

Box 2.4 | The GHK equations

Goldman (1943) and Hodgkin and Katz (1949) developed a formalism for describing the currents through and voltages across semipermeable membranes. This formalism models the diffusion of ions through a uniformly permeable membrane, predating the notion of channels or pores through the membrane. It is assumed that ions cross the membrane independently (the **independence principle**) and that the electric field within the membrane is constant. The flux or movement of ions within the membrane is governed by the internal concentration gradient and the electric field arising from the potential difference, calculated by the Nernst–Planck equation.

From these assumptions, the Goldman–Hodgkin–Katz current equation can be derived (Johnston and Wu, 1995):

$$I_X = P_X z_X F \frac{z_X F V}{RT} \left(\frac{[X]_{\text{in}} - [X]_{\text{out}} e^{-z_X F V / RT}}{1 - e^{-z_X F V / RT}} \right).$$

This equation predicts the net flow I_X per unit area of membrane, measured in cm^{-2} of an arbitrary ion type X with valency z_X . P_X is the **permeability** of the membrane to ion X, with units of cm s^{-1} . It characterises the ability of an ion X to diffuse through the membrane and is defined by the empirical relationship between molar flux J and the concentration difference across the membrane:

$$J_X = -P_X ([X]_{\text{in}} - [X]_{\text{out}}).$$

In the GHK model of the membrane, permeability is proportional to the diffusion coefficient, D_X , defined in Fick's first law (Equation 2.2). Hille (2001) discusses the relationship in more detail.

The GHK equation predates the notion of membrane channels and treats the membrane as homogeneous. In active membranes we can interpret the diffusion coefficient, D_X , as variable – an increase in the number of open channels in the membrane will increase the membrane permeability. Because of the assumption of a constant electric field in the membrane, the GHK equations are sometimes referred to as the **constant-field equations**.

2.4.1 An electrical circuit approximation of the GHK current equation

It is often sufficient to use a simpler equation in place of the GHK current equation. In the potassium characteristic shown in Figure 2.11a, the straight line that gives zero current at the equilibrium potential (-72 mV) is a close approximation of the I – V characteristic for membrane potentials between about -100 mV and 50 mV , the voltage range within which cells normally operate. The equation describing this line is:

$$I_X = g_X (V - E_X) \quad (2.10)$$

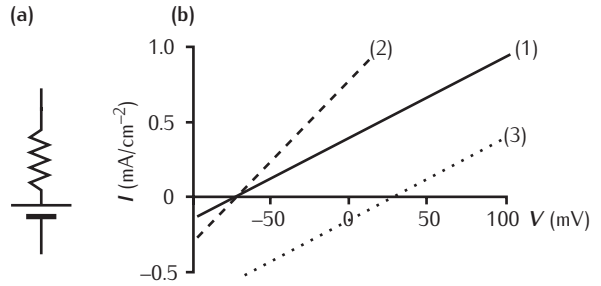
where X is the ion of interest, E_X its equilibrium potential, and g_X is the gradient of the line with the units of conductance per unit area, often mS cm^{-2} . The term in brackets ($V - E_X$) is called the **driving force**. When the membrane potential is at the equilibrium potential for X, the driving force is zero.

Fig. 2.12 Interpretation of the approximation of the GHK current equation. **(a)** The approximation can be viewed as a resistor, or conductance, in series with a battery. **(b)** The graph shows three different I - V characteristics from this circuit given different conductances and battery voltages.

(1) $g_X = 5.5 \text{ mS cm}^{-2}$,
 $E_X = -72 \text{ mV}$; this line is the same as the K^+ approximation in Figure 2.11a;

(2) $g_X = 11.0 \text{ mS cm}^{-2}$,
 $E_X = -72 \text{ mV}$;

(3) $g_X = 5.5 \text{ mS cm}^{-2}$,
 $E_X = 28 \text{ mV}$.



In some cases, such as for calcium in Figure 2.11b, the GHK I - V characteristic rectifies too much for a linear approximation to be valid.

Making this linear approximation is similar to assuming Ohm's law, $I = GV$, where conductance G is a constant. Since the straight line does not necessarily pass through the origin, the correspondence is not exact and this form of linear I - V relation is called **quasi-ohmic**. There is still a useful interpretation of this approximation in terms of electrical components. The I - V characteristic is the same as for a battery with **electromotive force** equal to the equilibrium potential in series with a resistor of resistance $1/g_X$ (Figure 2.12).

2.5 | The capacitive current

We now have equations that describe how the net flow of current I through the different types of channels depends on the membrane potential V . In order to complete the description of the system, we need to know how the current affects the voltage.

All the current passing through the membrane either charges or discharges the membrane capacitance. So the rate of change of charge on the membrane dq/dt is the same as the net current flowing through the membrane: $I = dq/dt$. By differentiating Equation 2.1 for the charge stored on a capacitor with respect to time, we obtain a differential equation that links V and I :

$$\frac{dV}{dt} = \frac{I}{C} = \frac{1}{C} \frac{dq}{dt}. \quad (2.11)$$

This shows that the rate of change of the membrane potential is proportional to the current flowing across the membrane. The change in voltage over time, during the charging or discharging of the membrane, is inversely proportional to the capacitance – it takes longer to charge up a bigger capacitor.

2.6 | The equivalent electrical circuit of a patch of membrane

We have seen how we can represent the permeable and impermeable properties of the membrane as electrical components. Figure 2.13 shows how these

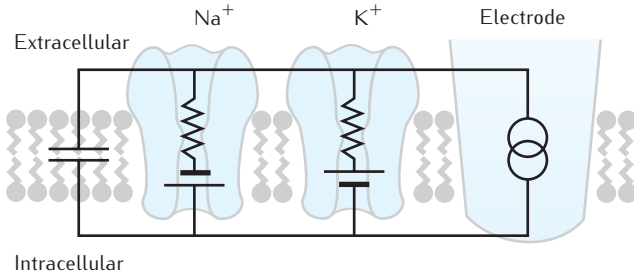


Fig. 2.13 The equivalent electrical circuit of a patch of membrane.

components fit together to form an **equivalent electrical circuit** of a small patch of membrane. It comprises the membrane capacitance in parallel with one resistor and battery in series for each type of ion channel. There is also a current source that represents an electrode that is delivering a constant amount of current. It is said to be in **current clamp mode**. The amount of current injected is denoted by I_e , and in electrophysiological applications is usually measured in nanoamps (nA).

For the remainder of this chapter, we consider a membrane that contains passive ion channels, with constant permeability or conductance. In general, ion channels are active, so their permeability changes in response to changes in membrane potential. It is useful to consider passive membranes as a first step towards understanding the behaviour of active membranes. In addition, for small deviations of the membrane potential from the resting potential, active channels can be treated as passive channels.

2.6.1 Simplification of the equivalent electrical circuit

We can simplify the electrical circuit representing a patch of passive membrane, such as the circuit shown in Figure 2.13, by lumping together all of the channel properties. Figure 2.14a shows this simplified circuit. In place of the two resistor/battery pairs in Figure 2.13, there is one pair with a resistance, which we call the specific membrane resistance R_m , measured in Ωcm^2 , and a membrane battery with an electromotive force of E_m .

We can derive these values from the conductances and reversal potentials of the individual ions using **Thévenin's theorem**. For channels X, Y and Z combined, the equivalent electromotive force and membrane resistance are:

$$E_m = \frac{g_X E_X + g_Y E_Y + g_Z E_Z}{g_X + g_Y + g_Z} \quad (2.12)$$

$$\frac{1}{R_m} = g_m = g_X + g_Y + g_Z.$$

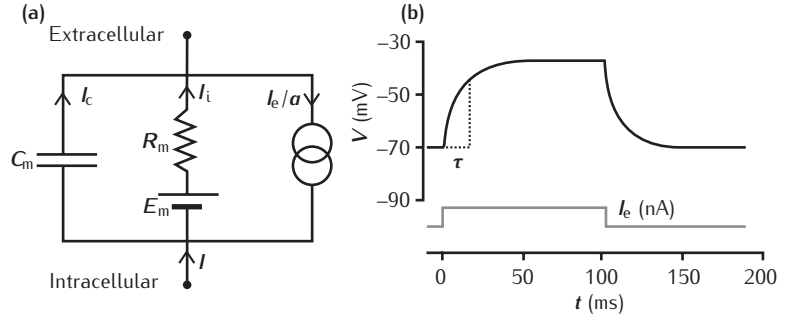
Note that Equation 2.12 is the ohmic equivalent of the GHK voltage equation Equation 2.9.

A summary of key passive quantities and their typical units is given in Table 2.3. It is usual to quote the parameters of the membrane as **intensive quantities**. To avoid adding extra symbols, we use intensive quantities in our electrical circuits and equations. Supposing that the area of our patch of membrane is a , its membrane resistance is proportional to the specific

Thévenin's theorem states that any combination of voltage sources and resistances across two terminals can be replaced by a single voltage source and a single series resistor. The voltage is the open circuit voltage E at the terminals and the resistance is E divided by the current with the terminals short circuited.

An **intensive quantity** is a physical quantity whose value does not depend on the amount or dimensions of the property being measured. An example of an intensive quantity is the specific membrane capacitance, the capacitance per unit area of membrane.

Fig. 2.14 (a) The electrical circuit representing a passive patch of membrane. (b) The behaviour of the membrane potential in an RC circuit in response to an injected current pulse, shown below.



Kirchhoff's current law is based on the principle of conservation of electrical charge. It states that at any point in an electrical circuit, the sum of currents flowing toward that point is equal to the sum of currents flowing away from that point.

membrane resistance divided by the area: R_m/a . Since conductance is the inverse of resistance, the membrane conductance of the patch is proportional to area: $g_m a$; its membrane capacitance is proportional to the specific membrane capacitance: $C_m a$. Current (for example, current crossing the membrane) is given by the current density I which has units $\mu\text{A cm}^{-2}$ multiplied by the area: Ia .

2.6.2 The RC circuit

The simplified circuit shown in Figure 2.14a is well known in electronics, where it is called an RC circuit, since its main elements are a resistor R and a capacitor C . In order to find out how the membrane potential changes when current is injected into the circuit, we need to know how current varies with voltage. By **Kirchhoff's current law**, the sum of the current Ia flowing through the membrane and the injected current I_e is equal to the sum of the capacitive current $I_c a$ and the ionic current $I_i a$:

$$\begin{aligned} Ia + I_e &= I_c a + I_i a \\ I + I_e/a &= I_c + I_i. \end{aligned} \quad (2.13)$$

The ionic current flowing through the resistor and battery is given by the quasi-ohmic relation in Equation 2.10:

$$\begin{aligned} I_i a &= \frac{V - E_m}{R_m/a} \\ I_i &= \frac{V - E_m}{R_m}. \end{aligned} \quad (2.14)$$

Finally, the capacitive current is given by the membrane capacitance multiplied by the rate of change of voltage (Section 2.5):

$$I_c = C_m \frac{dV}{dt}. \quad (2.15)$$

If this circuit is isolated, i.e. the membrane current Ia is zero, substituting for I_i , and I_c in Equation 2.13 for this RC circuit gives:

$$C_m \frac{dV}{dt} = \frac{E_m - V}{R_m} + \frac{I_e}{a}. \quad (2.16)$$

This is a first order **ordinary differential equation** (ODE) for the membrane potential V . It specifies how, at every instant in time, the rate of

change of the membrane potential is related to the membrane potential itself and the current injected. For any particular form of injected current pulse and initial membrane potential, it determines the time course of the membrane potential.

2.6.3 Behaviour of the RC circuit

Solving the differential equation is the process of using this equation to calculate how the membrane potential varies over time. We can solve Equation 2.16 using numerical methods. Appropriate numerical methods are programmed into neural simulation computer software, such as **NEURON** or **GENESIS**, so it is not strictly necessary to know the numerical methods in depth. However, a basic understanding of numerical methods is useful and we present an overview in Appendix B. Figure 2.14b shows the result of solving the equation numerically when the injected current is a square pulse of magnitude I_e and duration t_e . On the rising edge of the pulse the membrane potential starts to rise steeply. This rise away from the resting potential is referred to as **depolarisation**, because the amount of positive and negative charge on the membrane is reducing. As the pulse continues, the rise in voltage becomes less steep and the voltage gets closer and closer to a limiting value. On the falling edge of the pulse the membrane potential starts to fall quite steeply. The rate of fall decreases as the membrane potential gets close to its original value. As the charge on the membrane is building back up to resting levels, this phase is called **repolarisation**. By injecting negative current, it is possible to reduce the membrane potential below its resting level, which is referred to as **hyperpolarisation**.

Generally, it is difficult, and often not possible, to solve differential equations analytically. However, Equation 2.16 is sufficiently simple to allow an analytical solution. We assume that the membrane is initially at rest, so that $V = E_m$ at time $t = 0$. We then integrate Equation 2.16 to predict the response of the membrane potential during the current pulse, giving:

$$V = E_m + \frac{R_m I_e}{a} \left(1 - \exp \left(-\frac{t}{R_m C_m} \right) \right). \quad (2.17)$$

This is an inverted decaying exponential that approaches the steady state value $E_m + R_m I_e / a$ as time t gets very large. Defining V_0 as the value the membrane potential has reached at the end of the current pulse at $t = t_e$, the response of the membrane is given by:

$$V = E_m + (V_0 - E_m) \exp \left(-\frac{t - t_e}{R_m C_m} \right), \quad (2.18)$$

which is a decaying exponential.

In both rising and falling responses, the denominator inside the exponential is the product of the membrane resistance and membrane capacitance $R_m C_m$. This factor has the units of time, and it characterises the length of time taken for the membrane potential to get to $1/e$ (about one-third) of the way from the final value. For this reason the product $R_m C_m$ is defined as the **membrane time constant** τ . It is a measure of how long the membrane ‘remembers’ its original value. Typical values of τ for neurons range between

NEURON and **GENESIS** are two well known open source neural simulators which allow numerical solutions to the differential equations describing the spatiotemporal variation in the neuron membrane potential to be obtained. These simulators can be applied to a single neuron or a network of interconnected neurons. Appendix A.1 contains a comprehensive list of neural simulators.

Solving an equation analytically means that an expression for how the membrane potential (in this case) depends on position and time can be derived as a function of the various parameters of the system. The alternative is to solve the equation numerically.

Table 2.3 Passive quantities

Quantity	Description	Typical units	Relationships
d	Diameter of neurite	μm	
l	Length of compartment	μm	
R_m	Specific membrane resistance	$\Omega \text{ cm}^2$	
C_m	Specific membrane capacitance	$\mu\text{F cm}^{-2}$	
R_a	Specific axial resistance (resistivity)	$\Omega \text{ cm}$	
r_m	Membrane resistance per inverse unit length	$\Omega \text{ cm}$	$r_m = \frac{R_m}{\pi d}$
c_m	Membrane capacitance per unit length	$\mu\text{F cm}^{-1}$	$c_m = C_m \pi d$
r_a	Axial resistance per unit length	Ω/cm^{-1}	$r_a = \frac{4R_a}{\pi d^2}$
V	Membrane potential	mV	
E_m	Leakage reversal potential due to different ions	mV	
I	Membrane current density	$\mu\text{A cm}^{-2}$	
I_e	Injected current	nA	
I_c	Capacitive current	nA	
I_i	Ionic current	nA	

The units of R_m and R_a can often seem counter-intuitive. It can sometimes be more convenient to consider their inverse quantities, specific membrane conductance and specific intracellular conductance. These have units of S cm^{-2} and S cm^{-1} respectively. The quantities r_m , r_a , and c_m are useful alternatives to their specific counterparts. They express key electrical properties of a neurite of specific diameter and can clarify the equations representing a specific cable or neurite of arbitrary length.

1 and 20 ms. It is possible to measure the membrane time constant for use in a model RC type circuit. The assumptions that are made when doing this and the effects of measurement accuracy are discussed in Chapter 4.

Another important quantity that characterises the response of neurons to injected current is the **input resistance**, defined as the change in the steady state membrane potential divided by the injected current causing it (Koch, 1999). To determine the input resistance of any cell in which current is injected, the resting membrane potential is first measured. Next, a small amount of current I_e is injected, and the membrane potential is allowed to reach a steady state V_∞ . The input resistance is then given by:

$$R_{\text{in}} = \frac{V_\infty - E_m}{I_e}. \quad (2.19)$$

For a single RC circuit representation of a cell, the input resistance can be calculated from the properties of the cell. From Equation 2.16, by setting $dV/dt = 0$ the steady state membrane potential can be shown to be $V_\infty = E_m + (R_m/a)I_e$. By substituting this value of V_∞ into Equation 2.19, it can be seen that the input resistance $R_{\text{in}} = R_m/a$. This is a quasi-ohmic current-voltage relation where the constant of proportionality is the input resistance, given by R_m/a .

The input resistance measures the response to a steady state input. A more general concept is the **input impedance**, which measures the amplitude and phase lag of the membrane potential in response to a sinusoidal

injection current of a particular frequency. The input impedance of the RC circuit can be computed, and shows that the RC circuit acts as a low-pass filter, reducing the amplitude of high-frequency components of the input signal. The topic of input impedance and the frequency-response of neurons is covered in depth by Koch (1999).

2.7 | Modelling permeable properties in practice

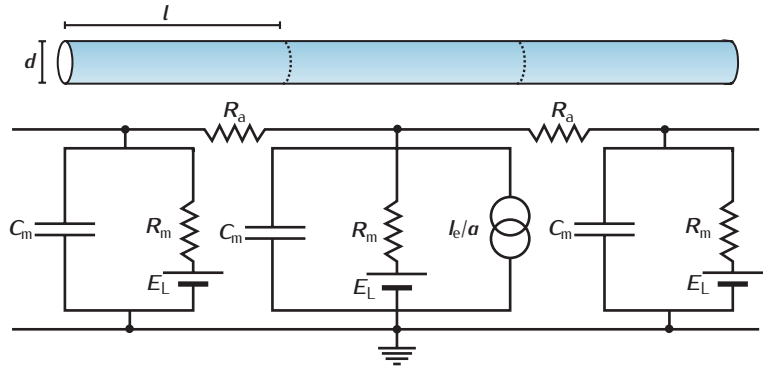
Both the approximations expressed by the GHK current equations and quasi-ohmic electrical circuit approximation are used in models. However, neither should be considered a perfect representation of currents through the membrane. The GHK equations were originally used to describe ion permeability through a uniform membrane, whereas today they are used primarily to describe the movement of ions through channels. Assumptions on which the equations are based, such as the independence of movement of ions through the membrane (the independence principle; Box 2.4 and Chapter 5) and of constant electric fields, are generally not valid within the restricted space of a single channel. It is therefore not surprising that experiments reveal that the flux through channels saturates at large ionic concentrations, rather than increasing without limit as the GHK equations would predict (Hille, 2001).

There are a number of models of the passage of ions through ion channels, which are more detailed than the GHK and quasi-ohmic descriptions (Hille, 2001), but these more detailed descriptions are not generally used in computational models of the electrical activity of neurons. We might ask how we can justify using a more inaccurate description when more accurate ones exist. In answer, modelling itself is the process of making approximations or simplifications in order to understand particular aspects of the system under investigation. A theme that will be visited many times in this book is: what simplifications or approximations are appropriate? The answer depends on the question that the model is designed to address. For certain questions, the level of abstraction offered by the quasi-ohmic approximation has proved extremely valuable, as we see in Chapter 3. Similarly, the GHK equation is used in many modelling and theoretical approaches to membrane permeability.

When choosing which of these approximations is most appropriate, there are a number of issues to consider. Most ion types do not have a strongly rectifying I - V characteristic in the region of typical membrane potentials, and so the quasi-ohmic approximation can be useful. However, if the I - V characteristic is very strongly rectifying (as in the example of calcium), the GHK current equation may give a better fit. Even with fairly weak rectification, the GHK can fit the data better than the quasi-ohmic approximation (Sah *et al.*, 1988).

We might want to model how changes in intracellular concentration affect the I - V characteristic. In this case, the GHK equations may be a more useful tool. This often applies to calcium, since its intracellular concentration is so low that relatively small influxes can change its concentration by an order of magnitude. Moreover, we may need to consider modelling imperfect

Fig. 2.15 A length of passive membrane described by a compartmental model.



(and more realistic) ion selective channels which have permeabilities to more than one ion. All ion selective channels allow some level of permeability to certain other ions, and so the GHK voltage equation can be used to calculate the reversal potential of these channels.

2.8 The equivalent electrical circuit of a length of passive membrane

So far, we have looked at the properties of a patch of membrane or small neuron. This is appropriate when considering an area of membrane over which the membrane potential is effectively constant, or **isopotential**. However, most neurons cannot be considered isopotential throughout, which leads to axial current flowing along the neurites. For example, during the propagation of action potentials, different parts of the axon are at different potentials. Similarly, dendrites cannot generally be treated as isopotential. This is evident from changes in the form of the excitatory postsynaptic potentials (EPSPs) as they move down a dendrite.

Fortunately, it is quite easy to extend the model of a patch of membrane to spatially extended neurites. In this chapter, we consider only an unbranched neurite, and in Chapter 4 we look at branched structures. Because of the similarity to an electrical cable, we often refer to this unbranched neurite as a cable.

2.8.1 The compartmental model

The basic concept is to split up the neurite into cylindrical **compartments** (Figure 2.15). Each compartment has a length l and a diameter d , making its surface area $a = \pi dl$. Within each compartment, current can flow onto the membrane capacitance or through the membrane resistance. This is described by the RC circuit for a patch of membrane, encountered in the last section. Additionally, current can flow longitudinally through the cytoplasm and the extracellular media. This is modelled by axial resistances that link the compartments.

Since it is usually assumed that the intracellular resistance is much greater than the extracellular resistance, it may be acceptable to consider the extracellular component of this resistance to be effectively zero (implying that the main longitudinal contribution is intracellular resistivity). We may then

model the extracellular medium as electrical ground, and it acts in an isopotential manner (as shown in Figure 2.15). For many research questions, such as modelling intracellular potentials, this assumption is valid. However, in any case it is straightforward to incorporate the non-zero extracellular resistance. In Chapter 9 the approach is extended to networks of resistances to model the field potentials in extended regions of extracellular space (Box 9.1).

We assume here a circuit as given in Figure 2.15, with the extracellular medium modelled as ground. The axial resistance of a compartment is proportional to its length l and inversely proportional to the cylinder's cross-sectional area $\pi d^2/4$. The **axial resistivity**, also known as the specific **axial resistance**, R_a , has units $\Omega \text{ cm}$ and gives the resistivity properties of the intracellular medium. The axial resistance of the cylindrical compartment is then $4R_a l/\pi d^2$. Compartments with longer lengths have larger axial resistance and those with larger cross-sectional areas have reduced resistances.

We can describe the electrical circuit representing the cable with one equation per compartment. We number the compartments in sequence using the subscript j . For example, V_j denotes the membrane potential in the j th compartment and $I_{e,j}$ is the current injected into the j th compartment. Following the procedure used in the previous section, we can use Kirchhoff's current law, the quasi-ohmic relation and the equation for the capacitive current (Equations 2.13 to 2.16) to derive our circuit equations. The main difference from the previous treatment is that, rather than the compartment being isolated, the membrane current $I_j a$ is now able to spread both leftwards and rightwards within the cytoplasm, i.e. the membrane current is equal to the sum of the leftwards and rightwards axial currents, each given by Ohm's law:

$$I_j a = \frac{V_{j+1} - V_j}{4R_a l/\pi d^2} + \frac{V_{j-1} - V_j}{4R_a l/\pi d^2}. \quad (2.20)$$

In this case, we are assuming all compartments have the same cylindrical dimensions. Substituting for this membrane current into Equation 2.13:

$$\begin{aligned} I_{c,j} a + I_{i,j} a &= I_j a + I_{e,j} \\ I_{c,j} a + I_{i,j} a &= \frac{V_{j+1} - V_j}{4R_a l/\pi d^2} + \frac{V_{j-1} - V_j}{4R_a l/\pi d^2} + I_{e,j}. \end{aligned} \quad (2.21)$$

This leads to an equation that is similar to Equation 2.16 for a patch of membrane, but now has two extra terms, describing the current flowing along the axial resistances into the two neighbouring compartments $j - 1$ and $j + 1$:

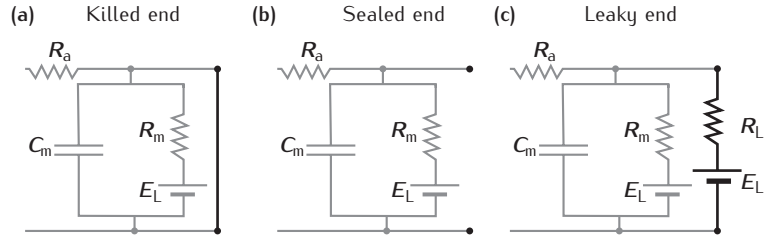
$$\pi d l C_m \frac{dV_j}{dt} = \frac{E_m - V_j}{R_m/\pi d l} + \frac{V_{j+1} - V_j}{4R_a l/\pi d^2} + \frac{V_{j-1} - V_j}{4R_a l/\pi d^2} + I_{e,j}. \quad (2.22)$$

We have used the surface area of the cylinder $\pi d l$ as the area a . Dividing through by this area gives a somewhat less complicated-looking equation:

$$C_m \frac{dV_j}{dt} = \frac{E_m - V_j}{R_m} + \frac{d}{4R_a} \left(\frac{V_{j+1} - V_j}{l^2} + \frac{V_{j-1} - V_j}{l^2} \right) + \frac{I_{e,j}}{\pi d l}. \quad (2.23)$$

This equation is the fundamental equation of a compartmental model.

Fig. 2.16 Circuit illustration of three types of cable terminal conditions.



2.8.2 Boundary conditions

The equations above assume that each compartment j has two neighbouring compartments $j - 1$ and $j + 1$, but this is not true in the compartments corresponding to the ends of neurites. Special treatment is needed for these compartments, which depends on the condition of the end of the neurite being modelled.

The simplest case is that of a **killed end**, in which the end of the neurite has been cut. This can arise in some preparations such as dissociated cells, and it means that the intracellular and extracellular media are directly connected at the end of the neurite. Thus the membrane potential at the end of the neurite is equal to the extracellular potential. To model this, in the equation for the membrane potential, V_0 in the first compartment is set to 0, as illustrated in Figure 2.16a. This allows Equation 2.23 to be used. The condition $V_0 = 0$ is called a **boundary condition** as it specifies the behaviour of the system at one of its edges. This type of boundary condition, where the value of a quantity at the boundary is specified, is called a **Dirichlet** boundary condition.

If the end of the neurite is intact, a different boundary condition is required. Here, because the membrane surface area at the tip of the neurite is very small, its resistance is very high. In this **sealed end** boundary condition, illustrated in electric circuit form in Figure 2.16b, we assume that the resistance is so high that a negligible amount of current flows out through the end. Since the axial current is proportional to the gradient of the membrane potential along the neurite, zero current flowing through the end implies that the gradient of the membrane potential at the end is zero. For reasons made clear in Appendix B.1 in the compartmental framework, this boundary condition is modelled by setting $V_{-1} = V_1$. This leads to a modified version of Equation 2.23 for compartment 0. This type of boundary condition, where the spatial derivative of a quantity at the boundary is specified, is called a **Neumann** boundary condition.

It can also be assumed that there is a **leaky end**; in other words, that the resistance at the end of the cable has a finite absolute value R_L (Figure 2.16c). In this case, the boundary condition is derived by equating the axial current, which depends on the spatial gradient of the membrane potential, to the current flowing through the end, $(V - E_m)/R_L$.

Box 2.5 Derivation of the cable equation

To derive the cable equation from the discrete equations for the compartmental model (Equation 2.23) we set the compartment length l to the small quantity δx . A compartment indexed by j is at a position $x = j\delta x$ along the cable, and therefore the membrane potentials in compartments $j - 1$, j and $j + 1$ can be written:

$$V_j = V(x, t) \quad V_{j-1} = V(x - \delta x, t) \quad V_{j+1} = V(x + \delta x, t).$$

Also, we define the current injected per unit length as $I_e(x, t) = I_{e,j}/\delta x$. This allows Equation 2.23 to be rewritten as:

$$\begin{aligned} C_m \frac{\partial V(x, t)}{\partial t} &= \frac{E_m - V(x, t)}{R_m} \\ &+ \frac{d}{4R_a} \left[\frac{1}{\delta x} \left(\frac{V(x + \delta x, t) - V(x, t)}{\delta x} - \frac{V(x, t) - V(x - \delta x, t)}{\delta x} \right) \right] + \frac{I_e(x, t)}{\pi d}. \end{aligned} \quad (a)$$

The derivative of V with respect to t is now a partial derivative to signify that the membrane potential is a function of more than one variable.

The length δx of each compartment can be made arbitrarily small, so that eventually there is an infinite number of infinitesimally short compartments. In the limit as δx goes to 0, the term in square brackets in the equation above becomes the same as the definition of the second partial derivative of distance:

$$\frac{\partial^2 V(x, t)}{\partial x^2} = \lim_{\delta x \rightarrow 0} \frac{1}{\delta x} \left(\frac{V(x + \delta x, t) - V(x, t)}{\delta x} - \frac{V(x, t) - V(x - \delta x, t)}{\delta x} \right).$$

Substituting this definition into Equation (a) leads to Equation 2.24, the cable equation.

In the case of discrete cables, the sealed end boundary condition is that:

$$\frac{d}{4R_a} \frac{V_1 - V_0}{\delta x^2} = \frac{I_{e,1}}{\pi d \delta x} + \frac{E_m - V_1}{\pi d \delta x R_L}.$$

In the limit of $\delta x \rightarrow 0$, at the $x = 0$ end of the cable, this is:

$$-\frac{d}{4R_a} \frac{\partial V}{\partial x} = \frac{I_e(0, t)}{\pi d} + \frac{E_m - V(0, t)}{\pi d R_L}.$$

At the $x = l$ end of the cable, this is

$$\frac{d}{4R_a} \frac{\partial V}{\partial x} = \frac{I_e(l, t)}{\pi d} + \frac{E_m - V(l, t)}{\pi d R_L},$$

assuming a sealed end means that the axial current at the sealed end is zero, and therefore that the gradient of the voltage at the end is also zero.

In Equation 2.26 we have introduced two diameter-specific constants, r_m and r_a , defined in Table 2.3. These are convenient quantities that express the key passive electrical properties of a specific cable of arbitrary length. They are often used to simplify the equations representing a neurite of specific diameter.

The value of λ determines the shape of the exponential voltage decay along the length of the cable. It is determined by the specific membrane resistance, the axial resistivity and the diameter of the cable:

$$\lambda = \sqrt{\frac{R_m d}{4R_a}} = \sqrt{\frac{r_m}{r_a}}. \quad (2.26)$$