A Simple and Practical Valuation Tree Calculus for First-Order Logic

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Abstract. We present a proof calculus for first-order logic with definitional extensions which is simple – it has only one rule. It is also practical, because it can be used in a intelligent proof assistant for verification of computer programs.

Keywords: formal systems, proof theory.

1 Introduction

The idea that boolean valuation trees can be used as formal calculus for first-order logic was certainly implicit in the work of Beth which lead to his development of semantic tableaux. In this paper we present a calculus for first-order logic with finite valuation trees being the proofs. The calculus has a flavor of Hilbert systems (one rule and many axioms), yet it exhibits, as it should, a close connection to semantic tableaux and Gentzen's style sequent calculi. To the best knowledge of authors, this idea is not directly present in any extant proof calculus, except perhaps in a rather rudimentary form, in the decision procedures based on the calculus of Davis and Putnam.

Our calculus seems to be of interest on its own in pure logic, especially in classroom situations and in self-contained expositions of (classical) first-order logic. This is because the role of syntax in our exposition is absolutely minimal – just the finite valuation trees – and even they are, per definitionem, of semantic character.

Yet, our main motivation for the development of the calculus is entirely pragmatical. It is to be a formal basis for a new version of our Intelligent Proof Assistant (IPA) for a programming language CL (Clausal Language) [CL97]. Programs in CL are just certain implications in extensions by definitions of Peano Arithmetic. We have successfully used CL for the last ten years in the teaching of first-order logic and also in our courses on program verification.

A typical work in an IPA is shown in Fig. 1. A theory is extended by a definition of the symbol f and a lemma (n) about it is proved. The lemma is then used in a proof of the theorem (m). The proof of (m) is itself done in the style of extensions. The theory is locally extended with the symbol g and a local lemma (i) about is is proved. Both lemmas are used to finish the proof of (m).

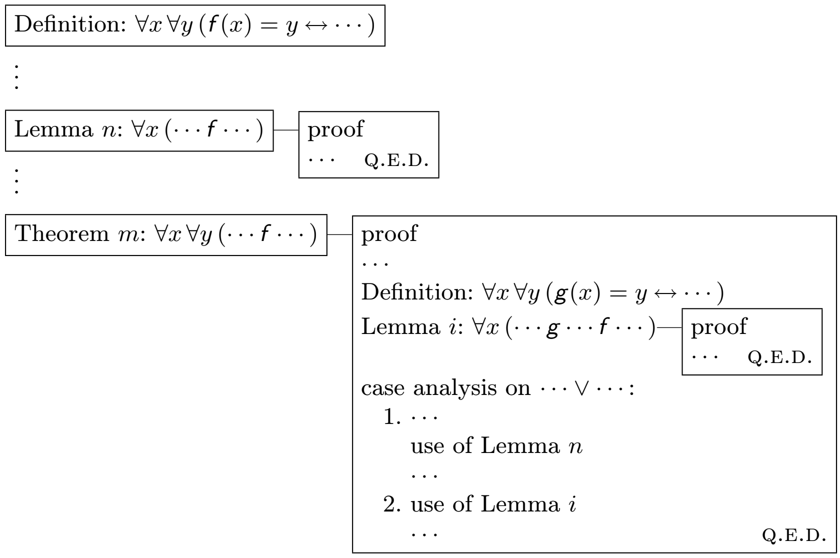


Fig. 1. A schema of development of proofs by a proof assistant in extensions of theories.

IPAs should offer the natural mathematical style of proofs because computer programming is a mentally exhausting activity. It is doubly so when coupled with the formal verification of programs. Good IPAs can certainly help with the development of correct programs (at least the mission critical ones), and so the search for them is a worthwhile research in applied logic. The reader will note that the cut elimination is not a central issue in IPAs because natural mathematical proofs without cuts (in the form of lemmas) are impossible.

This paper is developed as a purely logical exposition of proofs in extensions of first-order theories starting from propositional logic (Sect. 2). We then treat quantification (Sect. 3), equality (Sect. 4), and extensions by definitions (Sect. 5). We will not deal here with the all important pragmatical aspects of computer-assisted theorem proving, except briefly, in Par. 2.11. We, however, recommend that the reader pays a closer attention to the example given there.

2 Propositional Logic

2.1 Syntactic preliminaries. We use the standard notions of language (denoted by $\mathcal{L}$) for first-order logic with predicate symbols (P, R, …) and function symbols (f, g, …), terms (t, s, …). We will be concerned with countable languages only. Formulas (A, B, …) are built up from atomic formulas by the propositional connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, and the quantifiers $\forall$, $\exists$ binding variables (x, y, z, …). We use $A \leftrightarrow B$ as an abbreviation for $(A \rightarrow B) \wedge (B \rightarrow A)$.

A *sentence* is a formula with no free variables. A set of sentences is a *theory* if it can be recursively coded. Theories are denoted by the letters T and S. We write T, S and A, T as abbreviations of $T \cup S$ and $\{A\} \cup T$ respectively. Finite theories are denoted by the capital greek letters Γ and Δ.

2.2 Propositional semantics. A *truth assignment* is a set $\mathcal{I}$ of atomic and quantified sentences (i.e., $\forall x\, A[x]$ or $\exists x\, A[x]$). A truth assignment $\mathcal{I}$ *propositionally satisfies* an atomic or quantified sentence A, in writing $\mathcal{I} \models_{\mathrm{p}} A$, if $A \in \mathcal{I}$. We inductively extend the relation $\models_{\mathrm{p}}$ to all sentences using the classical interpretation of propositional connectives. We write $\mathcal{I} \models_{\mathrm{p}} T$ if $\mathcal{I} \models_{\mathrm{p}} A$ for all $A \in T$. We define $\mathcal{I} \mathrel{|\!\Rightarrow}_{\mathrm{p}} T$ to hold if $\mathcal{I} \models_{\mathrm{p}} A$ for some $A \in T$.

We say that T *propositionally implies* S, in writing $T \models_{\mathrm{p}} S$, if for all $\mathcal{I}$ s.t. $\mathcal{I} \models_{\mathrm{p}} T$ we have $\mathcal{I} \models_{\mathrm{p}} S$.

We say that T *propositionally implies one of* S, in writing $T \mathrel{|\!\Rightarrow}_{\mathrm{p}} S$, if for all $\mathcal{I}$ s.t. $\mathcal{I} \models_{\mathrm{p}} T$ we have $\mathcal{I} \mathrel{|\!\Rightarrow}_{\mathrm{p}} S$.

We call T the *antecedent* of the assertion $T \mathrel{|\!\Rightarrow}_{\mathrm{p}} S$, and S its *succedent*. Any assertion $T, T' \mathrel{|\!\Rightarrow}_{\mathrm{p}} S, S'$ is called a *weakening* of $T \mathrel{|\!\Rightarrow}_{\mathrm{p}} S$.

2.3 Initial propositional properties. The assertions listed in Fig. 2 are called the *initial propositional properties*. The properties are given parenthesized names. The names relate to their use in Lemma 2.14 for the treatment of eponymous rules of the propositional sequent calculus **G3cp** [TS00, page 77]. The rules of **G3cp** are listed in Fig. 4. Note that we use Dragalin's variant with $\neg$ instead of $\perp$ [Dra79].

(Ax)	$A \mathrel{|\!\Rightarrow}_{\mathrm{p}} A$	(L$\neg$)	$\neg A, A \mathrel{|\!\Rightarrow}_{\mathrm{p}} \emptyset$	(R$\neg$)	$\emptyset \mathrel{|\!\Rightarrow}_{\mathrm{p}} \neg A, A$
(L$\to$)	$A \to B, A \mathrel{|\!\Rightarrow}_{\mathrm{p}} B$	(R$\to_1$)	$\emptyset \mathrel{|\!\Rightarrow}_{\mathrm{p}} A \to B, A$	(R$\to_2$)	$B \mathrel{|\!\Rightarrow}_{\mathrm{p}} A \to B$
(L$\wedge_1$)	$A \wedge B \mathrel{|\!\Rightarrow}_{\mathrm{p}} A$	(L$\wedge_2$)	$A \wedge B \mathrel{|\!\Rightarrow}_{\mathrm{p}} B$	(R$\wedge$)	$A, B \mathrel{|\!\Rightarrow}_{\mathrm{p}} A \wedge B$
(L$\vee$)	$A \vee B \mathrel{|\!\Rightarrow}_{\mathrm{p}} A, B$	(R$\vee_1$)	$A \mathrel{|\!\Rightarrow}_{\mathrm{p}} A \vee B$	(R$\vee_2$)	$B \mathrel{|\!\Rightarrow}_{\mathrm{p}} A \vee B$

Fig. 2. Initial propositional properties.

It is decidable whether $\Gamma \mathrel{|\!\Rightarrow}_{\mathrm{p}} \Delta$ is a weakening of an initial property.

The following lemmas assert obvious properties of the relation $\mathrel{|\!\Rightarrow}_{\mathrm{p}}$. They form the basis of our proof calculus.

2.4 Lemma (Initial propositional properties). *The initial propositional properties are true.* □

2.5 Lemma (Weakening). *If $T \mathrel{|\!\Rightarrow}_{\mathrm{p}} S$, then $T, T' \mathrel{|\!\Rightarrow}_{\mathrm{p}} S, S'$.* □

2.6 Lemma (Cut). *$T \mathrel{|\!\Rightarrow}_{\mathrm{p}} S$ iff $A, T \mathrel{|\!\Rightarrow}_{\mathrm{p}} S$ and $T \mathrel{|\!\Rightarrow}_{\mathrm{p}} A, S$.* □

2.7 Valuation trees. A *valuation tree* (denoted by $\mathcal{D}$, $\mathcal{E}$) is a tree with each node either a leaf or an internal node with two ordered predecessors. A leaf is designated by the symbol $\circ$. Internal nodes are labeled with sentences, and written down as

$$\frac{\mathcal{D} \qquad \mathcal{E}}{A}$$

where $\mathcal{D}$ and $\mathcal{E}$ are the respective subtrees.

In Par. 2.8 we will fix the properties of valuation trees in a more general way, but the following informal interpretation should give the reader an intuition about them.

A valuation tree assigns the sentence A in an internal node the truth value "true" in the left subtree $\mathcal{D}$ and "false" in the right subtree $\mathcal{E}$. We will have

$\mathcal{D}$ proves A iff for every path p through $\mathcal{D}$ we have:

$$\{B \mid B \in p \text{ is true}\}, \{\neg B \mid B \in p \text{ is false}\} \models_{\mathrm{p}} A\ .$$

Moreover, for each p, the truth of the satisfaction relation will be decided only by the form of the sentences involved without involving semantics.

2.8 Proofs in propositional logic. We will now define a five-place relation $\mathcal{D}$ *p-witnesses (propositionally proves)* $T, \Gamma \Rrightarrow_{\mathrm{p}} S, \Delta$, in writing $\mathcal{D} \vdash_{\mathrm{p}} T; \Gamma \Rrightarrow_{\mathrm{p}} S; \Delta$, as the least relation satisfying:

- $\circ \vdash_{\mathrm{p}} T; \Gamma \Rrightarrow_{\mathrm{p}} S; \Delta$ if the assertion $(T \cap \Delta), \Gamma \Rrightarrow_{\mathrm{p}} (S \cap \Gamma), \Delta$ is a weakening of an initial propositional property,
- $\dfrac{\mathcal{D} \qquad \mathcal{E}}{A} \vdash_{\mathrm{p}} T; \Gamma \Rrightarrow_{\mathrm{p}} S; \Delta$ if $\mathcal{D} \vdash_{\mathrm{p}} T; A, \Gamma \Rrightarrow_{\mathrm{p}} S; \Delta$ and $\mathcal{E} \vdash_{\mathrm{p}} T; \Gamma \Rrightarrow_{\mathrm{p}} S; A, \Delta$.

We abbreviate $\mathcal{D} \vdash_{\mathrm{p}} \emptyset; \Gamma \Rrightarrow_{\mathrm{p}} \emptyset; \Delta$ and $\mathcal{D} \vdash_{\mathrm{p}} T; \emptyset \Rrightarrow_{\mathrm{p}} S; \emptyset$ to $\mathcal{D} \vdash_{\mathrm{p}} \Gamma \Rrightarrow_{\mathrm{p}} \Delta$ and $\mathcal{D} \vdash_{\mathrm{p}} T \Rrightarrow_{\mathrm{p}} S$ respectively.

2.9 Lemma (Soundness). *If* $\mathcal{D} \vdash_{\mathrm{p}} T; \Gamma \Rrightarrow_{\mathrm{p}} S; \Delta$*, then* $T, \Gamma \Rrightarrow_{\mathrm{p}} S, \Delta$.

Proof. By induction on $\mathcal{D}$. When $\mathcal{D} = \circ$, then $(T \cap \Delta), \Gamma \Rrightarrow_{\mathrm{p}} (S \cap \Gamma), \Delta$ holds by Lemmas 2.4 and 2.5 because it is a weakening of an initial property of $\Rrightarrow_{\mathrm{p}}$. By another weakening we get $T, \Gamma \Rrightarrow_{\mathrm{p}} S, \Delta$. The inductive case follows from Lemma 2.6. □

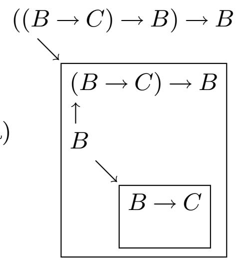
2.10 Remark. Note that our proofs, i.e., the valuation trees $\mathcal{D}$, are not, as it is usual, related to syntactic objects (i.e., to sequents), but by Lemma 2.9 they rather *witness the truth of* a semantic property. In order to avoid ambiguity, the informal phrase "$\mathcal{D}$ *p-witnesses* $T, \Gamma \Rrightarrow_{\mathrm{p}} S, \Delta$" should be understood as the assertion "$\mathcal{D} \vdash_{\mathrm{p}} T; \Gamma \Rrightarrow_{\mathrm{p}} S; \Delta$".

Also note that we could have introduced the proof relation as a three-place relation $\mathcal{D} \vdash_{\mathrm{p}} T \Rrightarrow_{\mathrm{p}} S$, but then, even with T and S recursive, the relation would not be decidable as expected, but only semi-decidable (r.e.). This is because $\circ \vdash_{\mathrm{p}} T \Rrightarrow_{\mathrm{p}} S$ would then mean an r.e. assertion that $T \Rrightarrow_{\mathrm{p}} S$ is a weakening of an initial property. The inclusion of finite sets Γ and Δ in the proof relation makes it decidable.

2.11 Example with discussion. In this paragraph we relate the valuation trees to the workings of an IPA from Fig. 1, and also to tableau and sequent proofs.

As an example, we demonstrate Pierce's Law $((B \to C) \to B) \to B$. We abbreviate the whole formula to P and its antecedent to A. In order to explain easier Fig. 3(a) we work first informally in a more detailed way than humans would need. We reason by contradiction where we assume $P{\downarrow}$ ("thumbs down" = false). For that it must be the case that $A{\uparrow}$ and $B{\downarrow}$. Note that the cases $A{\downarrow}$ or $B{\uparrow}$ cannot obtain because in the former we get a contradiction $P{\uparrow}$ from $\emptyset \mapsto A \to B, A$ (R$\to_1$), and in the latter we get the same contradiction from $B \mapsto A \to B$ (R$\to_2$). We now consider two cases. If $B \to C{\uparrow}$, then we get a contradiction $B{\uparrow}$ from $(B \to C) \to B, B \to C \mapsto B$ (L$\to$). If $B \to C{\downarrow}$, we get a contradiction $B{\uparrow}$ from $\emptyset \mapsto B \to C, B$ (R$\to_1$). Hence P is irrefutable.

Fig. 3(a) formally reflects exactly the above proof. The truth value arrows emanate from below the formulas as $\uparrow$ and $\searrow$. A missing arrow indicates a contradiction in the missing direction.

(a) 

(b)
$$\dfrac{\dfrac{\circ \quad \dfrac{\circ \quad \circ}{B \to C}}{B} \quad \circ}{(B \to C) \to B}$$

(c)
$$\begin{array}{ccc} & ((B \to C) \to B) \to B{\downarrow} & \\ & (B \to C) \to B{\uparrow} & \\ & B{\downarrow} & \\ / & & \backslash \\ B \to C{\downarrow} & & B{\uparrow} \\ B{\uparrow} & & \circ \\ \circ & & \end{array}$$

(d)
$$\dfrac{\dfrac{\stackrel{\backslash(\mathrm{R}{\to_2})/}{B,(B\to C)\to B\Rightarrow P} \qquad \dfrac{\stackrel{\backslash(\mathrm{L}{\to})/}{B\to C,(B\to C)\to B\Rightarrow B,P} \quad \stackrel{\backslash(\mathrm{R}{\to_1})/}{(B\to C)\to B\Rightarrow B\to C,B,P}}{(B\to C)\to B\Rightarrow B,P}}{(B\to C)\to B\Rightarrow P} \qquad \stackrel{\backslash(\mathrm{R}{\to_1})/}{\Rightarrow (B\to C)\to B,P}}{\Rightarrow P}$$

$$P \equiv ((B \to C) \to B) \to B$$

Fig. 3. Different renderings of a formal proof: (a) downwards growing proof tree, (b) valuation tree, (c) signed tableau, (d) **G3cp**.

A transformation where we take a node $\begin{array}{c} A \\ \uparrow \\ \mathcal{D}_1 \end{array} \searrow \boxed{\mathcal{D}_2}$ to the tree $\dfrac{\mathcal{D}_1^* \qquad \mathcal{D}_2^*}{A}$ transforms the tree (a) into the valuation tree p-witnessing $\emptyset \mapsto_{\mathrm{p}} P$ which is given in Fig. 3(b).

When we "unfold" the definition of p-witnessing (Par. 2.8), which can be viewed as connecting valuation trees to sequents, we transform the valuation tree (b) to a sequent proof of P in the calculus **G3cp** [TS00, page 77]. The proof is given in Fig. 3(d) where the reader will note that all inferences are cuts.

The "real" sequent rules are used only in the omitted upper branches where the indicated initial properties are derived.

The tree in Fig. 3(a) is somewhat similar to the derivation of P in the Smullyan's [Smu68] calculus of signed tableaux. The tableau is given in Fig. 3(c).

The beauty of the tree Fig. 3(a) is that it can be also read positively. This is the view taken by humans operating a proof assistant in the style of Fig. 1. The theorem to be proved is $((B \to C) \to B) \to B$. The box below it contains its proof where we first assume its antecedent $(B \to C) \to B$, and under this assumption we prove the "lemma" B inside its own box. The proof of the lemma consists of one case analysis on $B \to C$.

2.12 Lemma (Proof weakening). *If $\mathcal{D} \vdash_{\mathrm{p}} T; \Gamma \mapsto_{\mathrm{p}} S; \Delta$, then also $\mathcal{D} \vdash_{\mathrm{p}} (T, T'); (\Gamma, \Gamma') \mapsto_{\mathrm{p}} (S, S'); (\Delta, \Delta')$.*

Proof. By induction on $\mathcal{D}$. □

2.13 Free cut free valuation trees. A valuation tree $\mathcal{E}$ p-witnessing $T, \Gamma \mapsto_{\mathrm{p}} S, \Delta$ contains a *free cut* if it contains a non-leaf subtree such that

$$\frac{\mathcal{D}_1 \qquad \mathcal{D}_2}{A} \vdash_{\mathrm{p}} T; \Gamma' \mapsto_{\mathrm{p}} S; \Delta'$$

and the cut formula A is neither in T, S nor it is an immediate subformula of a formula from Γ', Δ'. If $\mathcal{E}$ contains no free cuts, it is *free cut free.*

2.14 Lemma (Reduction of G3cp proofs to $\vdash_{\mathrm{p}}$ trees). *If there is a derivation $\mathcal{D}$ of a closed sequent $\Gamma \Rightarrow \Delta$ in **G3cp**, then there is a valuation tree $\mathcal{D}^*$ p-witnessing $\Gamma \mapsto_{\mathrm{p}} \Delta$. If $\mathcal{D}$ is cut free, then $\mathcal{D}^*$ is free cut free.*

Proof. By induction on the derivation $\mathcal{D}$ of a sequent $\Gamma \Rightarrow \Delta$ in the calculus **G3cp**. If $\mathcal{D}$ is an axiom of the sequent calculus, there is $A \in \Gamma \cap \Delta$, and $\circ \vdash_{\mathrm{p}} \Gamma \mapsto_{\mathrm{p}} \Delta$ by the initial property (Ax) from Fig. 2.

If $\mathcal{D}$ derives $\Gamma \Rightarrow \Delta$ by application of a rule, then for $i = 1$ or $i = 1, 2$ the derivation $\mathcal{D}$ has predecessors $\mathcal{D}_i$ deriving $\Gamma_i \Rightarrow \Delta_i$. By induction hypotheses, there are valuation trees $\mathcal{D}_i^* \vdash \Gamma_i \mapsto_{\mathrm{p}} \Delta_i$, and we construct a tree $\mathcal{D}^* \vdash_{\mathrm{p}} \Gamma \mapsto_{\mathrm{p}} \Delta$ as indicated in Fig. 4. If, for instance, $\mathcal{D}$ is formed by the rule L$\to$, then we have $\Gamma = A \to B, \Gamma_1$; $\Gamma_2 = B, \Gamma_1$; $\Delta_1 = A, \Delta$; and $\Delta_2 = \Delta$. We construct the derivation $\mathcal{D}^*$ as follows:

$$\cfrac{\cfrac{\mathcal{D}_2^* \vdash B, A, A \to B, \Gamma_1 \mapsto_{\mathrm{p}} \Delta \quad \circ \vdash A, A \to B, \Gamma_1 \mapsto_{\mathrm{p}} B, \Delta}{B} \qquad \mathcal{D}_1^* \vdash A \to B, \Gamma_1 \mapsto_{\mathrm{p}} A, \Delta}{A} \vdash A \to B, \Gamma_1 \mapsto_{\mathrm{p}} \Delta$$

Note that by Lemma 2.12 both $\mathcal{D}_1^*$ and $\mathcal{D}_2^*$ witness also the indicated weakenings of $\Gamma_i \mapsto_{\mathrm{p}} \Delta_i$, and that the leaf in the construction witnesses a weakening of (L$\to$).

If $\mathcal{D}$ is cut free, then there are no free cuts in $\mathcal{D}^*$. □

	$\mathcal{D}$	$\mathcal{D}^*$
Ax	$A, \Gamma \Rightarrow A, \Delta$	$\circ \vdash (\mathrm{Ax})$
L$\neg$	$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \Rightarrow A, \Delta\end{array}}{\neg A, \Gamma \Rightarrow \Delta}$	$\dfrac{\circ \vdash (\mathrm{L}\neg) \qquad \mathcal{D}_1^*}{A}$
R$\neg$	$\dfrac{\begin{array}{c}\mathcal{D}_1\\ A, \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \neg A, \Delta}$	$\dfrac{\mathcal{D}_1^* \qquad \circ \vdash (\mathrm{R}\neg)}{A}$
L$\rightarrow$	$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \Rightarrow A, \Delta\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ B, \Gamma \Rightarrow \Delta\end{array}}{A \rightarrow B, \Gamma \Rightarrow \Delta}$	$\dfrac{\dfrac{\mathcal{D}_2^* \qquad \circ \vdash (\mathrm{L}{\rightarrow})}{B} \qquad \mathcal{D}_1^*}{A}$
R$\rightarrow$	$\dfrac{\begin{array}{c}\mathcal{D}_1\\ A, \Gamma \Rightarrow B, \Delta\end{array}}{\Gamma \Rightarrow A \rightarrow B, \Delta}$	$\dfrac{\dfrac{\circ \vdash (\mathrm{R}{\rightarrow_2}) \qquad \mathcal{D}_1^*}{B} \qquad \circ \vdash (\mathrm{R}{\rightarrow_1})}{A}$
L$\wedge$	$\dfrac{\begin{array}{c}\mathcal{D}_1\\ A, B, \Gamma \Rightarrow \Delta\end{array}}{A \wedge B, \Gamma \Rightarrow \Delta}$	$\dfrac{\dfrac{\mathcal{D}_1^* \qquad \circ \vdash (\mathrm{L}\wedge_2)}{B} \qquad \circ \vdash (\mathrm{L}\wedge_1)}{A}$
R$\wedge$	$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \Rightarrow A, \Delta\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma \Rightarrow B, \Delta\end{array}}{\Gamma \Rightarrow A \wedge B, \Delta}$	$\dfrac{\dfrac{\circ \vdash (\mathrm{R}\wedge) \qquad \mathcal{D}_1^*}{B} \qquad \mathcal{D}_2^*}{A}$
L$\vee$	$\dfrac{\begin{array}{c}\mathcal{D}_1\\ A, \Gamma \Rightarrow \Delta\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ B, \Gamma \Rightarrow \Delta\end{array}}{A \vee B, \Gamma \Rightarrow \Delta}$	$\dfrac{\mathcal{D}_1^* \qquad \dfrac{\mathcal{D}_2^* \qquad \circ \vdash (\mathrm{L}\vee)}{B}}{A}$
R$\vee$	$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \Rightarrow A, B, \Delta\end{array}}{\Gamma \Rightarrow A \vee B, \Delta}$	$\dfrac{\circ \vdash (\mathrm{R}\vee_1) \qquad \dfrac{\circ \vdash (\mathrm{R}\vee_2) \qquad \mathcal{D}_1^*}{B}}{A}$
Cut	$\dfrac{\begin{array}{c}\mathcal{D}_1\\ A, \Gamma \Rightarrow \Delta\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma \Rightarrow A, \Delta\end{array}}{\Gamma \Rightarrow \Delta}$	$\dfrac{\mathcal{D}_1^* \qquad \mathcal{D}_2^*}{A}$

Fig. 4. Translation of **G3cp** proofs to valuation trees.

2.15 Lemma (Completeness). *If $T \mathrel{|\!\Rightarrow}_{\mathrm{p}} S$, then $\mathcal{D} \vdash_{\mathrm{p}} T \mathrel{|\!\Rightarrow}_{\mathrm{p}} S$ for a free cut free $\mathcal{D}$.*

Proof. If $T \mathrel{|\!\Rightarrow}_{\mathrm{p}} S$, then by completeness of the calculus **G3cp** and by compactness there is a derivation $\mathcal{E}$ of a sequent $\Gamma \Rightarrow \Delta$ for some $\Gamma = C_1, \ldots, C_k \subseteq T$ and $\Delta = D_1, \ldots, D_n \subseteq S$. Cuts are eliminable in **G3cp** [TS00, page 92], so we can assume that $\mathcal{E}$ is cut free. By Lemma 2.14, there is a free cut free $\mathcal{E}^*$ s.t. $\mathcal{E}^* \vdash_{\mathrm{p}} \Gamma \mathrel{|\!\Rightarrow}_{\mathrm{p}} \Delta$. By Lemma 2.12, $\mathcal{E}^* \vdash_{\mathrm{p}} T; \Gamma \mathrel{|\!\Rightarrow}_{\mathrm{p}} S; \Delta$ holds. We construct a free cut free $\mathcal{D}^*$ s.t. $\mathcal{D}^* \vdash_{\mathrm{p}} T \mathrel{|\!\Rightarrow}_{\mathrm{p}} S$ as follows:

$$\dfrac{\begin{matrix}\dfrac{\dfrac{\circ \quad \begin{matrix}\dfrac{\circ \quad \mathcal{E}^*}{D_n}\\ \vdots\end{matrix}}{D_1} \quad \circ}{C_k}\\ \vdots\end{matrix} \quad \circ}{C_1} \vdash T \mathrel{|\!\Rightarrow}_{\mathrm{p}} S$$

Note that all indicated leaves witness a weakening of the identity axiom (Ax). □

2.16 Corollary (Soundness and completeness of $\vdash_{\mathrm{p}}$). *We have $T \mathrel{|\!\Rightarrow}_{\mathrm{p}} S$ iff $\mathcal{D} \vdash_{\mathrm{p}} T \mathrel{|\!\Rightarrow}_{\mathrm{p}} S$ for a free cut free $\mathcal{D}$.* □

3 Quantification Logic

In this section, we extend our calculus to the first-order quantification logic without equality.

3.1 Semantics of first-order quantification logic. We assume the standard notion of a structure $\mathcal{M}$ for a first-order language $\mathcal{L}$ not containing the equality symbol $=$. A structure has a non-empty domain M, assigns meaning to function and predicate symbols of $\mathcal{L}$, and extends the meaning in the usual way to terms and formulas. We write $\mathcal{M} \models A$ when $\mathcal{M}$ satisfies the sentence A.

We define $\mathcal{M} \models T$ ($\mathcal{M}$ is a *model* of T), $T \models S$, $\mathcal{M} \mathrel{|\!\Rightarrow} T$, and $T \mathrel{|\!\Rightarrow} S$ analogously to their propositional counterparts with the additional condition that if $T \models S$ or $T \mathrel{|\!\Rightarrow} S$, then T and S are theories in the same language.

3.2 Logical consequence in expansions of structures. We denote by $\mathcal{L}[\vec{\mathsf{F}}]$ a language which is just like $\mathcal{L}$, but contains new predicate or functions symbols $\vec{\mathsf{F}} = \mathsf{F}_1, \ldots, \mathsf{F}_n$. Let $\mathcal{M}$ be a structure for a language $\mathcal{L}$. As usual, a structure $\mathcal{M}'$ is an *expansion of* $\mathcal{M}$ *to* $\mathcal{L}[\vec{\mathsf{F}}]$ if $\mathcal{M}'$ is a structure for $\mathcal{L}[\vec{\mathsf{F}}]$ and coincides with $\mathcal{M}$ on all symbols of $\mathcal{L}$. This implies that the domains of $\mathcal{M}$ and $\mathcal{M}'$ are identical.

We write $T \mathrel{|\!\Rightarrow}_{\mathcal{L},\vec{\mathsf{F}}} S$ if all of the following conditions are satisfied: (a) $\mathcal{L}$ is a language, and $\vec{\mathsf{F}}$ are new symbols, (b) T is a theory in $\mathcal{L}$, (c) S is a theory in $\mathcal{L}[\vec{\mathsf{F}}]$, and (d) for each structure $\mathcal{M}$ for $\mathcal{L}$ such that $\mathcal{M} \models T$, there is an expansion $\mathcal{M}'$ of $\mathcal{M}$ to $\mathcal{L}[\vec{\mathsf{F}}]$ such that $\mathcal{M}' \mathrel{|\!\Rightarrow} S$.

3.3 Lemma.

(a) If $T \Rrightarrow_{\mathrm{p}} S$, and T, S are in the same language, then $T \Rrightarrow S$.
(b) $T \Rrightarrow S$ with T and S in $\mathcal{L}$ iff $T \Rrightarrow_{\mathcal{L},\emptyset} S$,
(c) (Weakening) *If $T \Rrightarrow_{\mathcal{L},\vec{\mathsf{F}}} S$, then $T, T' \Rrightarrow_{\mathcal{L},\vec{\mathsf{F}},\vec{\mathsf{G}}} S, S'$ for any new $\vec{\mathsf{G}}$, T' in $\mathcal{L}$, and S' in $\mathcal{L}[\vec{\mathsf{F}}, \vec{\mathsf{G}}]$.*

Proof. Straightforward. □

3.4 Initial quantification properties. The *initial quantification properties* are (i) the initial properties from Fig. 2 with $\Rrightarrow_{\mathrm{p}}$ replaced by $\Rrightarrow_{\mathcal{L},\emptyset}$ provided A and B are from $\mathcal{L}$; (ii) the following initial properties of quantifiers:

$$\begin{array}{llll} (\mathrm{R}\exists) & A[x/t] \Rrightarrow_{\mathcal{L},\emptyset} \exists x\, A[x], & (\mathrm{L}\forall) & \forall x\, A[x] \Rrightarrow_{\mathcal{L},\emptyset} A[x/t], \\ (\mathrm{L}\exists) & \exists x\, A[x] \Rrightarrow_{\mathcal{L},\mathsf{c}} A[x/\mathsf{c}], & (\mathrm{R}\forall) & \emptyset \Rrightarrow_{\mathcal{L},\mathsf{c}} \forall x\, A[x], \neg A[x/\mathsf{c}] \end{array}$$

provided x is the only free variable in $A[x]$ from $\mathcal{L}$, t is a closed term from $\mathcal{L}$, and $\mathsf{c} \notin \mathcal{L}$.

Note that by using the notion of logical consequence in expansions of structures we never have to deal with formulas containing free variables. We thus escape the annoying problem of a variable capture by a quantifier when substituting a term with free variables for a variable.

3.5 Lemma (Initial quantification properties). *Initial quantification properties are true.*

Proof. Propositional properties are true by Lemma 3.3(a)(b). Quantifier instantiation properties (R∃) and (L∀) hold trivially with $\models$, thus also with $\Rrightarrow$, and hence with $\Rrightarrow_{\mathcal{L},\emptyset}$ by Lemma 3.3(b). For the Henkin witnessing property (L∃) it suffices to expand any $\mathcal{M}$ for $\mathcal{L}$ s.t. $\mathcal{M} \models \exists x\, A[x]$ by interpreting c as a witness to satisfy $A[x/\mathsf{c}]$. The Henkin counterexample property (R∀) is made true by interpreting c in an expansion of $\mathcal{M}$ as a counterexample $\neg A[x/\mathsf{c}]$ whenever $\mathcal{M} \not\models \forall x\, A[x]$. □

3.6 Theorem (Expansion cuts).

$$T \Rrightarrow_{\mathcal{L},\emptyset} S \textit{ iff (i) } A[\vec{\mathsf{F}}], T \Rrightarrow_{\mathcal{L}[\vec{\mathsf{F}}],\emptyset} S \textit{ and (ii) } T \Rrightarrow_{\mathcal{L},\vec{\mathsf{F}}} A[\vec{\mathsf{F}}], S.$$

Proof. In the direction ($\rightarrow$), (i) take any $\mathcal{M}$ such that $\mathcal{M} \models A[\vec{\mathsf{F}}], T$. Contract it to $\mathcal{M}'$ for $\mathcal{L}$. We have $\mathcal{M}' \models T$, and we get $\mathcal{M}' \Rrightarrow S$, and hence $\mathcal{M} \Rrightarrow S$, from the assumption. (ii) Take any $\mathcal{M} \models T$, we get $\mathcal{M} \Rrightarrow S$ from the assumption. Arbitrary expansion of $\mathcal{M}$ to $\mathcal{M}'$ for $\mathcal{L}[\vec{\mathsf{F}}]$ will satisfy one of S.

In the direction ($\leftarrow$), take any $\mathcal{M}$ for $\mathcal{L}$ satisfying T. From the assumption (ii), we get an expansion $\mathcal{M}'$ satisfying one of $A[\vec{\mathsf{F}}], S$. If it is one of S, we also have $\mathcal{M} \Rrightarrow S$. Otherwise $\mathcal{M}' \models A[\vec{\mathsf{F}}]$. Now the assumption (i) applies, and we get $\mathcal{M}' \Rrightarrow S$, and hence $\mathcal{M} \Rrightarrow S$. □

3.7 Proofs in quantification logic. Thm. 3.6 can be strengthened by replacing in it everywhere $\emptyset$ with $\vec{G}$. As can be seen from the form of the initial properties in Par. 3.4 and from the following definition of proofs in $\vdash_{\mathrm{q}}$, we will use the notion of expansion consequence in an even weaker form with $\vec{F}$ containing at most one symbol (including none).

A seven-place relation $\mathcal{D}$ *q-witnesses* $T, \Gamma \mathrel{|\!\Rightarrow}_{\mathcal{L},\vec{F}} S, \Delta$, in writing $\mathcal{D} \vdash_{\mathrm{q}} T; \Gamma \mathrel{|\!\Rightarrow}_{\mathcal{L},\vec{F}} S; \Delta$, is defined as the least relation satisfying:

- $\circ \vdash_{\mathrm{q}} T; \Gamma \mathrel{|\!\Rightarrow}_{\mathcal{L},\vec{F}} S; \Delta$ if the assertion $(T \cap \Delta), \Gamma \mathrel{|\!\Rightarrow}_{\mathcal{L},\vec{F}} (S \cap \Gamma), \Delta$ is a weakening of an initial quantification property,
- $\dfrac{\mathcal{D} \qquad \mathcal{E}}{A[\vec{F}]} \vdash_{\mathrm{q}} T; \Gamma \mathrel{|\!\Rightarrow}_{\mathcal{L},\emptyset} S; \Delta$ if $\mathcal{D} \vdash_{\mathrm{q}} T; A[\vec{F}], \Gamma \mathrel{|\!\Rightarrow}_{\mathcal{L}[\vec{F}],\emptyset} S; \Delta$ and $\mathcal{E} \vdash_{\mathrm{q}} T; \Gamma \mathrel{|\!\Rightarrow}_{\mathcal{L},\vec{F}} S; A[\vec{F}], \Delta$.

3.8 Lemma (Reduction of G3c proofs to $\vdash_{\mathrm{q}}$ trees). *If there is a derivation $\mathcal{E}$ in **G3c** of a closed sequent $\Gamma \Rightarrow \Delta$ in language $\mathcal{L}$, then there is a valuation tree $\mathcal{E}^*$ q-witnessing $\Gamma \mathrel{|\!\Rightarrow}_{\mathcal{L},\emptyset} \Delta$. If $\mathcal{E}$ is cut free, then $\mathcal{E}^*$ is free cut free.*

Proof. The proof goes along the lines of the proof of Lemma 2.14 with the difference that we need to take into account the extensions of $\mathcal{L}$ with Henkin constants in the translated valuation tree $\mathcal{E}^*$. To that end, we assume that the eigen-variable rules (L$\exists$) and (R$\forall$) applied in $\mathcal{E}$ use new eigen-variables y_{c} corresponding to the Henkin constants c. For the duration of the proof, we write Γ^c for the set of formulas in Γ with every eigen-variable y_{c} replaced by the constant c. We prove by induction on $\mathcal{D}$ the following property:

> If $\mathcal{D}$ is a subtree of $\mathcal{E}$ proving in **G3c** the sequent $\Gamma \Rightarrow \Delta$, then there is a valuation tree $\mathcal{D}^*$ s.t. $\mathcal{D}^* \vdash_{\mathrm{q}} \Gamma^c \mathrel{|\!\Rightarrow}_{\mathcal{L}',\emptyset} \Delta^c$ where $\mathcal{L}'$ is the extension of $\mathcal{L}$ with Henkin constants corresponding to the eigen-variables y_{c} introduced on the branch leading from $\mathcal{D}$ to $\mathcal{E}$.

Note that for $\mathcal{D} = \mathcal{E}$ we have $\mathcal{L} = \mathcal{L}'$, $\Gamma^c = \Gamma$, and $\Delta^c = \Delta$.

The translation of rules of **G3c** in the inductive step is the same as in Lemma 2.14 for the propositional rules (see Fig. 4). The translation of quantification rules is in Fig. 5. In order not to clutter the figures we ignore in both of them the systematic replacement of eigen-variables by Henkin constants.

The most interesting case is the translation of (R$\forall$) where $\Delta = A[x/\mathsf{c}], \Delta_1$. We first cut on the sentence $\neg A^c[x/\mathsf{c}]$, and let the right leaf q-witness a weakening of the initial quantification property (R$\forall$). We then cut the left branch on $A^c[x/\mathsf{c}]$, let the left leaf q-witness a weakening of the initial propositional property $(L\neg)$: $A^c[x/\mathsf{c}], \neg A^c[x/\mathsf{c}] \mathrel{|\!\Rightarrow}_{\mathcal{L}'[\mathsf{c}],\emptyset} \emptyset$, and use the inductive hypothesis $\mathcal{D}_1^* \vdash_{\mathrm{q}} \Gamma^c \mathrel{|\!\Rightarrow}_{\mathcal{L}'[\mathsf{c}],\emptyset} A^c[x/\mathsf{c}], \Delta_1^c$ to q-witness by the same valuation tree $\mathcal{D}_1^*$ its weakening $\neg A^c[x/\mathsf{c}], \Gamma^c \mathrel{|\!\Rightarrow}_{\mathcal{L}'[\mathsf{c}],\emptyset} A^c[x/\mathsf{c}], \forall x\, A^c[x], \Delta_1^c$.

If $\mathcal{E}$ is cut free, then there are no free cuts in $\mathcal{E}^*$. □

3.9 Theorem (Soundness and completeness of $\vdash_{\mathrm{q}}$). *We have in quantification logic $T \mathrel{|\!\Rightarrow}_{\mathcal{L},\emptyset} S$ iff $\mathcal{D} \vdash_{\mathrm{q}} T \mathrel{|\!\Rightarrow}_{\mathcal{L},\emptyset} S$ for a free cut free $\mathcal{D}$.*

	$\mathcal{D}$	$\mathcal{D}^*$
R∃	$\dfrac{\overset{\mathcal{D}_1}{\Gamma \Rightarrow A[x/t], \exists x\, A[x], \Delta}}{\Gamma \Rightarrow \exists x\, A[x], \Delta}$	$\dfrac{\circ \vdash (\mathrm{R}\exists) \qquad \mathcal{D}_1^*}{A[x/t]}$
L∀	$\dfrac{\overset{\mathcal{D}_1}{A[x/t], \forall x\, A[x], \Gamma \Rightarrow \Delta}}{\forall x\, A[x], \Gamma \Rightarrow \Delta}$	$\dfrac{\mathcal{D}_1^* \qquad \circ \vdash (\mathrm{L}\forall)}{A[x/t]}$
L∃	$\dfrac{\overset{\mathcal{D}_1}{A[x/y_c], \Gamma \Rightarrow \Delta}}{\exists x\, A[x], \Gamma \Rightarrow \Delta}$	$\dfrac{\mathcal{D}_1^* \qquad \circ \vdash (\mathrm{L}\exists)}{A[x/c]}$
R∀	$\dfrac{\overset{\mathcal{D}_1}{\Gamma \Rightarrow A[x/y_c], \Delta}}{\Gamma \Rightarrow \forall x\, A[x], \Delta}$	$\dfrac{\dfrac{\circ \vdash (\mathrm{L}\neg) \qquad \mathcal{D}_1^*}{A[x/c]} \qquad \circ \vdash (\mathrm{R}\forall)}{\neg A[x/c]}$

Fig. 5. Translation of **G3c** proofs to valuation trees.

Proof. The soundness direction ($\leftarrow$) is proved similarly to Lemma 2.9, and the completeness direction ($\rightarrow$) follows from Lemma 3.8 by an auxiliary lemma corresponding to Lemma 2.15. ☐

4 First-order Logic with Equality

Had a language $\mathcal{L}$ of the quantification calculus from the previous section contained the binary equality predicate symbol $=$, it would have to be treated as a non-logical one, meaning that the usual properties of $=$ would have to be supplied by axioms in T.

For the rest of this paper, we treat the equality symbol as a logical one, and we will accordingly modify our proof calculus by strengthening the use of the initial properties without changing the definition of expansion cuts.

4.1 Semantics of equality. For the rest of this paper, we assign in structures the usual interpretation of the equality symbol always assumed to be in $\mathcal{L}$. This means that, although in the definition of $\models$ we add the clauses $\mathcal{M} \models s = t$, no other semantic definition needs to be explicitly changed. Thus, in particular, Lemma 3.3, Thm. 3.6, and Lemma 3.5 remain to hold.

4.2 Equality in sequent calculi. We have reduced the completeness proofs for the calculus $\vdash_{\mathrm{p}}$ and $\vdash_{\mathrm{q}}$ to the completeness of sequent calculi **G3cp** and **G3c** respectively. In this section we will reduce the equation calculus **G3c**$^=$ [NvP98]

[TS00, page 134] to the calculus $\vdash_{\mathrm{e}}$ defined in Par. 4.7. The former calculus contains two rules dealing with equality:

$$\text{Ref}\ \frac{t = t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad \text{Rep}\ \frac{P[x/s], P[x/t], t = s, \Gamma \Rightarrow \Delta}{P[x/t], t = s, \Gamma \Rightarrow \Delta}$$

where P is an atomic formula.

4.3 Lemma. *Every* **G3c**$^{=}$ *proof can be transformed by permuting its equational rules above the logical rules. The relative order of equational rules is not changed.*

Proof. The principal formulas of equation rules are atomic, and so the rules can be permuted with all logical rules of **G3c** including cuts (even cuts with principal formulas atomic). This is formally done by induction on the number of permutations needed. □

4.4 Equality closure. For a theory T and a set of closed terms D we define the *equality closure of T over D* as the smallest set $\mathrm{Eq}^*_D(T)$ satisfying:

- $T \cup \{(t = t) \mid t \in D\} \subseteq \mathrm{Eq}^*_D(T)$,
- $P[x/s] \in \mathrm{Eq}^*_D(T)$ provided $P[x/t], (t = s) \in \mathrm{Eq}^*_D(T)$ and $P[x]$ is an atomic formula with at most x free.

Let $\mathrm{Terms}(T)$ denote the set of all subterms occurring in the **atomic** sentences of T. We set $\mathrm{Eq}^*_{S,\vec{\mathsf{F}}}(T) := \mathrm{Eq}^*_D(T)$ where

$$D = \mathrm{Terms}(T) \cup (\mathrm{Terms}(S) \setminus \{s \mid \text{term } s \text{ contains any of } \vec{\mathsf{F}}\}).$$

4.5 Lemma. *$\mathcal{M} \models T$ iff $\mathcal{M} \models \mathrm{Eq}^*_{S,\vec{\mathsf{F}}}(T)$ for all structures $\mathcal{M}$ for $\mathcal{L}$ and theories T in $\mathcal{L}$, S in $\mathcal{L}[\vec{\mathsf{F}}]$.* □

4.6 Lemma. *If $T \models P$ and $T \cup \{P\}$ consists of atomic sentences, then $P \in \mathrm{Eq}^*_{\mathrm{Terms}(T,P)}(T)$.*

Proof. Let $D := \mathrm{Terms}(T, P)$ and $E := \mathrm{Eq}^*_D(T)$. By way of contradiction assume $P \notin E$. We will construct a model $\mathcal{M}$ of T falsifying P. Define $[t] = \{s \in D \mid (t = s) \in E\}$. It is easy to see that the sets $[s]$ form a partition $\mathcal{P}$ of D. Let $\mathcal{P} \cup \{\emptyset\}$ be the domain of $\mathcal{M}$. Interpret every function symbol f and predicate symbol R of $\mathcal{L}$ as follows:

$$\mathsf{f}^{\mathcal{M}}(d_1, \ldots, d_n) = \begin{cases} [\mathsf{f}(t_1, \ldots, t_n)] & \text{if } d_1{=}[t_1],\ \ldots,\ d_n{=}[t_n],\ \mathsf{f}(t_1, \ldots, t_n){\in}D, \\ \emptyset & \text{otherwise;} \end{cases}$$

$$\mathsf{R}^{\mathcal{M}}(d_1, \ldots, d_n) \text{ iff } d_1{=}[t_1],\ \ldots,\ d_n{=}[t_n],\ \mathsf{R}(t_1, \ldots, t_n){\in}E \text{ for some } t_1,\ \ldots,\ t_n\,.$$

It is easy to see that the interpretation is well-defined, and that $\mathcal{M} \models T$. If P is an equality $t = s$, then it must be the case that $t^{\mathcal{M}} = [t] \neq [s] = s^{\mathcal{M}}$, and so $\mathcal{M} \not\models t = s$. If P is $\mathsf{R}(t_1, \ldots, t_n)$, then not $\mathsf{R}^{\mathcal{M}}([t_1], \ldots, [t_n])$, i.e., $\mathcal{M} \not\models \mathsf{R}(t_1, \ldots, t_n)$. □

4.7 Proofs in first-order logic with equality. We now extend the relation of q-witnessing (see Par. 3.7) to first-order logic with equality without adding any new initial properties or changing the extension cuts. The difference is that instead of letting $\circ$ witness a set Γ in antecedents of $\Mapsto$, we saturate Γ with equalities by forming its equality closure.

The seven place relation $\mathcal{D}$ *e-witnesses* $T, \Gamma \Mapsto_{\mathcal{L},\vec{F}} S, \Delta$, in writing $\mathcal{D} \vdash_{\mathrm{e}} T; \Gamma \Mapsto_{\mathcal{L},\vec{F}} S; \Delta$, is the least relation satisfying:

- $\circ \vdash_{\mathrm{e}} T; \Gamma \Mapsto_{\mathcal{L},\vec{F}} S; \Delta$ if $(T \cap \Delta), \mathrm{Eq}^*_{\Delta,\vec{F}}(\Gamma) \Mapsto_{\mathcal{L},\vec{F}} (S \cap \Gamma), \Delta$ is a weakening of an initial quantification property,
- $\dfrac{\mathcal{D} \qquad \mathcal{E}}{A[\vec{F}]} \vdash_{\mathrm{e}} T; \Gamma \Mapsto_{\mathcal{L},\emptyset} S; \Delta$ if $\mathcal{D} \vdash_{\mathrm{e}} T; A[\vec{F}], \Gamma \Mapsto_{\mathcal{L}[\vec{F}],\emptyset} S; \Delta$, and $\mathcal{E} \vdash_{\mathrm{e}} T; \Gamma \Mapsto_{\mathcal{L},\vec{F}} S; A[\vec{F}], \Delta$.

Note that the relation is decidable because the equality closure for finite sets is finite.

4.8 Lemma (Reduction of G3c$^=$ proofs to $\vdash_{\mathrm{e}}$ trees). *If there is a derivation $\mathcal{E}$ in* **G3c**$^=$ *of a closed sequent $\Gamma \Rightarrow \Delta$ in a language $\mathcal{L}$, then there is a valuation tree $\mathcal{E}^*$ e-witnessing $\Gamma \Mapsto_{\mathcal{L},\emptyset} \Delta$. If $\mathcal{E}$ is cut free, then $\mathcal{D}^*$ is free cut free.*

Proof. For the duration of the proof we write Γ^c for the set of formulas in Γ but with every eigen-variable y_c replaced by the constant c. We write Γ^a for the set of atomic formulas in Γ. We may assume that $\mathcal{E}$ is in a form guaranteed by Lemma 4.3. We proceed by induction on $\mathcal{D}$ similar to that in Lemma 3.8:

> If $\mathcal{D}$ is a subtree of $\mathcal{E}$ proving in **G3c**$^=$ the sequent $\Gamma \Rightarrow \Delta$ and its immediate ancestor in $\mathcal{E}$ (if any) is not obtained by an equality rule (Ref) or (Rep), then there is a valuation tree $\mathcal{D}^*$ s.t. $\mathcal{D}^* \vdash_{\mathrm{e}} \Gamma^c \Mapsto_{\mathcal{L}',\emptyset} \Delta^c$ where $\mathcal{L}'$ is the extension of $\mathcal{L}$ with Henkin constants corresponding to the eigen-variables y_c introduced on the branch leading from $\mathcal{D}$ to $\mathcal{E}$.

In the base case, $\mathcal{D}$ is a maximal branch consisting of at most rules (Ref) or (Rep). Let the leaf of $\mathcal{D}$ be $\Gamma' \Rightarrow \Delta$. It must be an identity axiom and so there is a formula $A \in \Gamma_1 \cap \Delta$. We consider two cases: If A is not atomic, then we must have $A \in \Gamma$, and it suffices to set $\mathcal{D}^* := \circ$ because $\mathrm{Eq}^*_{\Delta^c,\emptyset}(\Gamma^c) \Mapsto_{\mathcal{L}',\emptyset} \Delta^c$ is a weakening of (Ax). If A is atomic, then we remove the non-atomic formulas from all sequents in $\mathcal{D}$ whereby we obtain a proof $\mathcal{D}^a$ of the sequent $\Gamma^a \Rightarrow \Delta^a$ with the leaf $\Gamma^a_1 \Rightarrow \Delta^a$ s.t. $A \in \Gamma^a_1 \cap \Delta^a$. Define $D := \mathrm{Terms}(\Gamma^c) \cup \mathrm{Terms}(\Delta^c)$ and

$$D' := D \cup \{t \mid (t = t) \text{ is principal in a (Ref) rule of } \mathcal{D}\}^c.$$

By the form of equational rules, we have $\Gamma^{ac}_1 \subseteq \mathrm{Eq}^*_{D'}(\Gamma^{ac})$. Thus $\mathrm{Eq}^*_{D'}(\Gamma^{ac}) \models A^c$, and by Lemma 4.5 $\Gamma^{ac} \models A^c$. By Lemma 4.6 we then get

$$A^c \in \mathrm{Eq}^*_{\mathrm{Terms}(\Gamma^{ac},A^c)}(\Gamma^{ac}) \subseteq \mathrm{Eq}^*_D(\Gamma^{ac}) \subseteq \mathrm{Eq}^*_D(\Gamma^c).$$

Thus it suffices to set $\mathcal{D}^* := \circ$ because $\circ \vdash_{\mathrm{e}} \Gamma^c \Mapsto_{\mathcal{L}',\emptyset} \Delta^c$. The inductive case goes through just as in the proof of Lemma 3.8.

If $\mathcal{E}$ is cut free, then there are no free cuts in $\mathcal{E}^*$. $\square$

4.9 Theorem (Soundness and completeness of $\vdash_{\mathrm{e}}$). *We have $T \mapsto_{\mathcal{L},\emptyset} S$ iff $\mathcal{D} \vdash_{\mathrm{e}} T \mapsto_{\mathcal{L},\emptyset} S$ for a free cut free $\mathcal{D}$.*

Proof. Soundness ($\leftarrow$) follows from the proposition

if $\mathcal{D} \vdash_{\mathrm{e}} T \mapsto_{\mathcal{L},\vec{F}} S$, then $T \mapsto_{\mathcal{L},\vec{F}} S$,

proved by induction on $\mathcal{D}$. In the base case, we have $(T \cap \Delta), \mathrm{Eq}^*_{\Delta,\vec{F}}(\Gamma) \mapsto_{\mathcal{L},\vec{F}} (S \cap \Gamma), \Delta$ because it is a weakening of an initial quantification property. We wish to show that also $T, \Gamma \mapsto_{\mathcal{L},\vec{F}} S, \Delta$ holds. So we take a structure $\mathcal{M}$ for $\mathcal{L}$ of T, Γ. By Lemma 4.5, we also have $\mathcal{M} \models \mathrm{Eq}^*_{\Delta,\vec{F}}(\Gamma)$, and thus $\mathcal{M}$ can be expanded to satisfy one of $(S \cap \Gamma), \Delta$ and hence one of S, Δ. Induction step follows directly from Thm. 3.6 since we must have $\vec{F} = \emptyset$.

Completeness ($\rightarrow$) follows from Lemma 4.8 by an auxiliary lemma corresponding to Lemma 2.15. ☐

4.10 Discussion. Our treatment of full first-order logic uses equality automatically. This is not only possible, but also feasible, by the existence of an efficient congruence closure algorithm of [DST80].

We have on purpose proved the completeness of our calculi by reduction of corresponding complete sequent calculi. The reason was that we wished to exhibit similarities between sequent proofs and valuation trees. In a self-contained exposition we would employ the well-known direct method used with sequent calculi. The method is even more natural with valuation trees. We must permit infinite valuation trees and assure that we systematically assign truth values to all sentences in T, S as well as to all immediate subformulas of sentences appearing on branches closer to the root of the constructed valuation tree. This means in particular, that we must use all terms of $\mathcal{L}$ for the instantiation of quantifiers. If such a systematic procedure fails to stop with a finite valuation tree, and thus fails to e-witness the property $T \mapsto S$, then the tree must contain an infinite consistent path. We then perform the well-known construction used also in the proof of Lemma 4.6 to construct a structure giving all sentences on the path their assigned values. The structure will thus satisfy T and falsify all of S.

5 Extensions of Theories by Definitions

In this section we treat extension of theories by definitions of predicate and function symbols. For that it suffices to extend the base case of the proof predicate $\vdash_{\mathrm{e}}$ with new initial properties justifying the extensions.

5.1 Initial extension properties. *Initial extension properties* are the initial quantification properties (see Par. 3.4) plus the following ones:

$$
\begin{array}{ll}
\text{(PDef)} & \emptyset \mapsto_{\mathcal{L},\mathsf{R}} \forall \vec{x}\,(\mathsf{R}(\vec{x}) \leftrightarrow A[\vec{x}]) \\
\text{(FDef)} & \forall \vec{x}\, \exists! y\, B[\vec{x}, y] \mapsto_{\mathcal{L},\mathsf{f}} \forall \vec{x}\, \forall y\, (\mathsf{f}(\vec{x}) = y \leftrightarrow B[\vec{x}, y])\,.
\end{array}
$$

for any formulas $A[x_1, \ldots, x_n]$, $B[x_1, \ldots, x_n, y]$ in $\mathcal{L}$ with only the indicated variables free and any symbols R, f not in $\mathcal{L}$.

5.2 Lemma. *Initial extension properties are true.*

Proof. See [Sho67]. □

5.3 Proofs in extension logic. We define define a new relation $\mathcal{D}$ *d-witnesses* $T, \Gamma \Rrightarrow_{\mathcal{L},\vec{F}} S, \Delta$, in writing $\mathcal{D} \vdash_{\mathrm{d}} T; \Gamma \Rrightarrow_{\mathcal{L},\vec{F}} S; \Delta$. The relation is just like the $\vdash_{\mathrm{e}}$ relation (see Par. 4.7) but we permit initial extension properties instead of quantificational ones.

Thm. 5.4 asserts the well-known conservativity of extensions by definitions where any use of new symbols in a $\vdash_{\mathrm{d}}$ proof of a property not containing new symbols can be eliminated so the property has an $\vdash_{\mathrm{e}}$ proof. Thm. 5.5 asserts that that the $\vdash_{\mathrm{d}}$ calculus is sound. We cannot have completeness unless we are willing to examine all possible definitional extensions. For that we would have to restrict the definition of $\Rrightarrow_{\mathcal{L},\vec{F}}$ in Par. 3.2 to Henkin and definitional extensions. To examine all extensions is, however, not the intent of introducing new symbols. They are introduced in order to manage the complexity of proofs by abbreviating long formulas.

5.4 Theorem (Conservativity). *If $\mathcal{D} \vdash_{\mathrm{d}} T \Rrightarrow_{\mathcal{L},\emptyset} S$, then $\mathcal{D}^* \vdash_{\mathrm{e}} T \Rrightarrow_{\mathcal{L},\emptyset} S$ for some $\mathcal{D}^*$.*

Proof. $\mathcal{D}^*$ is obtained from $\mathcal{D}$ by a translation A^* eliminating introduced predicate and function symbols from formulas A. This is described by Shoenfield [Sho67] and also by Troelstra and Schwichtenberg [TS00, page 124]. □

5.5 Theorem (Soundness of extensions). *If $\mathcal{D} \vdash_{\mathrm{d}} T \Rrightarrow_{\mathcal{L},\vec{F}} S$, then $T \Rrightarrow_{\mathcal{L},\vec{F}} S$.*

Proof. Similarly to the proof of soundness in Thm. 4.9 except that we use Lemma 5.2 in the base case instead of Lemma 3.5. □

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