Computer Graphics Course

Three-Dimensional Modeling

Plan of lectures

1. Objects in Three-Dimensional Space

- Points, curves, surfaces and solids
- Parametric and implicit geometric objects
- Quadrics and superquadrics
- Blobby objects
- Volumetric objects
- Isosurfaces

2. Three-Dimensional Transformations

- Affine transformations (translation, rotation, scaling)
- Deformations (twisting, bending, tapering)
- Set-theoretic operations
- Offsetting and blending
- Metamorphosis
- Collision detection

3. Representations of Solids

- Boundary representation
- Constructive Solid Geometry (CSG)
- Octrees
- Sweeps
- Function representation

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Dimension of objects

We use the term "**geometric object**" or simply "**object**" to denote a point set in **n**-dimensional Euclidean space.

An object is **k**-dimensional if there is a continuous one-to-one mapping of the **k**-dimensional square on this object.

Dimension of an object $\mathbf{n} = 3, \mathbf{k} \le \mathbf{n}$	Object
0	Point
1	Curve
2	Surface
3	Solid

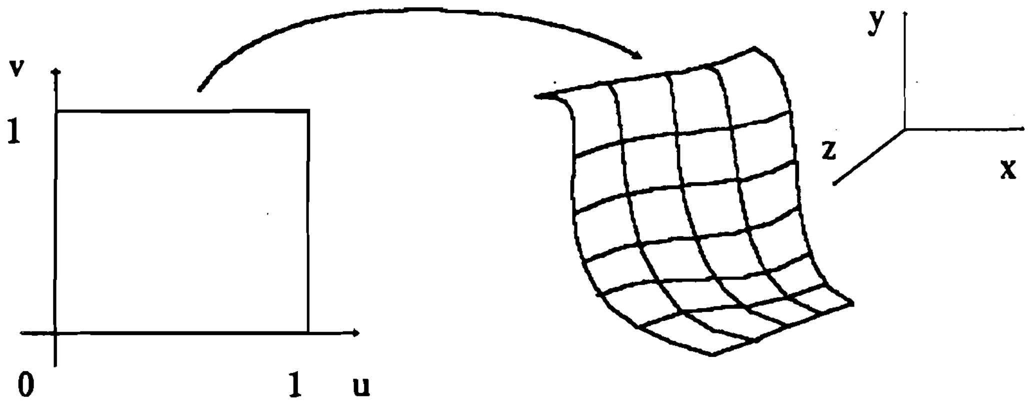
Parametric geometric objects

Curve

$x = x(t)$
$y = y(t)$
$z = z(t)$

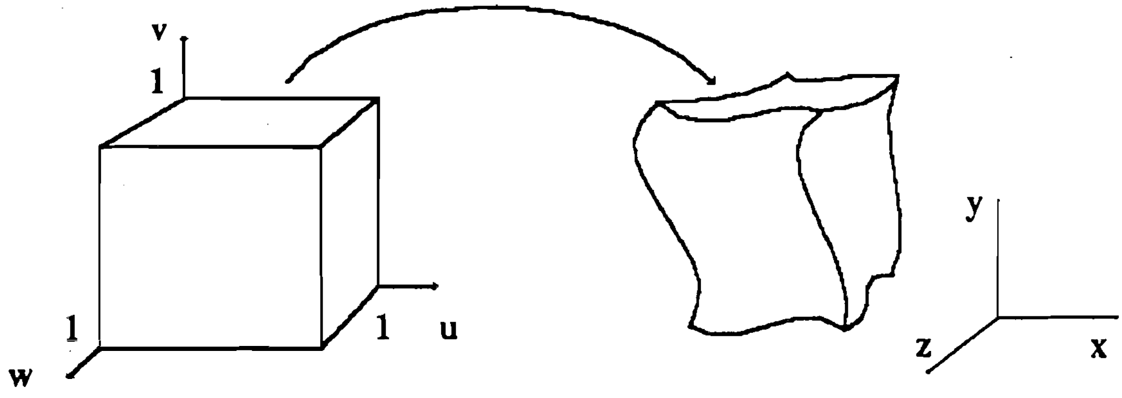
Surface

$x = x(u, v)$
$y = y(u, v)$
$z = z(u, v)$



Solid

$x = x(u, v, w)$
$y = y(u, v, w)$
$z = z(u, v, w)$



Implicit geometric objects

Let $\mathbf{f(x_1,x_2, ..., x_n)}$ be a continuous real function of n variables. Implicit objects are defined in n-dimensional space as follows:

Solid (k = n) $f(x_1,x_2, ..., x_n) \geq 0$

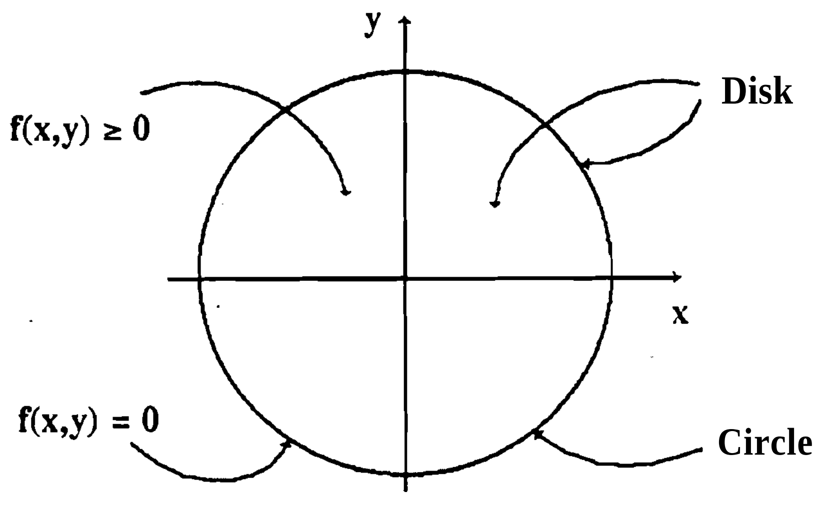
Others (k < n) $f(x_1,x_2, ..., x_n) = 0$

<u>Two-dimensional example</u>

$$f(x,y) = R^2 - x^2 - y^2$$

Disk (k=2) $f(x,y) \geq 0$

Circle(k=1) $f(x,y) = 0$



Planes and planar halfspaces

$$f(x,y,z) = Ax + By + Cz + D$$

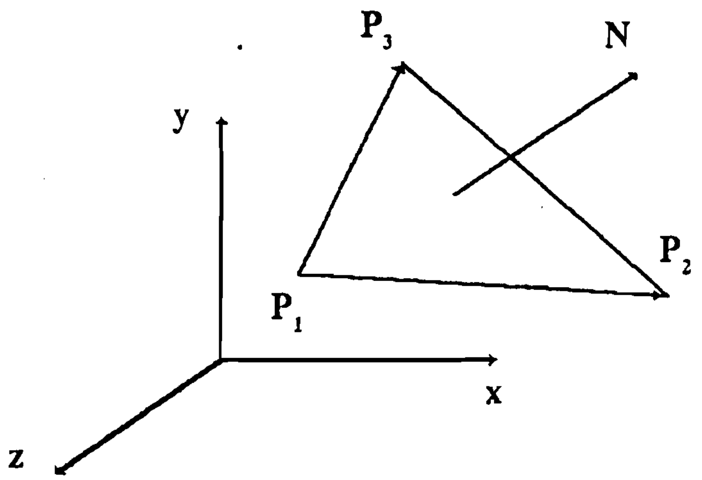
Plane (k = 2) $f(x,y,z) = 0$

Planar halfspace (k = 3) $f(x,y,z) \geq 0$

The normal vector N orthogonal to the plane:

$$\mathbf{N} = (A, B, C)$$

<u>Plane equation for a triangle</u>



Triangle $P_1P_2P_3$ with $P_i = (x_i, y_i, z_i)$

1) Vectors $\mathbf{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$

and $P_1P_3 = (x_3 - x_1, y_3 - y_1, z_3 - z_1)$

2) $N = \mathbf{P_1P_2} \times \mathbf{P_1}P_3$ is the vector cross product

3) With obtained **A,** B, C get the equation for D:

$$Ax_1 + By_1 + Cz_1 + D = 0$$

Quadrics

General equation

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + Gx + Hy + Pz + R = 0$$

Types

(1)	Real ellipsoid	$x^2/a^2+y^2/b^2+z^2/c^2=1$
(2)	Imaginary ellipsoid	$x^2/a^2+y^2/b^2+z^2/c^2=-1$
(3)	Hyperboloid of one sheet	$x^2/a^2+y^2/b^2-z^2/c^2=1$
(4)	Hyperboloid of two sheets	$x^2/a^2+y^2/b^2-z^2/c^2=-1$
(5)	Real quadric cone	$x^2/a^2+y^2/b^2-z^2/c^2=0$
(6)	Imaginary quadric cone	$x^2/a^2+y^2/b^2+z^2/c^2=0$
(7)	Elliptic paraboloid	$x^2/a^2+y^2/b^2+2z=0$
(8)	Hyperbolic paraboloid	$x^2/a^2-y^2/b^2+2z=0$
(9)	Real elliptic cylinder	$x^2/a^2+y^2/b^2=1$
(10)	Imaginary elliptic cylinder	$x^2/a^2+y^2/b^2=-1$
(11)	Hyperbolic cylinder	$x^2/a^2-y^2/b^2=1$
(12)	Real intersecting planes	$x^2/a^2-y^2/b^2=0$
(13)	Imaginary intersecting planes	$x^2/a^2+y^2/b^2=0$
(14)	Parabolic cylinder	$x^2+2y=0$
(15)	Real parallel planes	$x^2=1$
(16)	Imaginary parallel planes	$x^2=-1$
(17)	Coincident planes	$x^2=0$

Quadrics

Ellipsoid

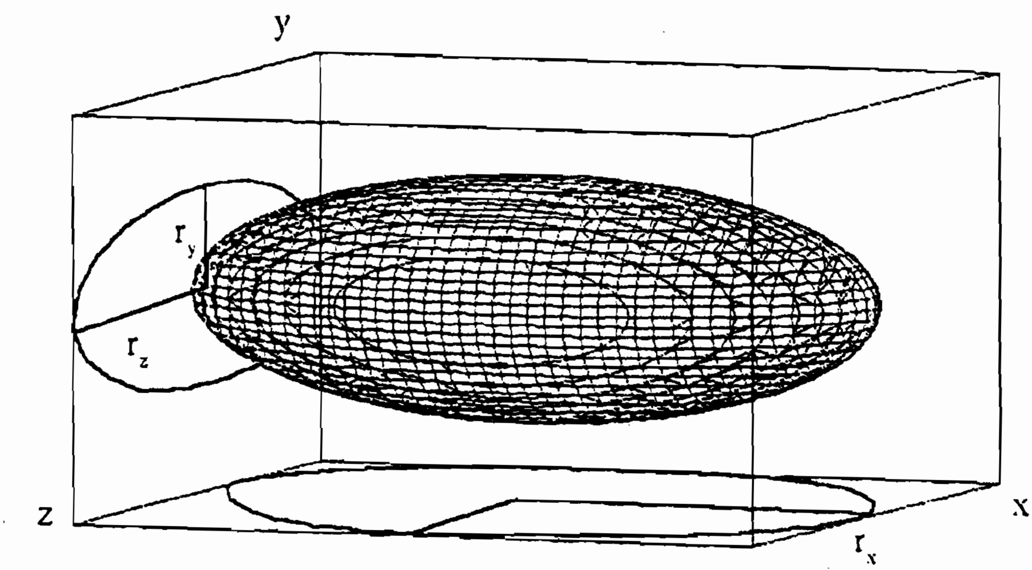
Implicit form:

$$f(x,y,z) = 1 - \left(\frac{x}{r_x}\right)^2 - \left(\frac{y}{r_y}\right)^2 - \left(\frac{z}{r_z}\right)^2$$

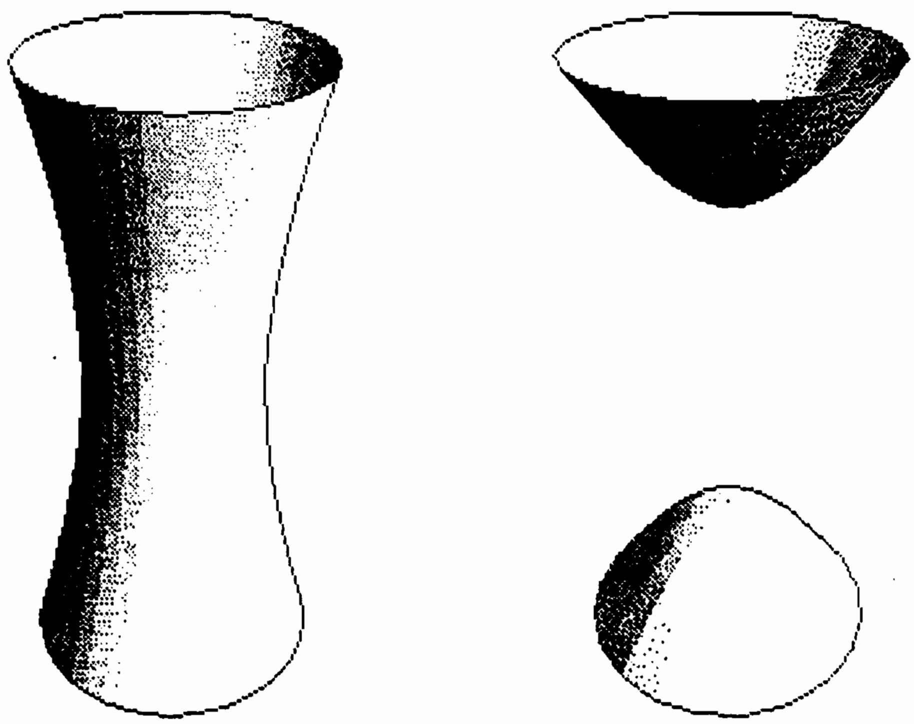
Parametric form:

$$\begin{aligned} x &= r_x \cos\phi \cos\theta, \quad -\pi/2 \le \phi \le \pi/2 \\ y &= r_y \cos\phi \sin\theta, \quad -\pi \le \theta \le \pi \\ z &= r_x \sin\phi \end{aligned}$$

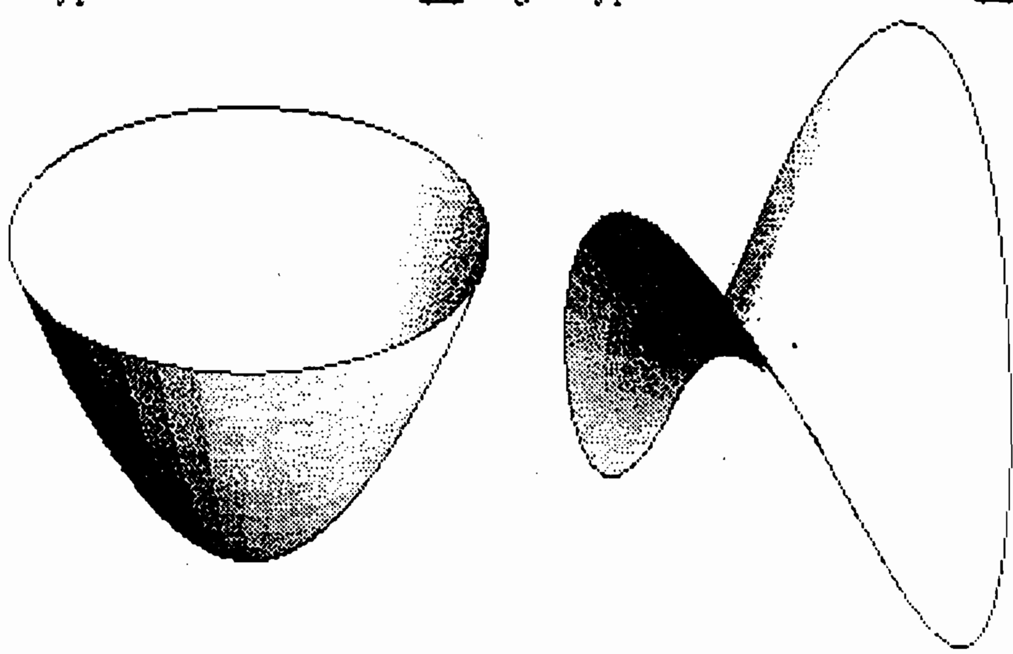
Sphere $r_x = r_y = r_z$



Quadrics



Left: hyperboloid of one sheet (3). Right: hyperboloid of two sheets (4).



Left: elliptic paraboloid (7). Right: hyperbolic paraboloid (8).

Superquadrics

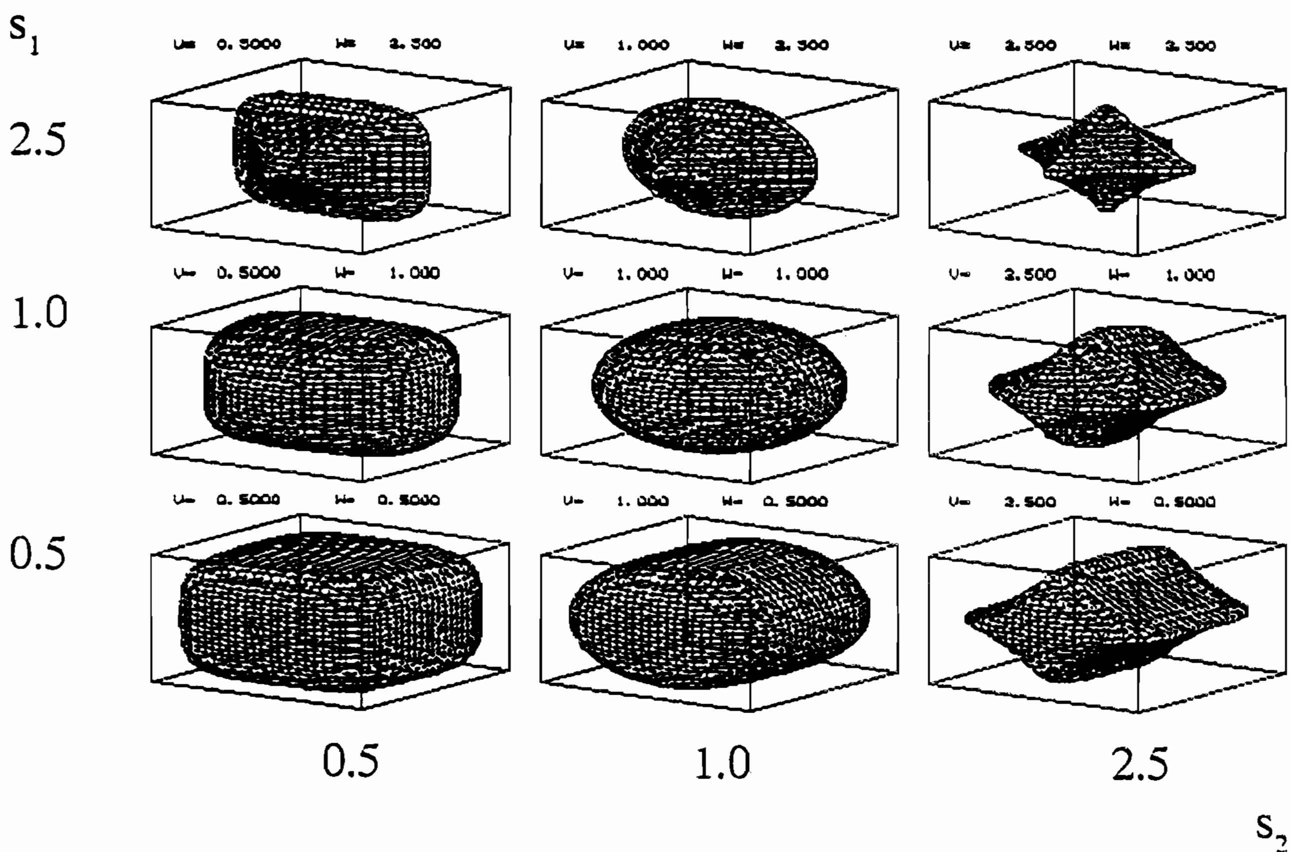
Superellipsoid

Implicit form:

$$f(x,y,z) = 1 - \left[\left(\frac{x}{r_x}\right)^{2/s_2} + \left(\frac{y}{r_y}\right)^{2/s_2}\right]^{s_2/s_1} - \left(\frac{z}{r_z}\right)^{2/s_1}$$

Parametric form:

$$\begin{aligned} x &= r_x \cos^{s_1}\phi \cos^{s_2}\theta, \quad -\pi/2 \le \phi \le \pi/2 \\ y &= r_y \cos^{s_1}\phi \sin^{s_2}\theta, \quad -\pi \le \theta \le \pi \\ z &= r_x \sin^{s_1}\phi \end{aligned}$$



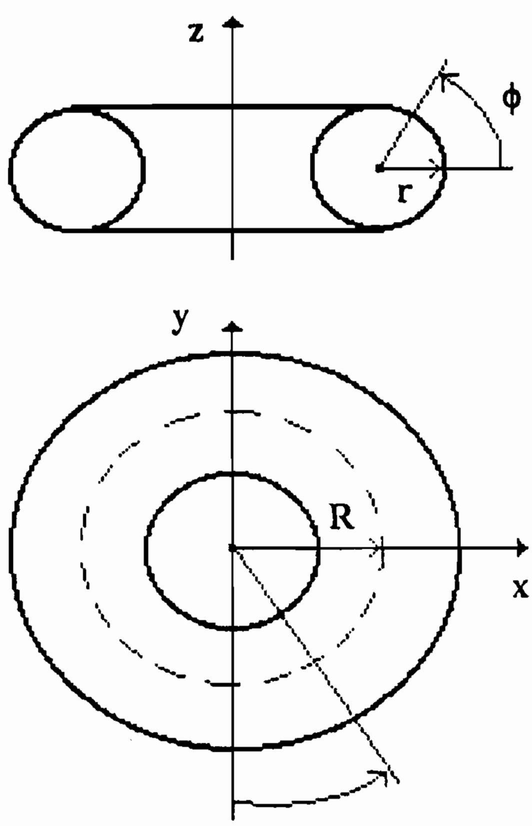
Torus

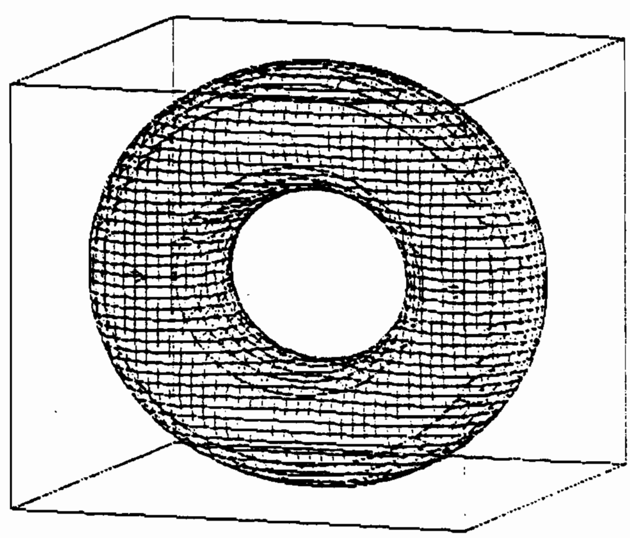
Implicit form:

$$f(x,y,z) = r^2 - x^2 - y^2 - z^2 - R^2 + 2R\sqrt{x^2 + y^2}$$

Parametric form:

$$x = (R + r\cos\phi)\cos\theta, \quad -\pi \le \phi \le \pi$$
$$y = (R + r\cos\phi)\sin\theta, \quad -\pi \le \theta \le \pi$$
$$z = r\sin\phi$$





Blobby objects

- Implicit form only
- Natural shape blending
- Used to represent molecular shapes, liquid and melting objects, human body

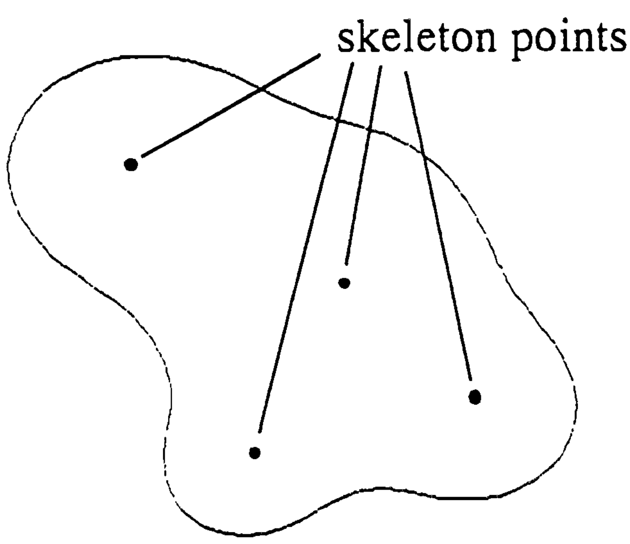
General formulation:

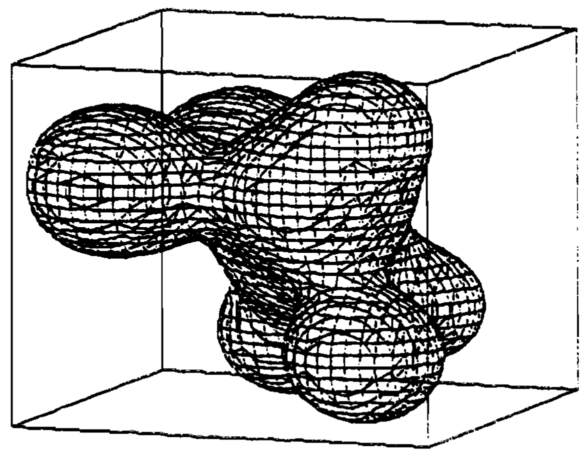
$$f(x,y,z) = \sum_k f(r_k) - T$$

$$r_k = \sqrt{(x-x_k)^2 + (y-y_k)^2 + (z-z_k)^2}$$

Original "blobby model" (Blinn 1982):

$$f(r_k) = b_k e^{-a_k r_k^2} \quad \text{or} \quad f(r_k) = b_k e^{-a_k r_k}$$



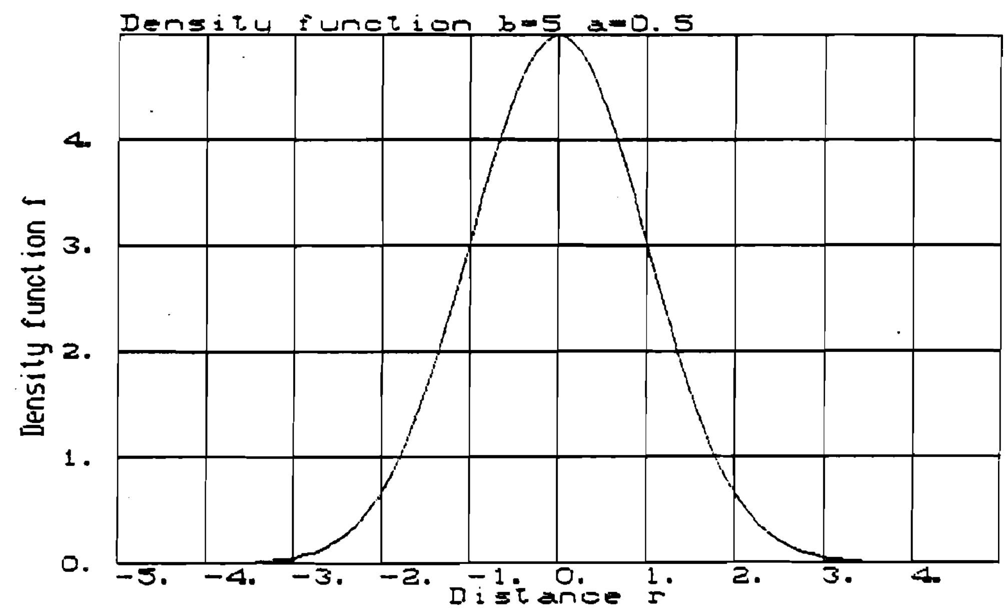


Blobby objects

Density functions

Original "blobby model" (Blinn 1982):

$$f(r) = be^{-ar^2}$$



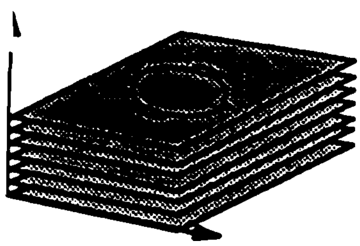
"Metaballs" (Nishimura et al. 1985):

$$f(r) = \begin{cases} b(1 - 3r^2/d^2), & 0 < r \le d/3 \\ 1.5b(1 - r/d), & d/3 < r \le d \\ 0, & r > d \end{cases}$$

"Soft objects" (Wyvill et al. 1.986):

$$f(r) = \begin{cases} 1 - \dfrac{22r^2}{9d^2} + \dfrac{17r^4}{9d^4} - \dfrac{4r^6}{9d^6} & 0 < r \le d \\ 0, & r > d \end{cases}$$

Volumetric objects



x_i	y_i	z_i	Density
0.00	0.00	0.00	243
0.00	0.00	0.12	175
•	•	•	•
0.00	0.00	1.00	186
0.00	0.12	0.00	187
•	•	•	•

$$F_{ijk} = F(X_i, Y_j, Z_k)$$

$i = 1, \ldots, N \quad j = 1, \ldots, N \quad k = 1, \ldots, N$

Medical Scanners, MRI, PET, ect.

Numerical value is defined in the nodes of 3D grid: density, temperature, pressure, and so on. Object is defined by F>O.

Source of data:

- Computer tomography
  Numerical simulation
  Manual sculpting (like 3D drawing with a brush)
  Voxelized geometric primitives:

$$F_{ijk} = 1, \text{ if } f(x_i, y_j, z_k) \geq 0$$
$$F_{ijk} = 0, \text{ if } f(x_i, y_j, z_k) < 0$$

Isosurfaces

A surface defined as

$$F(x,y,z) = C$$

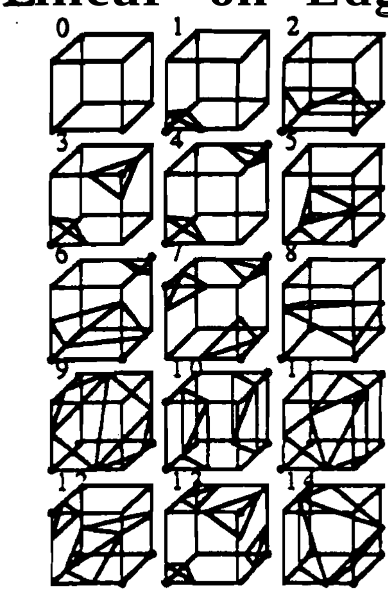
is called the "isosurface" of the function of three variables. The case C = 0 - implicit surfaces.

Extraction of isosurfaces from volume data and conversion them to polygon meshes is called "isosurface polygonization".

Well-known polygonization algorithm:

'Marching Cubes

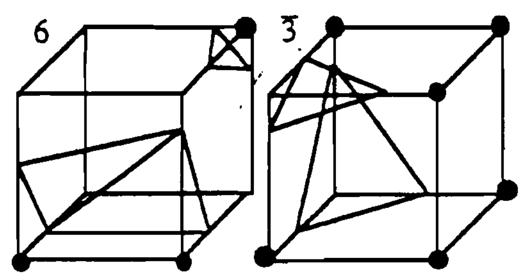
Linear on Edges



Volume Visualization

Lists all possible cases and forms a look-up table.

Problems:



The main problem: intersection of an isosurface with four edges of a cell face - ambiguous solutions.