Restricted Unification in the Description Logic EL

**SYNTAX**
Let $N_C$ and $N_R$ are disjoint infinite sets of concept and role names, the sets of EL and FL₀ concepts is inductively defined as follows:

**EL:**
- $\forall r.C$ (conjunction),
- $r \in N_R$ (top concept),
- $\exists r.C$ (existential restriction).

**FL₀:**
- $\forall r.C$ (conjunction),
- $r \in N_R$ (top concept),
- $\forall r.C$ (value restriction).

**SEMANTICS**
An interpretation $I$ has a domain $\Delta^I$ and associates:
- concepts $C$ with set $C^I$, and
- roles $r$ with binary relations $r^I$.

The semantics of the constructors in EL and FL₀ are defined through identities:

**EL:**
- $\forall r.I = \Delta^I$,
- $(C \sqcap D)^I = C^I \cap D^I$,
- $(\exists r.C)^I = \{ x \in \Delta^I \mid \exists y \in \Delta^I : (x,y) \in r^I, y \in C^I \}$.

**FL₀:**
- $\forall r.I = \Delta^I$,
- $(C \sqcap D)^I = C^I \cap D^I$,
- $(\forall r.C)^I = \{ x \in \Delta^I \mid \forall y \in \Delta^I : (x,y) \in r^I, y \in C^I \}$.

**UNIFICATION PROBLEM**
-A substitution $\sigma$ is a mapping from $N_r$ into the set of all EL or FL₀ concept patterns such that $\text{dom}(\sigma) \cap N_r \neq \emptyset$ and $\{X \mid \sigma(X) = X\}$ is finite. This mapping is extended to concept patterns in the obvious way:

$\sigma(A) = \sigma$ for all $A \in N_C \cup \{ \top \}$,
$\sigma(C \sqcap D) = \sigma(C) \sqcap \sigma(D)$,
$\sigma(\exists r.C) = \exists r. \sigma(C)$,
$\sigma(\forall r.C) = \forall r. \sigma(C)$.

- An EL (or FL₀) unification problem is an equation of the form $C \equiv D$ where $C$, $D$ are EL or FL₀ concept patterns. A unifier (or solution) of this equation is a substitution $\sigma$ such that $(C \equiv D) \sigma$ holds in EL (or FL₀).

**UNIFICATION IN THE DL EL AND FL₀**
For the classical setting of unification, both EL and FL₀ behave similarly:

- Both have “bad” unification type 0.
- Both have “intractable” (i.e., not polynomial) complexity of unification.

**SYNTACTICALLY RESTRICTED UNIFICATION**
 syntactically k-restricted unification problem is an equation of the form $C \equiv_D^k D$, where $C$, $D$ are EL (or FL₀) concept patterns. A unifier of this equation (also called syntactically k-restricted unifier) is a substitution $\sigma$ such that $(C \equiv D) \sigma$ in EL (or FL₀).

**SEMANTICALLY RESTRICTED UNIFICATION**
Semantically k-restricted unification problem is an equation of the form $C \equiv_D^k D$, where $C$, $D$ are EL (or FL₀) concept patterns. A unifier of this equation (also called semantically k-restricted unifier) is a substitution $\sigma$ such that $(C \equiv D) \sigma$ holds in EL (or FL₀).

- The unified concepts needs to have a role depth $\leq k$.

**SYNTACTICALLY RESTRICTING THE ROLE DEPTH**
-The role depth of an EL concept is the maximal nesting of existential restrictions in this concept. Hence,
- $\text{rd}(T) = \text{rd}(A) = 0$ for all $A \in N_C$,
- $\text{rd}(C \sqcap D) = \max(\text{rd}(C), \text{rd}(D))$,
- $\text{rd}(\exists r.C) = 1 + \text{rd}(C)$.

- Subsumption and equivalence restricted to concepts of role depth $\leq k$ as follows:

$\exists r.C \equiv_D^k D$ if $C \sqsubseteq D$ and $\text{rd}(C) \leq k$,
$C \equiv_D^k D$ if $C \equiv_D^k D$ and $D \equiv_D^k D$.

**SEMANTICALLY RESTRICTING THE LENGTH OF ROLE PATHS**
-The role path of length $n$ is a sequence $d_0, r_1, d_2, \ldots, d_{n-1}, r_n, d_n$.
- The interpretation $I$ is called $k$-restricted if it does not admit any role paths of length $> k$.

- Subsumption and equivalence restricted to interpretations with role paths of length $\leq k$ defined as follows:

$\exists r.C \equiv_D^k D$ if $C \sqsubseteq D$ holds for all $k$-restricted interpretations $I$,
$C \equiv_D^k D$ if $C \equiv_D^k D$ and $D \equiv_D^k D$.

**CONCLUSION**
- We have investigated both a semantically and a syntactically restricted variant of unification in EL, where either the role depth of concepts or the length of role paths in interpretations is restricted by a natural number $k \geq 1$.
- The decision problem for both a semantically and a syntactically restricted variant of unification in EL is NP-complete.

**RESTRIC TED UNIFICATION IN EL**

- The unification type of FL₀ improves when going to the restricted setting of unification.
- In the k-restricted setting, finite complete sets of unifiers always exist.
- Both syntactically and semantically k-restricted unification in FL₀ is of type finitary (unitary) for unification with constant (without constant).
- The complexity of deciding k-restricted unifiability depends on the coding of the number $k$.

**SYNTACTIC EXP. TIME**
- In ExpTime-
- In PSpace-complete

**SEMANTIC EXP. TIME**
- In ExpTime-
- In PSpace-complete

**RESTRIC TED UNIFICATION IN FL₀**
- The unification type of FL₀ does not improve when going to the restricted setting of unification.
- In the k-restricted setting.
- Both syntactically and semantically k-restricted unification in FL₀ is of type Zero.
- The complexity of deciding k-restricted unifiability stays NP-complete for both syntactically and semantically cases.

**why unification is relevant for DLs?**
Unification of concept patterns has been proposed as a non-standard inference service in DL that can for example be used to detect redundancies in ontologies.

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