in \mathcal{L}'_\mathbf{e}(s)$;

(11). assume $\langle s, t\rangle \in \mathcal{E}'_\mathbf{e}(R_\mathbf{f})$. As $s, t \in \mathcal{S}'_\mathbf{e}$, from the construction we have $s, t \in V_\mathbf{e}$, neither of them blocked. We need to distinguish two cases.
If $\langle s, t\rangle \in E$, then from the construction we have $R_\mathbf{f} \in \mathcal{L}_\mathbf{e}(\langle s, t\rangle)$. This implies $\top_\mathbf{f} \in \mathcal{L}_\mathbf{e}(s)$ and $\top_\mathbf{f} \in \mathcal{L}_\mathbf{e}(t)$, as $T$ is complete and otherwise the $\top_R$-rule would be applicable. Finally $\top_\mathbf{f} \in \mathcal{L}'_\mathbf{e}(s)$ and $\top_\mathbf{f} \in \mathcal{L}'_\mathbf{e}(t)$ follows from the construction.
If $\langle s, t\rangle \notin E$, then due to the construction there must exist $u \in V_\mathbf{e}$ such that $\langle s, u\rangle \in E$ and $u$ is blocked by $t$. In this case $R_\mathbf{f} \in \mathcal{L}_\mathbf{e}(\langle s, u\rangle)$ and hence $\top_\mathbf{f} \in \mathcal{L}_\mathbf{e}(s)$ and $\top_\mathbf{f} \in \mathcal{L}_\mathbf{e}(u)$ (otherwise $\top_R$-rule is applicable). Finally, as $t$ blocks $u$, we have $\mathcal{L}_\mathbf{e}(u) = \mathcal{L}_\mathbf{e}(t)$ and hence $\top_\mathbf{f} \in \mathcal{L}_\mathbf{e}(t)$ as well. Now for the construction $\top_\mathbf{f} \in \mathcal{L}_\mathbf{e}(s)$ and $\top_\mathbf{f} \in \mathcal{L}_\mathbf{e}(t)$ implies $\top_\mathbf{f} \in \mathcal{L}'_\mathbf{e}(s)$ and $\top_\mathbf{f} \in \mathcal{L}'_\mathbf{e}(t)$;

(12). suppose that $s \in \mathcal{S}'_\mathbf{e} \cap \mathcal{S}'_\mathbf{f}$, $A_\mathbf{g} \in \mathcal{L}'_\mathbf{e}(s)$ and either $\mathbf{e} \prec \mathbf{f}$ or $\mathbf{f} \prec \mathbf{e}$. Directly from the construction $s \in V_\mathbf{e} \cap V_\mathbf{f}$ and $A_\mathbf{g} \in \mathcal{L}_\mathbf{e}(s)$. From the completeness $T$ and subsequent non applicability of the $A$-rule on $s$ it follows that $A_\mathbf{g} \in \mathcal{L}_\mathbf{f}(s)$. Then from the construction $A_\mathbf{g} \in \mathcal{L}'_\mathbf{f}(s)$;

(13). suppose that $s, t \in \mathcal{S}'_{\mathbf{e}} \cap \mathcal{S}'_{\mathbf{f}}$, $\langle s, t\rangle \in \mathcal{E}'_{\mathbf{e}}(R_{\mathbf{g}})$ and either $\mathbf{e} \prec \mathbf{f}$ or $\mathbf{f} \prec \mathbf{e}$. Again we get straight from the construction that $s, t \in V_{\mathbf{e}} \cap V_{\mathbf{f}}$, neither node blocked. Similarly to the case (11), we must distinguish two cases.
If $\langle s, t\rangle \in E$, then from the construction $R_{\mathbf{g}} \in \mathcal{L}_{\mathbf{e}}(\langle s, t\rangle)$. From the completeness of $T$ we get $R_{\mathbf{g}} \in \mathcal{L}_{\mathbf{f}}(\langle s, t\rangle)$ as otherwise the $R$-rule would be applicable. Since $t$ is not blocked, the construction implies that $\langle s, t\rangle \in \mathcal{E}'_{\mathbf{f}}(R_{\mathbf{g}})$.
If $\langle s, t\rangle \notin E$, then there must be $u \in V_{\mathbf{e}}$ which is blocked by $t$ such that $R_{\mathbf{g}} \in \mathcal{L}_{\mathbf{e}}(\langle s, u\rangle)$. From the completeness of $T$ we get $R_{\mathbf{g}} \in \mathcal{L}_{\mathbf{f}}(\langle s, u\rangle)$ as otherwise the $R$-rule would be applicable. Finally, since $u$ is blocked by $t$ then the construction implies $\langle s, t\rangle \in \mathcal{E}'_{\mathbf{f}}(R_{\mathbf{g}})$.

**Completeness.** Given a concept $C$ that is $\mathbf{d}$-satisfiable w.r.t. a CKR $\mathfrak{K}$ without ABoxes, we have to show that if the $C_{\mathcal{T}}$ algorithm constructs a complete and clash free completion tree $T$ on the input $\mathfrak{K}$, $\mathbf{d}$ and $C$.

Since $C$ is $\mathbf{d}$-satisfiable w.r.t. $\mathfrak{K}$, a model $\mathfrak{I}$ of $\mathfrak{K}$ with $C^{\mathcal{I}_{\mathbf{d}}} \neq \emptyset$ must exist. The proof is by bi-simulation: we will simulate the run of the algorithm on the given input and inductively construct a mapping $\pi : V \rightarrow \bigcup_{\mathbf{e} \in \mathfrak{D}_\Gamma} \Delta_{\mathbf{e}}$ for which we will show the following property $(*)$:

> For each node $x \in V$ and for each $\mathbf{e} \in \mathfrak{D}_\Gamma$: (a) if $x \in V_{\mathbf{e}}$ then $\pi(x) \in \Delta_{\mathbf{e}}$; (b) if $C \in \mathcal{L}_{\mathbf{e}}(x)$ then $\pi(x) \in C^{\mathcal{I}_{\mathbf{e}}}$; (c) if $R \in \mathcal{L}_{\mathbf{e}}(\langle x, y\rangle)$ then $\langle \pi(x), \pi(y)\rangle \in R^{\mathcal{I}_{\mathbf{e}}}$.

First step of the algorithm is the initialization. During this step a node $s_0$ is created with $\mathcal{L}_{\mathbf{d}}(s_0) = \{C\}$. Since $C^{\mathcal{I}_{\mathbf{d}}} \neq \emptyset$, we know that there is at least one node $x \in C^{\mathcal{I}_{\mathbf{d}}}$. Let us set $\pi(s_0) = x$. Obviously $(*)$ is satisfied for $x$.

We will inductively proceed with the proof based on the sequence in which the tableaux expansion rules are applied. We distinguish multiple cases based on which rule was applied:

- **$\sqcap$-rule:** if this rule was applied on some node $x \in V_{\mathbf{e}}$ and some complex concept of the form $C_1 \sqcap C_2$, then it was the case that $C_1 \sqcap C_2 \in \mathcal{L}_{\mathbf{e}}(x)$ prior to the application. From the induction hypothesis $(*)$ was satisfied before the rule was applied. Hence $\pi(x) \in (C_1 \sqcap C_2)^{\mathcal{I}_{\mathbf{e}}}$. As $\mathfrak{I}$ is a model of $\mathfrak{K}$, in this case both $\pi(x) \in C_1^{\mathcal{I}_{\mathbf{e}}}$ and $\pi(x) \in C_2^{\mathcal{I}_{\mathbf{e}}}$ must hold. Therefore the property $(*)$ still holds after $C_1$ and $C_2$ have been added to $\mathcal{L}_{\mathbf{e}}(x)$. Since the rule did not do any other change on $T$, $(*)$ is satisfied after the rule is applied;
- **$\sqcup$-rule:** before the rule was applied we had $x \in V_{\mathbf{e}}$ and $C_1 \sqcup C_2 \in \mathcal{L}_{\mathbf{e}}(x)$ for some complex concept of the form $C_1 \sqcup C_2$. From the induction hypothesis $(*)$ was satisfied and hence $\pi(x) \in (C_1 \sqcup C_2)^{\mathcal{I}_{\mathbf{e}}}$. As $\mathfrak{I} \models \mathfrak{K}$, we either have $\pi(x) \in C_1^{\mathcal{I}_{\mathbf{e}}}$ or $\pi(x) \in C_2^{\mathcal{I}_{\mathbf{e}}}$. Without loss of generality we may assume that $\pi(x) \in C_1^{\mathcal{I}_{\mathbf{e}}}$ (as the other case is symmetric). Then the rule is applied, it is nondeterministically decided if $C_1$ or $C_2$ is added to $\mathcal{L}_{\mathbf{e}}(x)$. If $C_2$ was picked, the algorithm either eventually proves satisfiability (which closes the proof) or backtracks and $C_1$ is tried next. Therefore we may assume that $C_1$ was picked and added to $\mathcal{L}_{\mathbf{e}}(x)$. In this case however $(*)$ is satisfied as $\pi(x) \in C_1^{\mathcal{I}_{\mathbf{e}}}$;
- **$\exists$-rule:** before the rule was applied we had $x \in V_{\mathbf{e}}$ and $\exists R.D \in \mathcal{L}_{\mathbf{e}}(x)$ for some complex concept of the form $\exists R.D$. From the induction hypothesis $(*)$ was satisfied and hence $\pi(x) \in \exists R.D^{\mathcal{I}_{\mathbf{e}}}$. Therefore there must be $y \in \Delta_{\mathbf{e}}$ such that $\langle \pi(x), d\rangle \in R^{\mathcal{I}_{\mathbf{e}}}$ and $d \in D^{\mathcal{I}_{\mathbf{e}}}$. Let us map $\pi(z) = d$ where $z$ is the node newly added to $V_{\mathbf{e}}$ by the rule application. Indeed $(*)$ is satisfied after $z$, $\mathcal{L}_{\mathbf{e}}(\langle x, z\rangle)$ and $\mathcal{L}_{\mathbf{e}}(z)$ were added, as we have just seen that $\pi(z) \in \Delta_{\mathbf{e}}$ and $\langle \pi(x), \pi(z)\rangle \in R^{\mathcal{I}_{\mathbf{e}}}$ and $\pi(z) \in C^{\mathcal{I}}_{\mathbf{e}}$;
- **$\forall$-rule:** before the rule was applied we had $x, y \in V_{\mathbf{e}}$ with $\forall R.D \in \mathcal{L}_{\mathbf{e}}(x)$ and $R \in \mathcal{L}_{\mathbf{e}}(\langle x, y\rangle)$, for some complex concept $\forall R.D$. From the induction hypothesis $(*)$ is satisfied and hence $\pi(x) \in \forall R.D^{\mathcal{I}_{\mathbf{e}}}$ and $\langle \pi(x), \pi(y)\rangle \in R^{\mathcal{I}_{\mathbf{e}}}$. This implies $\pi(y) \in D^{\mathcal{I}_{\mathbf{e}}}$ as $\mathfrak{I}$ is a model of $\mathfrak{K}$. Therefore $(*)$ is satisfied after the rule application adds $D$ to $\mathcal{L}_{\mathbf{e}}(y)$;
- **$\mathcal{T}$-rule:** whenever the rule is applied $x \in V_{\mathbf{e}}$ we know from induction hypothesis that $(*)$ was satisfied at this point. The rule adds $\mathrm{nnf}(\neg D \sqcup E)$ into $\mathcal{L}_{\mathbf{e}}(x)$ for some $D \sqsubseteq E \in \mathrm{K}(\mathcal{C}_{\mathbf{e}})$. The fact that $\mathfrak{I} \models \mathfrak{K}$ implies $\mathcal{I}_{\mathbf{e}} \models D \sqsubseteq E$. By DL equivalence properties and definition of NNF it holds that for every $z \in \top^{\mathcal{I}_{\mathbf{e}}} = \Delta_{\mathbf{e}}$ we have $z \in \mathrm{nnf}(\neg D \sqcup E)^{\mathcal{I}_{\mathbf{e}}}$. From $(*)$ we know that $\pi(x) \in \Delta_{\mathbf{e}}$ and therefore $(*)$ is still satisfied after $\mathrm{nnf}(\neg D \sqcup E)$ was added into $\mathcal{L}_{\mathbf{e}}(x)$;
- **$\Delta\!\uparrow$-rule:** if this rule was applied then some node $x \in V_{\mathbf{e}}$ was added to $V_{\mathbf{f}}$ with $\mathbf{e} \prec \mathbf{f}$. From the induction hypothesis $x \in \Delta_{\mathbf{e}}$. Since $\Delta_{\mathbf{e}} \subseteq \Delta_{\mathbf{f}}$ whenever $\mathbf{e} \prec \mathbf{f}$ in any CKR model, then $x \in \Delta_{\mathbf{f}}$. Before the application $x \notin V_{\mathbf{e}}$, hence $\mathcal{L}_{\mathbf{e}}(x) = \emptyset$ as all tableaux rules add concepts into $\mathcal{L}_{\mathbf{e}}$ only if $x \in V_{\mathbf{e}}$. Therefore parts (a) and (b) of $(*)$ hold also after the rule is applied. Part (c) trivially holds since no edge was added or labeled;
- **$\Delta\!\downarrow$-rule:** if this rule was applied on some node $x \in V_{\mathbf{f}}$ then it was the case that $\top_{\mathbf{e}} \in \mathcal{L}_{\mathbf{f}}$ and $x \notin V_{\mathbf{e}}$, $\mathbf{e} \prec \mathbf{f}$ before the application. From the induction hypothesis, $\pi(x) \in \top_{\mathbf{e}}^{\mathcal{I}_{\mathbf{f}}}$. As $\mathbf{e} \prec \mathbf{f}$, from definition of CKR models, condition 6 we have $\pi(x) \in \top_{\mathbf{e}}^{\mathcal{I}_{\mathbf{e}}}$ and hence $\pi(x) \in \Delta_{\mathbf{e}}$. Therefore part (a) of $(*)$ is satisfied. Similarly to the previous case $\mathcal{L}_{\mathbf{e}}(x) = \emptyset$ and therefore also part (b) of $(*)$ is satisfied for $x$ and $\mathbf{e}$. Part (c) is satisfied trivially as no edge was added or labeled;
- **$\top_{\sqsubseteq}$-rule:** if this rule was applied on some node $x \in V_{\mathbf{e}}$ with the result of adding $(\neg\top_{\mathbf{f}} \sqcup \top_{\mathbf{g}})$ into $\mathcal{L}_{\mathbf{e}}(x)$, then as $(*)$ was satisfied before the application, we have $\pi(x) \in \Delta_{\mathbf{e}}$. As $\mathfrak{I}$ is a CKR model, $\top_{\mathbf{f}}^{\mathcal{I}_{\mathbf{e}}} \subseteq \top_{\mathbf{g}}^{\mathcal{I}_{\mathbf{e}}}$ must hold. This implies $\mathcal{I}_{\mathbf{e}} \models \top_{\mathbf{f}} \sqsubseteq \top_{\mathbf{g}}$ which in turn implies $\pi(x) \in (\neg\top_{\mathbf{f}} \sqcup \top_{\mathbf{g}})^{\mathcal{I}_{\mathbf{e}}}$ by similar reasoning as in case of the $\mathcal{T}$-rule above. Therefore $(*)$ is satisfied after the rule is applied;
- **$\top_A$-rule:** in this case we had $x \in V_{\mathbf{f}}$, $A_{\mathbf{e}} \in \mathcal{L}_{\mathbf{f}}(x)$ before the rule was applied. From induction hypothesis $(*)$ was satisfied at this point. Therefore $\pi(x) \in A_{\mathbf{e}}^{\mathcal{I}_{\mathbf{f}}}$. From definition of CKR models, condition 2, this implies $\pi(x) \in \top_{\mathbf{e}}^{\mathcal{I}_{\mathbf{f}}}$ and therefore $(*)$ is satisfied also after the rule is applied.

**$\top_R$-rule:** similarly to the previous case we had $x, y \in V_{\mathbf{f}}$, $R_{\mathbf{e}} \in \mathcal{L}_{\mathbf{f}}(\langle x, y\rangle)$ and from the induction hypothesis $(*)$ was satisfied. Therefore $\langle \pi(x), \pi(y)\rangle \in R_{\mathbf{e}}^{\mathcal{I}_{\mathbf{f}}}$. From definition of CKR models, condition 3, this implies both $\pi(x), \pi(y) \in \top_{\mathbf{e}}^{\mathcal{I}_{\mathbf{f}}}$. Therefore $(*)$ is still satisfied after $\top_{\mathbf{e}}$ is added to both $\mathcal{L}_{\mathbf{f}}(x)$ and $\mathcal{L}_{\mathbf{f}}(y)$ during the rule application;

**$A$-rule:** before the application we had $x \in V_{\mathbf{e}} \cap V_{\mathbf{f}}$, $A_{\mathbf{g}} \in \mathcal{L}_{\mathbf{e}}(x)$ and either $\mathbf{e} \prec \mathbf{f}$ or $\mathbf{f} \prec \mathbf{e}$. We distinguish these two cases.

If $\mathbf{e} \prec \mathbf{f}$, then from the induction hypothesis $\pi(x) \in A_{\mathbf{g}}^{\mathcal{I}_{\mathbf{e}}}$ and from the definition of CKR models, condition 7 this gives us $\pi(x) \in A_{\mathbf{g}}^{\mathcal{I}_{\mathbf{f}}} \cap \Delta_{\mathbf{e}}$, i.e., certainly $\pi(x) \in A_{\mathbf{g}}^{\mathcal{I}_{\mathbf{f}}}$, therefore $(*)$ is satisfied also after adding $A_{\mathbf{g}}$ into $\mathcal{L}_{\mathbf{f}}(x)$.

If $\mathbf{f} \prec \mathbf{e}$, then we get $\pi(x) \in \Delta_{\mathbf{f}}$ and $\pi(x) \in A_{\mathbf{g}}^{\mathcal{I}_{\mathbf{e}}}$ from the induction hypothesis. Hence $\pi(x) \in A_{\mathbf{g}}^{\mathcal{I}_{\mathbf{e}}} \cap \Delta_{\mathbf{f}}$ and consequently we obtain $\pi(x) \in A_{\mathbf{g}}^{\mathcal{I}_{\mathbf{f}}}$ from condition 7 of the definition of CKR models, and so $(*)$ still holds after the rule adds $A_{\mathbf{g}}$ into $\mathcal{L}_{\mathbf{f}}(x)$.

**$R$-rule:** before the application we had $x, y \in V_{\mathbf{e}} \cap V_{\mathbf{f}}$, $\langle x, y\rangle \in E$, $R_{\mathbf{g}} \in \mathcal{L}_{\mathbf{e}}(\langle x, y\rangle)$ and either $\mathbf{e} \prec \mathbf{f}$ or $\mathbf{f} \prec \mathbf{e}$.

If $\mathbf{e} \prec \mathbf{f}$, then from the induction hypothesis $\langle \pi(x), \pi(y)\rangle \in R_{\mathbf{g}}^{\mathcal{I}_{\mathbf{e}}}$ and from the definition of CKR models, condition 8 this gives us $\langle \pi(x), \pi(y)\rangle \in R_{\mathbf{g}}^{\mathcal{I}_{\mathbf{f}}} \cap (\Delta_{\mathbf{e}} \times \Delta_{\mathbf{e}})$, i.e., certainly $\langle \pi(x), \pi(y)\rangle \in R_{\mathbf{g}}^{\mathcal{I}_{\mathbf{f}}}$, and therefore $(*)$ is satisfied also after adding $R_{\mathbf{g}}$ into $\mathcal{L}_{\mathbf{f}}(\langle x, y\rangle)$.

If $\mathbf{f} \prec \mathbf{e}$, then we get $\pi(x), \pi(y) \in \Delta_{\mathbf{f}}$ and $\langle \pi(x), \pi(y)\rangle \in R_{\mathbf{g}}^{\mathcal{I}_{\mathbf{e}}}$, from the induction hypothesis. Hence $\langle \pi(x), \pi(y)\rangle \in R_{\mathbf{g}}^{\mathcal{I}_{\mathbf{e}}} \cap (\Delta_{\mathbf{f}} \times \Delta_{\mathbf{f}})$ and consequently we obtain $\langle \pi(x), \pi(y)\rangle \in R_{\mathbf{g}}^{\mathcal{I}_{\mathbf{f}}}$ from the definition of CKR models, condition 7, and so $(*)$ still holds after the rule adds $R_{\mathbf{g}}$ into $\mathcal{L}_{\mathbf{f}}(\langle x, y\rangle)$.

Once no more rules can be applied, the completion tree $T$ is complete. Let us show by contradiction that it is also clash-free: suppose that from some $x \in V$, for some $\mathbf{e} \in \mathfrak{D}_\Gamma$ and for some concept $D$ we have both $D \in \mathcal{L}_{\mathbf{e}}(x)$ and $\neg D \in \mathcal{L}_{\mathbf{e}}(x)$. From $(*)$ this implies that $\pi(x) \in D^{\mathcal{I}_{\mathbf{e}}} \cap \neg D^{\mathcal{I}_{\mathbf{e}}}$ which contradicts the fact that $\mathfrak{I}$ is a CKR interpretation. Therefore $T$ is clash-free and the algorithm outputs "$C$ is $\mathbf{d}$-satisfiable". □

**Theorem 2** (Correctness with ABoxes). *Given a CKR $\mathfrak{K}$, $\mathbf{d} \in \mathfrak{D}_\Gamma$, and a concept $C$ in NNF on the input, the tableaux algorithm $C_{\mathcal{T}}$ always terminates and it is sound and complete.*

*Proof.* **Termination.** The completion tree $T$ generated during the initialization is a finite graph with finite number of nodes. Hence the initialization step terminates. The algorithm consequently exhaustively applies the completion rules on each of the nodes of $T$, that is, it tries to prove separately for each constant appearing in $\mathfrak{K}$ (plus for $s_0$), that a model exists in which the constant belongs to all concepts in its label. As we proved in Theorem 1 (termination), in the original algorithm this process terminates in finite time. We have extended the algorithm with M-rule. This rule however always merges two of the finite number of nodes generated during the initialization step. Hence also this rule can be applied only finitely many times and so the algorithm terminates.

**Soundness.** Assume that a complete and clash-free completion tree was constructed. Let us construct a CKR interpretation $\mathfrak{I} = \{\mathcal{I}_{\mathbf{e}}\}_{\mathbf{e} \in \mathfrak{D}_\Gamma}$ where $\mathcal{I}_{\mathbf{e}}$ is constructed as follows:

1. $\Delta_{\mathbf{e}} = \{x \mid x \in V_{\mathbf{e}}\}$;
2. $a^{\mathcal{I}_{\mathbf{e}}} = a^{\mathbf{g}}$ if $a^{\mathbf{g}} \in V_{\mathbf{e}}$ for some $\mathbf{g} \in \mathfrak{D}_\Gamma$ otherwise $a^{\mathcal{I}_{\mathbf{e}}}$ is undefined;
3. $A^{\mathcal{I}_{\mathbf{e}}} = \{x \mid x \in V_{\mathbf{e}}, A \in \mathcal{L}_{\mathbf{e}}(x),$ and $x$ is not blocked$\}$, $A \in N_{\mathrm{I}}$;
4. $R^{\mathcal{I}_{\mathbf{e}}} = \{\langle x, y\rangle \mid x, y \in V_{\mathbf{e}}, R \in \mathcal{L}_{\mathbf{e}}(\langle x, y\rangle)$ and $y$ is not blocked or there exists $z$ s.t. $R \in \mathcal{L}_{\mathbf{e}}(\langle x, z\rangle)$ and $z$ is blocked by $y\}$;

Interpretations of complex concepts are inductively defined according to semantic constraints on DL-interpretations. We will show that the following statement denoted by $(*)$ holds:

$$C \in \mathcal{L}_{\mathbf{e}}(x) \implies x \in C^{\mathcal{I}_{\mathbf{e}}}$$

This is showed by structural induction:

- If $C = A$ is atomic this follows from the construction;
- If $C = \neg C_1$ then $C_1$ must be atomic due to NNF. Since $T$ is clash-free, if $\neg C_1 \in \mathcal{L}_{\mathbf{e}}(x)$ then $C_1 \notin \mathcal{L}_{\mathbf{e}}(x)$ and hence from the construction $x \notin C_1^{\mathcal{I}_{\mathbf{e}}}$ which implies $x \in \neg C_1^{\mathcal{I}_{\mathbf{e}}}$;
- If $C = \exists R.C_1$ then since $T$ is complete, there must be an $R$-successor $y$ of $x$ in $V_{\mathbf{e}}$ s.t. $C_1 \in \mathcal{L}_{\mathbf{e}}(y)$. Now we distinguish two cases:

  If $y$ is not blocked, from induction hypothesis $y \in C_1^{\mathcal{I}_{\mathbf{e}}}$ and from the construction $\langle x, y\rangle \in R^{\mathcal{I}_{\mathbf{e}}}$. Hence in fact $x \in \exists R.C_1^{\mathcal{I}_{\mathbf{e}}}$.

  If $y$ is blocked, then from the construction there is $z \in V_{\mathbf{e}}$ which is not blocked, it blocks $y$ and $\langle x, z\rangle \in R^{\mathcal{I}_{\mathbf{e}}}$. Since $z$ blocks $y$ and $C_1 \in \mathcal{L}_{\mathbf{e}}(y)$, we have $C_1 \in \mathcal{L}_{\mathbf{e}}(z)$ and from the induction hypothesis from induction hypothesis $x \in C_1^{\mathcal{I}_{\mathbf{e}}}$. Hence in fact $x \in \exists R.C_1^{\mathcal{I}_{\mathbf{e}}}$.
- Cases $C = C_1 \sqcap C_2$, $C = C_1 \sqcup C_2$, and $C = \forall R.C_1$ are analogous and each follows from the fact that $T$ is complete and the respective tableaux rule is not applicable.

We will show that $\mathfrak{I}$ is a CKR-model of $\mathfrak{K}$, that is, it satisfies all relevant conditions of Definition 6:

1. $(\top_{\mathbf{d}})^{\mathcal{I}_{\mathbf{f}}} \subseteq (\top_{\mathbf{e}})^{\mathcal{I}_{\mathbf{f}}}$ if $\mathbf{d} \prec \mathbf{e}$: Let $x \in (\top_{\mathbf{d}})^{\mathcal{I}_{\mathbf{f}}}$ from the construction $\top_{\mathbf{d}} \in \mathcal{L}_{\mathbf{f}}(x)$. Since the $\top_{\sqsubseteq}$ rule is not applicable, we have $\neg\top_{\mathbf{d}} \sqcup \top_{\mathbf{e}} \in \mathcal{L}_{\mathbf{f}}(x)$. Since $\sqcup$-rule is not applicable we have that either $\neg\top_{\mathbf{d}}$ or $\top_{\mathbf{e}}$ belong to $\mathcal{L}_{\mathbf{f}}(x)$ but since $T$ is clash-free it must be the case that $\top_{\mathbf{e}} \in \mathcal{L}_{\mathbf{f}}(x)$. Hence from the construction $x \in \top_{\mathbf{e}}^{\mathcal{I}_{\mathbf{f}}}$.
2. $(A_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} \subseteq (\top_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}}$: if $x \in (A_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}}$ then from the construction $A_{\mathbf{f}} \in \mathcal{L}_{\mathbf{d}}(x)$. Since $\top_A$-rule is not applicable it must be the case that $\top_{\mathbf{f}} \in \mathcal{L}_{\mathbf{d}}(x)$. Now directly from the construction $x \in \top_{\mathbf{f}}^{\mathcal{I}_{\mathbf{d}}}$.
3. $(R_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} \subseteq (\top_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} \times (\top_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}}$: this follows analogously from the fact that the $\top_R$-rule is not applicable.

4. $a^{\mathcal{I}_\mathbf{e}} = a^{\mathcal{I}_\mathbf{d}}$ given $\mathbf{d} \prec \mathbf{e}$, either $a^{\mathcal{I}_\mathbf{d}}$ is defined, or $a^{\mathcal{I}_\mathbf{e}}$ is defined and $a^{\mathcal{I}_\mathbf{e}} \in \Delta_\mathbf{d}$:
   If $a^{\mathcal{I}_\mathbf{d}}$ is defined, then form the construction there is some node $a^{\mathcal{I}_\mathbf{d}} = a^\mathbf{g} \in V_\mathbf{d}$. Since $\Delta\uparrow$-rule is not applicable then also $a^\mathbf{g} \in V_\mathbf{e}$. Therefore $a^{\mathcal{I}}_\mathbf{e}$ is defined as well. If however $a^{\mathcal{I}_\mathbf{e}} = a^\mathbf{h} \in V_\mathbf{d}$ and $a^\mathbf{g} \neq a^\mathbf{h}$ then there must be also $a^\mathbf{h} \in V_\mathbf{e}$. In this case however the M-rule is applicable and hence this case never happens. Therefore $a^{\mathcal{I}_\mathbf{d}} = a^\mathbf{g} = a^{\mathcal{I}_\mathbf{e}}$.
   If $a^{\mathcal{I}_\mathbf{e}} = a^\mathbf{h}$ is defined and $a^{\mathcal{I}_\mathbf{e}} \in \Delta_\mathbf{d}$, then from the construction $a^\mathbf{h} \in V_\mathbf{e}$ and $\top_\mathbf{d} \in \mathcal{L}_\mathbf{e}(a^\mathbf{h})$. Then since the $\Delta\downarrow$-rule is not applicable we have $a^\mathbf{h} \in V_\mathbf{d}$. Then however due to the construction also $a^{\mathcal{I}_\mathbf{d}}$ is defined to some $a^\mathbf{g}$. Analogously to the previous case it follows that $a^\mathbf{h} = a^\mathbf{g}$ because otherwise the M-rule would be applicable.

6. $(X_\mathbf{d})^{\mathcal{I}_\mathbf{e}} = (X_\mathbf{d})^{\mathcal{I}_\mathbf{d}}$ if $\mathbf{d} \prec \mathbf{e}$: we will prove this for $X_\mathbf{d} = R_\mathbf{d}$ being a role. For concepts the proof is analogous: suppose $\langle x, y\rangle \in (R_\mathbf{d})^{\mathcal{I}_\mathbf{d}}$. Then we distinguish two cases:
   If $\langle x, y\rangle \in E$ then from the construction $x, y \in V_\mathbf{d}$ and $R_\mathbf{d} \in \mathcal{L}_\mathbf{d}(\langle x, y\rangle)$. Since the $\Delta\uparrow$ rule is not applicable also $x, y \in V_\mathbf{e}$. Then however since $R$-rule is not applicable also $R_\mathbf{d} \in \mathcal{L}_\mathbf{e}(\langle x, y\rangle)$ and from the construction $\langle x, y\rangle \in R_\mathbf{d}^{\mathcal{I}_\mathbf{e}}$.
   If $\langle x, y\rangle \notin E$ then from the construction there is $z \in V_\mathbf{d}$ such that $y$ blocks $z$ and $R_\mathbf{d} \in \mathcal{L}_\mathbf{d}(\langle x, z\rangle)$. Since $\Delta\uparrow$- and $R$-rules are not applicable we derive that $x, z \in V_\mathbf{e}$ and $R_\mathbf{d} \in \mathcal{L}_\mathbf{e}(\langle x, z\rangle)$. From the construction this gives us that $\langle x, y\rangle \in R_\mathbf{d}^{\mathcal{I}_\mathbf{e}}$.
   Suppose now that $\langle x, y\rangle \in (R_\mathbf{d})^{\mathcal{I}_\mathbf{e}}$. Then again have to distinguish two cases:
   If $\langle x, y\rangle \in E$ then from the construction $x, y \in V_\mathbf{e}$ and $R_\mathbf{d} \in \mathcal{L}_\mathbf{e}(\langle x, y\rangle)$. Since the $\top_R$-rule is not applicable we have $\top_\mathbf{d} \in \mathcal{L}_\mathbf{e}(x)$ and $\top_\mathbf{d} \in \mathcal{L}_\mathbf{e}(y)$. Since the $\Delta\downarrow$ rule is not applicable also $x, y \in V_\mathbf{d}$. Then however since $R$-rule is not applicable also $R_\mathbf{d} \in \mathcal{L}_\mathbf{d}(\langle x, y\rangle)$ and from the construction $\langle x, y\rangle \in R_\mathbf{d}^{\mathcal{I}_\mathbf{d}}$.
   If $\langle x, y\rangle \notin E$ then from the construction there is $z \in V_\mathbf{e}$ such that $y$ blocks $z$ and $R_\mathbf{d} \in \mathcal{L}_\mathbf{e}(\langle x, z\rangle)$. Since the $\top_R$-rule is not applicable we have $\top_\mathbf{d} \in \mathcal{L}_\mathbf{e}(x)$ and $\top_\mathbf{d} \in \mathcal{L}_\mathbf{e}(z)$. Since $\Delta\uparrow$- and $R$-rules are not applicable we derive that $x, z \in V_\mathbf{d}$ and $R_\mathbf{d} \in \mathcal{L}_\mathbf{d}(\langle x, z\rangle)$. From the construction this gives us that $\langle x, y\rangle \in R_\mathbf{d}^{\mathcal{I}_\mathbf{d}}$.

7. $(A_\mathbf{f})^{\mathcal{I}_\mathbf{d}} = (A_\mathbf{f})^{\mathcal{I}_\mathbf{e}} \cap \Delta_\mathbf{d}$ if $\mathbf{d} \prec \mathbf{e}$: this case is analogous to the next one.

8. $(R_\mathbf{f})^{\mathcal{I}_\mathbf{d}} = (R_\mathbf{f})^{\mathcal{I}_\mathbf{e}} \cap (\Delta_\mathbf{d} \times \Delta_\mathbf{d})$ if $\mathbf{d} \prec \mathbf{e}$:
   Suppose $\langle x, y\rangle \in (R_\mathbf{f})^{\mathcal{I}_\mathbf{d}}$. Directly from the construction we have $\langle x, y\rangle \in \Delta_\mathbf{d} \times \Delta_\mathbf{d}$, it remains to prove $\langle x, y\rangle \in (R_\mathbf{f})^{\mathcal{I}_\mathbf{e}}$. We again distinguish two cases:
   If $\langle x, y\rangle \in E$ then from the construction $x, y \in V_\mathbf{d}$ and $R_\mathbf{f} \in \mathcal{L}_\mathbf{d}(\langle x, y\rangle)$. Since the $\Delta\uparrow$ rule is not applicable also $x, y \in V_\mathbf{e}$. Then however since $R$-rule is not applicable also $R_\mathbf{f} \in \mathcal{L}_\mathbf{e}(\langle x, y\rangle)$ and from the construction $\langle x, y\rangle \in R_\mathbf{f}^{\mathcal{I}_\mathbf{e}}$.
   If $\langle x, y\rangle \notin E$ then from the construction there is $z \in V_\mathbf{d}$ such that $y$ blocks $z$ and $R_\mathbf{f} \in \mathcal{L}_\mathbf{d}(\langle x, z\rangle)$. Since $\Delta\uparrow$- and $R$-rules are not applicable we derive that $x, z \in V_\mathbf{e}$ and $R_\mathbf{f} \in \mathcal{L}_\mathbf{e}(\langle x, z\rangle)$. From the construction this gives us that $\langle x, y\rangle \in R_\mathbf{f}^{\mathcal{I}_\mathbf{e}}$.
   Suppose now that $\langle x, y\rangle \in (R_\mathbf{f})^{\mathcal{I}_\mathbf{e}} \cap (\Delta_\mathbf{d} \times \Delta_\mathbf{d})$. Then again have to distinguish two cases:
   If $\langle x, y\rangle \in E$ then from the construction $x, y \in V_\mathbf{e}$ and $R_\mathbf{f} \in \mathcal{L}_\mathbf{e}(\langle x, y\rangle)$. Since $x, y \in \Delta_\mathbf{d}$, then directly from the construction we have $\top_\mathbf{d} \in \mathcal{L}_\mathbf{d}(x)$ and $\top_\mathbf{d} \in \mathcal{L}_\mathbf{d}(y)$ and since $\Delta\uparrow$-rule is not applicable then also $\top_\mathbf{d} \in \mathcal{L}_\mathbf{e}(x)$ and $\top_\mathbf{d} \in \mathcal{L}_\mathbf{e}(y)$. Then however since $R$-rule is not applicable also $R_\mathbf{f} \in \mathcal{L}_\mathbf{d}(\langle x, y\rangle)$ and from the construction $\langle x, y\rangle \in R_\mathbf{f}^{\mathcal{I}_\mathbf{d}}$.
   If $\langle x, y\rangle \notin E$ then from the construction there is $z \in V_\mathbf{e}$ such that $y$ blocks $z$ and $R_\mathbf{f} \in \mathcal{L}_\mathbf{e}(\langle x, z\rangle)$. Since $x, y \in \Delta_\mathbf{d}$, then directly from the construction we have $\top_\mathbf{d} \in \mathcal{L}_\mathbf{d}(x)$ and $\top_\mathbf{d} \in \mathcal{L}_\mathbf{d}(y)$, and since $y$ blocks $z$ then also $z \in \Delta_\mathbf{d}$, $\top_\mathbf{d} \in \mathcal{L}_\mathbf{d}(z)$. Since $\Delta\uparrow$-rule is not applicable then also $\top_\mathbf{d} \in \mathcal{L}_\mathbf{e}(x)$ and $\top_\mathbf{d} \in \mathcal{L}_\mathbf{e}(z)$. Since $R$-rule is not applicable we derive that $x, z \in V_\mathbf{d}$ and $R_\mathbf{f} \in \mathcal{L}_\mathbf{d}(\langle x, z\rangle)$. From the construction this gives us that $\langle x, y\rangle \in R_\mathbf{f}^{\mathcal{I}_\mathbf{d}}$.

9. $\mathcal{I}_\mathbf{d} \models \mathrm{K}(\mathcal{C}_\mathbf{d})$: we have to prove that $\mathcal{I}_\mathbf{d} \models \phi$ for all axioms $\phi \in \mathrm{K}(\mathcal{C}_\mathbf{d})$. Depending on the type of axiom:
   If $\phi = C \sqsubseteq D$. Suppose $x \in C^{\mathcal{I}_\mathbf{d}}$. Since $\mathcal{T}$-rule is not applicable $\mathrm{nnf}(\neg C) \sqcup D \in \mathcal{L}_\mathbf{d}(x)$ and since $\sqcup$-rule is not applicable the either $\mathrm{nnf}(\neg C) \in \mathcal{L}_\mathbf{d}(x)$ or $D \in \mathcal{L}_\mathbf{d}(x)$. If $\mathrm{nnf}(\neg C) \in \mathcal{L}_\mathbf{d}(x)$ then due to $(*)$ $x \in \neg C^{\mathcal{I}_\mathbf{d}}$ which contradicts the assumption and hence it must be the case that $D \in \mathcal{L}_\mathbf{d}(x)$ and from $(*)$ also $x \in D^{\mathcal{I}_\mathbf{d}}$, that is, $\mathcal{I}_\mathbf{d} \models \phi$.
   If $\phi = C(a)$, then in the initialization step we have added a node $a^\mathbf{d}$ into $V_\mathbf{d}$ and $C$ to its label $\mathcal{L}_\mathbf{d}(a^\mathbf{d})$. This implies that after the algorithm is over there is $a^\mathbf{g}$ in $V_\mathbf{d}$ with $a^{\mathcal{I}_\mathbf{d}} = a^\mathbf{g} \in C^{\mathcal{I}_\mathbf{d}}$, where $a^\mathbf{g}$ is the resulting node after possibly multiple merging operations that merged $a^\mathbf{d}$ with some other nodes and finally with $a^\mathbf{g}$, however $C$ is surely in the label of this node as implied by the definition of the merging operation.
   If $\phi = R(a, b)$ the proof is analogous.

Hence $\mathfrak{I}$ is a model of $\mathfrak{K}$ and $C^{\mathcal{I}_\mathbf{d}}$ is nonempty as $s_0 \in C^{\mathcal{I}_\mathbf{d}}$.

**Completeness.** Let $\mathfrak{I}$ be a model of $\mathfrak{K}$ with $C^{\mathcal{I}_\mathbf{d}} \neq \emptyset$. Let us first observe that no clash is introduced during the initialization step of the algorithm: if there is a clash in some node $a^\mathbf{g}$ of $V_\mathbf{e}$, i.e, $\mathcal{L}_\mathbf{e}(a^\mathbf{g})$ contains both $D$ and $\neg D$ then necessarily there were both axioms $D(a)$ and $\neg D(a)$ in the ABox of $C_\mathbf{e}$. In this case however $\mathfrak{K}$ has no models which contradicts the assumptions.

Afterwards the algorithm constructs a completion tree rooted in each of the nodes generated during the initialization step. We have showed in Theorem 1 (completeness) that the completion rule can be applied in such a way that no clash is introduced. In this Lemma we did not consider the M-rule, whose completeness we have to show in addition.

Note that only the nodes that were generated during the initialization step and correspond to individuals in the ABox can possibly be merged. Immediately before $\mathrm{merge}(a^\mathbf{g}, a^\mathbf{h})$

was applied, the algorithm was working independently on two completion trees, one rooted in $a^{\mathbf{g}}$ and the other one in $a^{\mathbf{h}}$. Let us denote the completion tree rooted in $a^{\mathbf{g}}$ by $T^{\mathbf{g}} = \langle V^{\mathbf{g}}, E^{\mathbf{g}}, \mathcal{L}^{\mathbf{g}} \rangle$ and the other one $T^{\mathbf{h}} = \langle V^{\mathbf{h}}, E^{\mathbf{h}}, \mathcal{L}^{\mathbf{h}} \rangle$. Using the CKR model $\mathfrak{I}$, in Theorem 1 (completeness) we were inductively constructing two mappings $\pi^{\mathbf{g}}$ and $\pi^{\mathbf{h}}$ such that property $(*)$ always holds for any $\pi \in \{\pi^{\mathbf{g}}, \pi^{\mathbf{h}}\}$. We will extend this property with one additional condition as follows:

> For each node $x \in V$ and for each $\mathbf{e} \in \mathfrak{D}_\Gamma$: (a) if $x \in V_{\mathbf{e}}$ then $\pi(x) \in \Delta_{\mathbf{e}}$; (b) if $C \in \mathcal{L}_{\mathbf{e}}(x)$ then $\pi(x) \in C^{\mathcal{I}_{\mathbf{e}}}$; (c) if $R \in \mathcal{L}_{\mathbf{e}}(\langle x, y \rangle)$ then $\langle \pi(x), \pi(y) \rangle \in R^{\mathcal{I}_{\mathbf{e}}}$; (d) if $a^{\mathbf{g}} \in V_{\mathbf{e}}$ then $a^{\mathcal{I}_{\mathbf{e}}} = \pi(a^{\mathbf{g}})$.

Observe that every time we initialize $\pi$ on some node $a^{\mathbf{f}} \in V_{\mathbf{f}}$ we pick $\pi(a^{\mathbf{f}}) = a^{\mathcal{I}_{\mathbf{f}}}$ in order to satisfy condition (d). Such a pick trivially satisfies also conditions (a–c) and at this point the completion tree $T$ encodes exactly the ABox which must be satisfied in $\mathfrak{I}$.

After calling $\mathrm{merge}(a^{\mathbf{g}}, a^{\mathbf{h}})$, let us extend $\pi^{\mathbf{g}}$ as follows: for every node $x \in V$ s.t. $\pi^{\mathbf{h}}(x)$ is defined and $\pi^{\mathbf{g}}$ is not defined, assign $\pi^{\mathbf{g}}(x) := \pi^{\mathbf{h}}(x)$.

Observe that after the merging $(*)$ is trivially satisfied for all nodes apart from $a^{\mathbf{g}}$ (note that node $a^{\mathbf{h}}$ was removed). In order to show that it is also satisfied for $a^{\mathbf{g}}$ we will first show that if $\mathrm{merge}(a^{\mathbf{g}}, a^{\mathbf{h}})$ is called, then $\pi^{\mathbf{g}}(a^{\mathbf{g}}) = \pi^{\mathbf{h}}(a^{\mathbf{h}})$: since $\mathrm{merge}(a^{\mathbf{g}}, a^{\mathbf{h}})$ was called from the M-rule and for some $\mathbf{d} \preceq \mathbf{e}$, it was the case that $a^{\mathbf{g}} \in V_{\mathbf{d}}$ and $a^{\mathbf{h}} \in V_{\mathbf{e}}$; then, since $(*)$ was satisfied before merging, from (d) we have $a^{\mathcal{I}_{\mathbf{d}}} = \pi^{\mathbf{g}}(a^{\mathbf{g}})$ and $a^{\mathcal{I}_{\mathbf{e}}} = \pi^{\mathbf{g}}(a^{\mathbf{g}})$. From condition 4 of CKR models we have $\pi^{\mathbf{g}}(a^{\mathbf{g}}) = a^{\mathcal{I}_{\mathbf{d}}} = a^{\mathcal{I}_{\mathbf{e}}} = \pi^{\mathbf{h}}(a^{\mathbf{h}})$.

The fact that conditions (a–d) of $(*)$ are satisfied for $\pi^{\mathbf{g}}$ after the merging is now an easy consequence of what we just showed.

We have thus showed that $(*)$ holds after application of all rules, and this implies that clash is not introduced. Therefore the algorithm is complete.

□

**Theorem 3** (Complexity)**.** *The complexity of the $C_{\mathcal{T}}$ algorithm is in* NEXPTIME *with respect to the combined size of the input.*

*Proof.* In the proof of Theorem 1 (termination) we have showed that the algorithm terminates after applying at most $O(2^{2 \times c \times n})$ tableaux rules, where $n = |\mathrm{clos}_{\mathfrak{K}}(C)|$, and $c$ is the number of contexts which is bounded by a constant. Observing that $n$ is bounded by the size of the combined input (measured by its combined string length), we obtain that the algorithm is in NEXPTIME.

Let us now consider ABoxes as well. During the initialization step the algorithm takes $O(n)$ steps Then the algorithm is independently executes for each of the remaining nodes, i.e., at most $O(n)$ times. Each of the runs previously took $O(2^{2 \times c \times n})$ however we did not consider the M-rule. This rule however merges nodes, hence it can be executed only once for each pair of nodes generated during the initialization step (in all runs combined). There is $O(n)$ of such nodes hence the rule is executed at most $O(n^2)$ times overall. Therefore the combined time complexity under $O(n) + O(n^2) + O(n) \times O(2^{2 \times c \times n})$ which is still in NEXPTIME. □

**Lemma 3** (Correctness of optimized rules)**.** *Given a CKR $\mathfrak{K}$, $\mathbf{d} \in \mathfrak{D}_\Gamma$, and a concept $C$ in NNF on the input, the algorithm ${C_{\mathcal{T}}}^*$ always terminates and it is sound and complete.*

*Proof.* To prove the correctness of the ${C_{\mathcal{T}}}^*$ algorithm, we mostly have to repeat the presented proofs for tableaux interpretation (Lemma 1), soundness and completeness (Theorem 1) by adapting the proofs with respect to new rules and related new tableaux conditions. Thus, in the following we will mainly report how the presented proofs are modified in order to show their validity with respect to ${C_{\mathcal{T}}}^*$ conditions.

Remark that the termination result (from Theorem 1) is not influenced by the definition of the new rules. More precisely, with respect to this result, the new rules $(\top_A^*, \top_R^*, \top_{\sqsubseteq}^*)$ behave in the same way as the ones $(\top_A, \top_R, \top_{\sqsubseteq})$ that they replace: they are part of the rules adding concepts to node labels whenever these are not already present in the target labels, and their number of applications has already been proved as bounded.

**Tableaux interpretation.** We have to prove the following proposition, now considering the new conditions on tableaux definition.

> Given a CKR $\mathfrak{K}$ with no ABoxes and some $\mathbf{d} \in \mathfrak{D}_\Gamma$, a concept $C$ is $\mathbf{d}$-satisfiable w.r.t. $\mathfrak{K}$ iff there exists a $\mathbf{d}$-tableau for $C$.

As in Lemma 1, we prove the assertion by showing the two inclusion directions, basically repeating the same constructions.

*If direction.* As $C$ is $\mathbf{d}$-satisfiable, there is a CKR-interpretation $\mathfrak{I} = \{\mathcal{I}_{\mathbf{f}}\}_{\mathbf{f} \in \mathfrak{D}_\Gamma}$ such that $C^{\mathcal{I}_{\mathbf{d}}} \neq \emptyset$. We construct $\mathbf{T} = \{\mathbf{T}_{\mathbf{f}}\}_{\mathbf{f} \in \mathfrak{D}_\Gamma}$ as in the proof above for Lemma 1. To prove the if-direction of the assertion, we only have to show that $\mathbf{T}$ satisfies the CKR-tableaux definition also with respect to the newly defined conditions (9')-(11'): as the proof for all of the other conditions are not modified, we only show the cases related to these new conditions. Assume $\mathbf{e}, \mathbf{f}, \mathbf{g}$ are arbitrary dimensional vectors of $\mathfrak{D}_\Gamma$; $s, t$ are any two elements of $\mathcal{S}_{\mathbf{e}}$; $A, C_1, C_2$ are any concepts from $\mathrm{clos}_{\mathfrak{K}}(C)$ such that $A$ is atomic; and $R$ is arbitrary role of $\mathcal{R}_{\mathfrak{K},C}$.

(9') let $\mathbf{f} \preceq \mathbf{g}$ and $A_{\mathbf{f}} \in \mathcal{L}_{\mathbf{e}}(s)$. In this case, in every CKR-interpretation we have $\top_{\mathbf{f}}^{\mathcal{I}_{\mathbf{e}}} \subseteq \top_{\mathbf{g}}^{\mathcal{I}_{\mathbf{e}}}$. Since $A_{\mathbf{f}} \in \mathcal{L}_{\mathbf{e}}(s)$, by construction we have $s \in A_{\mathbf{f}}^{\mathcal{I}_{\mathbf{e}}}$: moreover, since $A_{\mathbf{f}}^{\mathcal{I}_{\mathbf{e}}} \subseteq \top_{\mathbf{f}}^{\mathcal{I}_{\mathbf{e}}}$ we also have that $s \in \top_{\mathbf{f}}^{\mathcal{I}_{\mathbf{e}}}$. Finally, this implies that $s \in \top_{\mathbf{g}}^{\mathcal{I}_{\mathbf{e}}}$ and, by construction, $\top_{\mathbf{g}} \in \mathcal{L}_{\mathbf{e}}(s)$.

(10') let $\mathbf{f} \preceq \mathbf{g}$ and $\langle s, t \rangle \in \mathcal{E}_{\mathbf{e}}(R_{\mathbf{f}})$. As in the case above, in every CKR-interpretation we have $\top_{\mathbf{f}}^{\mathcal{I}_{\mathbf{e}}} \subseteq \top_{\mathbf{g}}^{\mathcal{I}_{\mathbf{e}}}$. Moreover, from the construction $\langle s, t \rangle \in R_{\mathbf{f}}^{\mathcal{I}_{\mathbf{e}}}$. From $\mathfrak{I}$ being a CKR-interpretation (condition 3 of CKR model definition), we know that $R_{\mathbf{f}}^{\mathcal{I}_{\mathbf{e}}} \subseteq \top_{\mathbf{f}}^{\mathcal{I}_{\mathbf{e}}} \times \top_{\mathbf{f}}^{\mathcal{I}_{\mathbf{e}}}$: thus we have that $s, t \in \top_{\mathbf{f}}^{\mathcal{I}_{\mathbf{e}}}$. Finally, this implies that $s, t \in \top_{\mathbf{g}}^{\mathcal{I}_{\mathbf{e}}}$ and, by construction, $\top_{\mathbf{g}} \in \mathcal{L}_{\mathbf{e}}(s)$ and $\top_{\mathbf{g}} \in \mathcal{L}_{\mathbf{e}}(t)$.

(11') let $\mathbf{f} \prec \mathbf{g}$ and $\neg\top_{\mathbf{g}} \in \mathcal{L}_{\mathbf{e}}(s)$. In every CKR-interpretation we have $\top_{\mathbf{f}}^{\mathcal{I}_{\mathbf{e}}} \subseteq \top_{\mathbf{g}}^{\mathcal{I}_{\mathbf{e}}}$. Since $\neg\top_{\mathbf{g}} \in \mathcal{L}_{\mathbf{e}}(s)$, by construction we have $s \notin \top_{\mathbf{g}}^{\mathcal{I}_{\mathbf{e}}}$. This implies that $s \notin \top_{\mathbf{f}}^{\mathcal{I}_{\mathbf{e}}}$ and hence $s \in \neg\top_{\mathbf{f}}^{\mathcal{I}_{\mathbf{e}}}$. By construction, this implies that $\neg\top_{\mathbf{f}} \in \mathcal{L}_{\mathbf{e}}(s)$.

*Only-if direction.* Let us assume that a $\mathbf{d}$-tableau $\mathbf{T} = \{\mathbf{T_d}\}_{\mathbf{d}\in\mathfrak{D}_\Gamma}$ exists for $C$. We assume the same construction of the CKR-interpretation $\overline{\mathfrak{I}} = \{\overline{\mathcal{I}}_{\mathbf{f}}\}_{\mathbf{d}\in\mathfrak{D}_\Gamma}$ as presented in Lemma 1. We can show that the resulting structure is again a CKR model of $\mathfrak{K}$: note that (†) can be directly verified as in proof for Lemma 1.

Thus, we only need to verify that conditions 1–9 on the CKR-model definition still hold with respect to new tableaux conditions. We only present the cases which need to be modified (intuitively, the ones that used conditions (9)-(11) of the original tableaux definition), while the proof for other cases can be shown to be analogous.

1. we must prove that $\top_{\mathbf{e}}^{\overline{\mathcal{I}}_{\mathbf{g}}} \subseteq \top_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}}$ if $\mathbf{e} \prec \mathbf{f}$, for any $\mathbf{g} \in \mathfrak{D}_\Gamma$. Assume $x \in \top_{\mathbf{e}}^{\overline{\mathcal{I}}_{\mathbf{g}}}$. From the construction, we have $x \in \mathcal{S}_{\mathbf{g}}$ and $\top_{\mathbf{e}} \in \mathcal{L}_{\mathbf{g}}(x)$ (since $\top_{\mathbf{e}} \in \mathrm{clos}_{\mathfrak{K}}(C)$). As $\mathbf{e} \prec \mathbf{f}$, from condition (9') of CKR tableaux we have $\top_{\mathbf{f}} \in \mathcal{L}_{\mathbf{g}}(x)$: thus, by construction, it holds that $x \in \top_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}}$.

2. we must prove that $A_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}} \subseteq \top_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}}$ for any $\mathbf{g}, \mathbf{f} \in \mathfrak{D}_\Gamma$ and for any atomic concept $A_{\mathbf{f}}$. Let $x \in A_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}}$: from the construction this implies $x \in \mathcal{S}_{\mathbf{g}}$ and $A_{\mathbf{f}} \in \mathcal{L}_{\mathbf{g}}(x)$. From condition (9') of CKR-tableau this directly implies $\top_{\mathbf{f}} \in \mathcal{L}_{\mathbf{g}}(x)$. Finally due to the construction $x \in \top_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}}$. Thus it holds that $A_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}} \subseteq \top_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}}$;

3. we must prove that $R_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}} \subseteq \top_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}} \times \top_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}}$ for any $\mathbf{g}, \mathbf{f} \in \mathfrak{D}_\Gamma$ and for any role $R_{\mathbf{f}}$. Let $\langle x, y\rangle \in R_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}}$ be arbitrary. From the construction this implies $x, y \in \mathcal{S}_{\mathbf{g}}$ and $\langle x, y\rangle \in \mathcal{E}_{\mathbf{g}}(R_{\mathbf{f}})$. From condition (10') of CKR-tableau this implies $\top_{\mathbf{f}} \in \mathcal{L}_{\mathbf{g}}(x)$ and $\top_{\mathbf{f}} \in \mathcal{L}_{\mathbf{g}}(y)$. This in turn implies $x, y \in \top_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}}$ due to the construction. Hence we have $R_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}} \subseteq \top_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}} \times \top_{\mathbf{f}}^{\overline{\mathcal{I}}_{\mathbf{g}}}$;

5. we must prove that $X_{\mathbf{e}}^{\overline{\mathcal{I}}_{\mathbf{f}}} = X_{\mathbf{e}}^{\overline{\mathcal{I}}_{\mathbf{e}}}$ for any $\mathbf{e}, \mathbf{f} \in \mathfrak{D}_\Gamma$ with $\mathbf{e} \prec \mathbf{f}$ and for any atomic concept or role $X_{\mathbf{e}}$.

   Let $\underline{X_{\mathbf{e}} = A_{\mathbf{e}}\text{ be an atomic concept}}$ and let $x \in A_{\mathbf{e}}^{\overline{\mathcal{I}}_{\mathbf{e}}}$. From the construction $x \in \mathcal{S}_{\mathbf{e}}$, and as $\mathbf{e} \prec \mathbf{f}$ then from condition (7) of the $\mathbf{d}$-tableaux also $x \in \mathcal{S}_{\mathbf{f}}$, hence $x \in \mathcal{S}_{\mathbf{e}} \cap \mathcal{S}_{\mathbf{f}}$. This together with the fact that $A_{\mathbf{e}} \in \mathcal{L}_{\mathbf{e}}(x)$ (which follows from the construction) implies that $A_{\mathbf{e}} \in \mathcal{L}_{\mathbf{f}}(x)$ by condition (12) on $\mathbf{d}$-tableaux. Finally, from the construction we have $x \in A_{\mathbf{e}}^{\overline{\mathcal{I}}_{\mathbf{f}}}$. Vice versa, assume $x \in A_{\mathbf{e}}^{\overline{\mathcal{I}}_{\mathbf{f}}}$. From the construction $x \in \mathcal{S}_{\mathbf{f}}$ and $A_{\mathbf{e}} \in \mathcal{L}_{\mathbf{f}}(x)$. By condition (9') of $\mathbf{d}$-tableaux we get that $\top_{\mathbf{e}} \in \mathcal{L}_{\mathbf{f}}(x)$ and consequently by condition (8) we have $x \in \mathcal{S}_{\mathbf{e}}$. Hence in fact $x \in \mathcal{S}_{\mathbf{e}} \cap \mathcal{S}_{\mathbf{f}}$ and by condition (12) we derive $A_{\mathbf{e}} \in \mathcal{L}_{\mathbf{e}}(x)$ which by construction implies $x \in A_{\mathbf{e}}^{\overline{\mathcal{I}}_{\mathbf{e}}}$.

   Let $\underline{X_{\mathbf{e}} = R_{\mathbf{e}}\text{ be a role}}$ and let $\langle x, y\rangle \in R_{\mathbf{e}}^{\overline{\mathcal{I}}_{\mathbf{e}}}$. From the construction $x, y \in \mathcal{S}_{\mathbf{e}}$, and as $\mathbf{e} \prec \mathbf{f}$ then from condition (7) of the $\mathbf{d}$-tableaux $x, y \in \mathcal{S}_{\mathbf{e}} \cap \mathcal{S}_{\mathbf{f}}$. This together with the fact that $\langle x, y\rangle \in \mathcal{E}_{\mathbf{e}}(R_{\mathbf{e}})$ (which follows from the construction) implies that $\langle x, y\rangle \in \mathcal{E}_{\mathbf{f}}(R_{\mathbf{e}})$ by condition (13) of $\mathbf{d}$-tableaux. Finally, from the construction we have $\langle x, y\rangle \in R_{\mathbf{e}}^{\overline{\mathcal{I}}_{\mathbf{f}}}$. Vice versa, assuming $\langle x, y\rangle \in R_{\mathbf{e}}^{\overline{\mathcal{I}}_{\mathbf{f}}}$, we know from the construction that $x, y \in \mathcal{S}_{\mathbf{f}}$ and $\langle x, y\rangle \in \mathcal{E}_{\mathbf{f}}(R_{\mathbf{e}})$. By condition (10') of $\mathbf{d}$-tableaux we get both $\top_{\mathbf{e}} \in \mathcal{L}_{\mathbf{f}}(x)$ and $\top_{\mathbf{e}} \in \mathcal{L}_{\mathbf{f}}(y)$. Then, by condition (8) we have $x, y \in \mathcal{S}_{\mathbf{e}}$. Hence $x, y \in \mathcal{S}_{\mathbf{e}} \cap \mathcal{S}_{\mathbf{f}}$ and hence by condition (13) we derive $\langle x, y\rangle \in \mathcal{E}_{\mathbf{e}}(R_{\mathbf{e}})$ which by construction implies $\langle x, y\rangle \in R_{\mathbf{e}}^{\overline{\mathcal{I}}_{\mathbf{e}}}$;

**Soundness.** We need to prove the following proposition:

> If the $C_{\mathcal{T}}{}^*$ algorithm generates a complete and clash-free completion tree for a given input CKR $\mathfrak{K}$ (without ABoxes), $\mathbf{d} \in \mathfrak{D}_\Gamma$, and a concept $C$, then there exists a $\mathbf{d}$-tableau for $\mathfrak{K}$ and $C$.

Let $T = \langle V, E, \mathcal{L}\rangle$ be a clash-free and complete completion tree generated by the $C_{\mathcal{T}}{}^*$ algorithm on the input $\mathfrak{K}$, $\mathbf{d}$, and $C$. We assume the construction of the CKR-tableau $\mathbf{T}' = \{\mathbf{T}'_{\mathbf{e}}\}_{\mathbf{e}\in\mathfrak{D}_\Gamma}$ as presented in the proof for Theorem 1 (soundness).

In order to prove that $\mathbf{T}'$ is a $\mathbf{d}$-tableaux, we ought to show that the conditions placed on $\mathbf{d}$-tableaux by Definition 7 are satisfied. Let $\mathbf{e}, \mathbf{f}$ be any two dimensional vectors of $\mathfrak{D}_\Gamma$ and let $s, t$ be any elements of $\mathcal{S}'_{\mathbf{e}}$, let $A$ be any atomic concept, $C_1$ and $C_2$ any possibly complex concepts and $R$ any role. We will consider only the cases for modified rules, since the proof for other cases is analogous to the one presented for Theorem 1 (soundness).

(9'). assume $\mathbf{f} \preceq \mathbf{g}$ and $A_{\mathbf{f}} \in \mathcal{L}'_{\mathbf{e}}(s)$. As $s \in \mathcal{S}'_{\mathbf{e}}$, from the construction $s \in V_{\mathbf{e}}$ and it is not blocked. As $T$ is complete we have $\top_{\mathbf{g}} \in \mathcal{L}_{\mathbf{e}}(s)$, otherwise $\top_A^*$-rule is applicable. From the construction $\top_{\mathbf{g}} \in \mathcal{L}'_{\mathbf{e}}(s)$;

(10'). assume $\mathbf{f} \preceq \mathbf{g}$ and $\langle s, t\rangle \in \mathcal{E}'_{\mathbf{e}}(R_{\mathbf{f}})$. As $s, t \in \mathcal{S}'_{\mathbf{e}}$, from the construction we have $s, t \in V_{\mathbf{e}}$, neither of them blocked. Following the construction of $T'$, we need to distinguish two cases.

If $\underline{\langle s, t\rangle \in E}$ then by construction we have $R_{\mathbf{f}} \in \mathcal{L}_{\mathbf{e}}(\langle s, t\rangle)$. As $T$ is complete, we have $\top_{\mathbf{g}} \in \mathcal{L}_{\mathbf{e}}(s)$ and $\top_{\mathbf{g}} \in \mathcal{L}_{\mathbf{e}}(t)$ otherwise the $\top_R^*$-rule would be applicable. Finally $\top_{\mathbf{g}} \in \mathcal{L}'_{\mathbf{e}}(s)$ and $\top_{\mathbf{g}} \in \mathcal{L}'_{\mathbf{e}}(t)$ follows from the construction.

If $\underline{\langle s, t\rangle \notin E}$ then due to the construction there must exist $u \in V_{\mathbf{e}}$ such that $\langle s, u\rangle \in E$ and $u$ is blocked by $t$. In this case $R_{\mathbf{f}} \in \mathcal{L}_{\mathbf{e}}(\langle s, u\rangle)$ and hence $\top_{\mathbf{g}} \in \mathcal{L}_{\mathbf{e}}(s)$ and $\top_{\mathbf{g}} \in \mathcal{L}_{\mathbf{e}}(u)$, otherwise $\top_R^*$-rule would be applicable. Finally, as $t$ blocks $u$, we have $\mathcal{L}_{\mathbf{e}}(u) = \mathcal{L}_{\mathbf{e}}(t)$ and hence $\top_{\mathbf{g}} \in \mathcal{L}_{\mathbf{e}}(t)$ as well. Now for the construction $\top_{\mathbf{g}} \in \mathcal{L}_{\mathbf{e}}(s)$ and $\top_{\mathbf{g}} \in \mathcal{L}_{\mathbf{e}}(t)$ implies $\top_{\mathbf{g}} \in \mathcal{L}'_{\mathbf{e}}(s)$ and $\top_{\mathbf{g}} \in \mathcal{L}'_{\mathbf{e}}(t)$;

(11'). assume $\mathbf{f} \prec \mathbf{g}$ and $\neg\top_{\mathbf{g}} \in \mathcal{L}'_{\mathbf{e}}(s)$. As $s \in \mathcal{S}'_{\mathbf{e}}$, from the construction $s \in V_{\mathbf{e}}$ and it is not blocked. As $T$ is complete we have $\neg\top_{\mathbf{f}} \in \mathcal{L}_{\mathbf{e}}(s)$, otherwise $\top^*_{\sqsubseteq}$-rule is applicable. Finally, by construction, we have $\neg\top_{\mathbf{f}} \in \mathcal{L}'_{\mathbf{e}}(s)$;

**Completeness.** We need to verify the assertion:

> Given any concept $C$ that is $\mathbf{d}$-satisfiable w.r.t. a CKR $\mathfrak{K}$, for some $\mathbf{d} \in \mathfrak{D}_\Gamma$, the $C_{\mathcal{T}}{}^*$ algorithm constructs a complete and clash free completion tree $T$ on the input $\mathfrak{K}$, $\mathbf{d}$ and $C$.

We adapt the proof by bi-simulation for Theorem 1 (completeness) to the newly defined rules of $C_{\mathcal{T}}{}^*$. The initialization step of the algorithm is not modified: we only ought to show that, in the inductive step, applications of the new rules in the tableaux expansion still verify the property $(*)$ stated for Theorem 1 (completeness). The assertion then follows by the same considerations of the analogous proof.

$\top^*_A$**-rule:** in this case we had $x \in V_{\mathbf{f}}$, $A_{\mathbf{d}} \in \mathcal{L}_{\mathbf{f}}(x)$ before the rule was applied, with $\mathbf{d} \preceq \mathbf{e}$. By induction hypothesis, $(*)$ was satisfied at this point. therefore $\pi(x) \in A_{\mathbf{d}}^{\mathcal{I}_{\mathbf{f}}}$. By condition 2 of CKR model definition, we have that $\pi(x) \in \top_{\mathbf{d}}^{\mathcal{I}_{\mathbf{f}}}$. By condition 1, $\top_{\mathbf{d}}^{\mathcal{I}_{\mathbf{f}}} \subseteq \top_{\mathbf{e}}^{\mathcal{I}_{\mathbf{f}}}$ and thus $\pi(x) \in \top_{\mathbf{e}}^{\mathcal{I}_{\mathbf{f}}}$: therefore, $(*)$ is satisfied also after the rule is applied.

$\top^*_R$**-rule:** in this case we had $x, y \in V_{\mathbf{f}}$, $R_{\mathbf{d}} \in \mathcal{L}_{\mathbf{f}}(\langle x, y\rangle)$ before the rule was applied, with $\mathbf{d} \preceq \mathbf{e}$. By induction hypothesis $(*)$ was satisfied at this point, therefore $\langle \pi(x), \pi(y)\rangle \in R_{\mathbf{d}}^{\mathcal{I}_{\mathbf{f}}}$. By condition 3 of CKR model definition, we have that $\pi(x) \in \top_{\mathbf{d}}^{\mathcal{I}_{\mathbf{f}}}$ and $\pi(y) \in \top_{\mathbf{d}}^{\mathcal{I}_{\mathbf{f}}}$. By condition 1, $\top_{\mathbf{d}}^{\mathcal{I}_{\mathbf{f}}} \subseteq \top_{\mathbf{e}}^{\mathcal{I}_{\mathbf{f}}}$ and thus $\pi(x) \in \top_{\mathbf{e}}^{\mathcal{I}_{\mathbf{f}}}$ and $\pi(y) \in \top_{\mathbf{e}}^{\mathcal{I}_{\mathbf{f}}}$. Therefore, $(*)$ is satisfied also after the rule is applied.

$\top^*_{\sqsubseteq}$**-rule:** in this case we had $x \in V_{\mathbf{f}}$, $\neg\top_{\mathbf{e}} \in \mathcal{L}_{\mathbf{f}}(x)$ before the rule was applied, with $\mathbf{d} \prec \mathbf{e}$. By induction hypothesis $(*)$ was satisfied at this point. therefore $\pi(x) \notin \top_{\mathbf{e}}^{\mathcal{I}_{\mathbf{f}}}$. Given that $\mathbf{d} \prec \mathbf{e}$, we have $\mathcal{I}_{\mathbf{f}} \models \neg\top_{\mathbf{d}} \sqcup \top_{\mathbf{e}}$: thus, it holds that $\pi(x) \in (\neg\top_{\mathbf{d}} \sqcup \top_{\mathbf{e}})^{\mathcal{I}_{\mathbf{f}}}$ and hence $\pi(x) \in \neg\top_{\mathbf{d}}^{\mathcal{I}_{\mathbf{f}}}$. Therefore, $(*)$ is satisfied also after the rule is applied.

□