Computational Logic

First-Order Logic

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Alphabet

An *alphabet* contains

- Variables
  $x, y, z, \ldots$
- Constants
  $c, d, e, \ldots$
- Function symbols
  $f, g, h, \ldots$
- Predicate symbols
  $p, q, r, \ldots$
- Logical connectives
  $\neg, \vee, \wedge, \Rightarrow, \Leftrightarrow, \ldots$
- Quantifiers
  $\forall \quad \exists$
- Punctuation symbols
  $(\quad) \quad ,$

Term and Atom

A *term* is

- a variable
- a constant
- an expression $f(t_1, \ldots, t_n)$ if f is a function symbol with arity n and $t_1, \ldots, t_n$ are terms

A *atom* is an expression $p(t_1, \ldots, t_n)$ where p is a predicate symbol with arity n and $t_1, \ldots, t_n$ are terms.

Formula

A *formula* is

- an atom
- $\neg\Phi$ if Φ is a formula
- $(\Phi \wedge \Psi)$ if Φ and Ψ are formulas
- $(\Phi \vee \Psi)$ if Φ and Ψ are formulas
- $(\Pi \rightarrow \Psi)$ if Φ and Ψ are formulas
- $(\Pi \leftrightarrow \Psi)$ if Φ and Ψ are formulas
- $\ldots$
- $(\forall x)\Phi$ if x is a variable and Φ is a formula
- $(\exists x)\Phi$ if x is a variable and Φ is a formula

A *language* is a set $\mathcal{L}$ of all formulas.

Structure

A *domain* is a set of individuals D.

A *signature* is a tripple $\sigma = (F, P, arity)$ where

- F is a set of function symbols
- P is a set of predicate symbols
- $arity : F \cup P \mapsto N$ is an arity function

An *interpretation* is a function I such that

- $I(f)$ is a function $f^I : D^{arity(f)} \mapsto D$
- $I(p)$ is a relation $p^I \subseteq D^{arity(p)}$

A *structure* is a tripple $\mathcal{D} = (D, \sigma, I)$ where

- D is a domain
- σ is a signature
- I is an interpretation function

Example

$$(\forall x)p(c, x, x)$$
$$(\forall x)(\forall y)(\forall z)(p(x, g(y), z) \Leftrightarrow p(f(x), y, z))$$

- Domain $D = N$
- Signature
  $\sigma = (\{c, f, g\}, \{p\}, \{c \mapsto 0, f \mapsto 1, g \mapsto 1, p \mapsto 3\})$
- Interpretation
  $I(c) = 0$
  $I(f) = x \mapsto x + 1$
  $I(g) = x \mapsto x + 1$
  $I(p) = \{(x, y, z) \mid x + y = z\}$

Variable Assignment

A *variable assignment* is a mapping $e\colon X \mapsto D$ where X is a set of variables and D is a domain.
If $x \in X$ is a variable and $d \in D$ is an individual, then by $e(x \mapsto d)$ we will denote a variable assignment satisfying

$$e(x \mapsto d)(y) = \begin{cases} d & \text{if } x = y \\ e(y) & \text{if } x \neq y \end{cases}$$

Valuation

Let $\mathcal{D}$ be a struture and e be a variable assignment.
The *value of* a term t (denoted by $t[e]$) is

- $e(t)$ if t is a variable
- c^I if t is a constant
- $f^I(t_1[e], \ldots, t_n[e])$ if $t = f(t_1, \ldots, t_n)$ is a compound term

A formula Φ is *true* w.r.t. $\mathcal{D}$ and e (denoted by $\mathcal{D} \models \Phi[e]$) iff

- $\mathcal{D} \models p(t_1, \ldots, t_n)[e]$ iff $(t_1[e], \ldots, t_n[e]) \in p^I$
- $\mathcal{D} \models \neg\Phi[e]$ iff $\mathcal{D} \not\models \Phi[e]$
- $\mathcal{D} \models (\Phi \wedge \Psi)[e]$ iff $\mathcal{D} \models \Phi[e]$ and $\mathcal{D} \models \Psi[e]$
- $\mathcal{D} \models (\Phi \vee \Psi)[e]$ iff $\mathcal{D} \models \Phi[e]$ or $\mathcal{D} \models \Psi[e]$
- $\mathcal{D} \models (\Phi \rightarrow \Psi)[e]$ iff $\mathcal{D} \not\models \Phi[e]$ or $\mathcal{D} \models \Psi[e]$
- $\mathcal{D} \models (\Phi \leftrightarrow \Psi)[e]$ iff $\mathcal{D} \models \Phi[e]$ iff $\mathcal{D} \models \Psi[e]$
- $\mathcal{D} \models (\forall x)\Phi[e]$ iff $\mathcal{D} \models \Phi[e(x \mapsto d)]$ for all $d \in D$
- $\mathcal{D} \models (\exists x)\Phi[e]$ iff $\mathcal{D} \models \Phi[e(x \mapsto d)]$ for some $d \in D$

Entailment

A formula Φ is *true* w.r.t. a structure $\mathcal{D}$ (denoted by $\mathcal{D} \models \Phi$) iff $\mathcal{D} \models \Phi[e]$ for all variable assignments e.
A set of formulas T *entails* a formula Φ (denoted by $T \models \Phi$) iff for all structures $\mathcal{D}$ holds $\mathcal{D} \models \Phi$ whenever $\mathcal{D} \models \Psi$ for all Ψ in T.

A formula is in *negation normal form* iff if $\{\neg, \wedge, \vee\}$ are are the only allowed connectives and literals are the only negated subformulas.

A formula is in *prenex normal form* iff it is of the form $(Q_1 x_1) \dots (Q_n x_n) F$, $n \geq 0$, where Q_i is a quantifier, x_i is a variable and F is quantifier-free formula.

A formula is in *Skolem normal form* iff it is in prenex normal form with only universal quantifiers.

A formula is in *conjunctive normal form* iff it is conjunction of disjunctive clauses, where a *disjunctive clause* is a disjunction of literals.

A formula is in *disjunctive normal form* iff it is disjunction of conjunctive clauses, where a *conjunctive clause* is a conjunction of literals.

Negation Normal Form

- Double negative law:
  $\neg\neg P / P$
- De Morgan's law:
  $\neg(P \land Q) / (\neg P \lor \neg Q)$
  $\neg(P \lor Q) / (\neg P \land \neg Q)$
- Quantifiers:
  $\neg(\forall x)P / (\exists x)\neg P$
  $\neg(\exists x)P / (\forall x)\neg P$

Prenex Normal Form

- Negation:
  $\neg(\exists x)P/(\forall x)\neg P$
  $\neg(\forall x)P/(\exists x)\neg P$
- Conjunction:
  $((\forall x)P \wedge Q)/(\forall x)(P \wedge Q) \quad (Q \wedge (\forall x)P)/(\forall x)(Q \wedge P)$
  $((\exists x)P \wedge Q)/(\exists x)(P \wedge Q) \quad (Q \wedge (\exists x)P)/(\exists x)(Q \wedge P)$
  if x does not appear as free variable in Q
- Disjunction:
  $((\forall x)P \vee Q)/(\forall x)(P \vee Q) \quad (Q \vee (\forall x)P)/(\forall x)(Q \vee P)$
  $((\exists x)P \vee Q)/(\exists x)(P \vee Q) \quad (Q \vee (\exists x)P)/(\exists x)(Q \vee P)$
  if x does not appear as free variable in Q
- Implication:
  $((\forall x)P \rightarrow Q)/(\exists x)(P \rightarrow Q) \quad (Q \rightarrow (\forall x)P)/(\forall x)(Q \rightarrow P)$
  $((\exists x)P \rightarrow Q)/(\forall x)(P \rightarrow Q) \quad (Q \rightarrow (\exists x)P)/(\exists x)(Q \rightarrow P)$
  if x does not appear as free variable in Q

Skolem Normal Form

Formulas P and Q are *equisatisfiable* if P is satisfiable if and only if Q is satisfiable.

Given a formula F:

1. If F is already in Skolem normal form, we are done.
2. If not, then F is of the form

$$(\forall x_1)\dots(\forall x_m)(\exists y)F'(x_1,\dots,x_m,y,z_1,\dots,z_n)$$

   where each z_i is a free variable and F' is in prenex normal form. Replace y with $f(x_1,\dots,x_m,z_1,\dots,z_n)$ where f is a new function symbol.

Conjunctive Normal Form

1. Negation Normal Form
2. Prenex Normal Form
3. Skolem Normal Form
4. Distributive law ($\vee$ over $\wedge$):
   $((P \wedge Q) \vee R)/(P \vee R) \wedge (Q \vee R)$
   $(P \vee (Q \wedge R))/(P \vee Q) \wedge (P \vee R)$

Disjunctive Normal Form

1. Negation Normal Form
2. Prenex Normal Form
3. Skolem Normal Form
4. Distributive law ($\wedge$ over $\vee$):
   $((P \vee Q) \wedge R) / (P \wedge R) \vee (Q \wedge R)$
   $(P \wedge (Q \vee R)) / (P \wedge Q) \vee (P \wedge R)$