Answer Set Programming

Properties

Martin Baláž

Department of Applied Informatics
Comenius University in Bratislava

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Outline

1. Existence of Stable Model
2. Minimality
3. Support

Classes of Logic Programs

- Default Negation
  - without default negation
  - default negation in the bodies
  - default negation also in the heads
- Disjunction
  - without disjunction
  - disjunction in the heads
- Explicit Negation
  - without explicit negation
  - with explicit negation
- Constraints
  - without constraints
  - with constrains

Stable Models

Definition (Program Reduct)

Let I be an interpretation. A *reduct* of a generalized disjunctive extended logic program with constraints Π (denoted by Π^I) is a positive disjunctive extended logic program with constraints obtained from Π by deleting

- rules containing a default literal $L, I \not\models L$ in the body
- rules containing a default literal $L, I \models L$ in the head
- default literals $L, I \models L$ in the bodies of remaining rules
- default literals $L, I \not\models L$ in the heads of remaining rules

Definition (Stable Model)

An interpretation I is a *stable model of* a generalized disjunctive extended logic program with constraints Π if I is a min. model of Π^I.

Problems

- Existence of Stable Model
  - For given logic program, does exist a stable model?
  - Does exist a logic program with at least two stable models?
- Minimality
  - Is a stable model of a logic program minimal?
- Support
  - For any objective atom, does exist a supporting rule?

Logic Programs without Default Negation

Theorem

Let Π be a positive logic program. Then $T_\Pi \uparrow \omega$ is the only stable model of Π.

Theorem

Let Π be a positive disjunctive logic program. Then there exists a stable model of Π.

Example (Two stable models)

```
a v b.
```

$$M_1 = \{a\}$$
$$M_2 = \{b\}$$

Remaining Classes of Logic Programs

Example (Normal Logic Programs)

```
a :- not a.
```

Example (Extended Logic Programs)

```
a.
-a.
```

Example (Logic Programs with Constraints)

```
a.
:- a.
```

Brave Reasoning

Example

```
a v b.
a?

$ dlv -brave test1
a is bravely true, evidenced by {a}
```

Example

```
b v c.
a?

$ dlv -brave test2
a is bravely false
```

Cautious Reasoning

Example

```
a v b.
a?

$ dlv -cautious test1
a is cautiously false, evidenced by {b}
```

Example

```
a :- not a.
a?

$ dlv -cautious test3
a is cautiously true
```

Minimality

Theorem

Stable model of a normal disjunctive extended logic program is minimal.

Theorem

Stable model of a general extended logic program is minimal.

Example (Generalized Disjunctive Logic Programs)

```
a v not a.
```

$$
\begin{aligned}
M_1 &= \emptyset \\
M_2 &= \{a\}
\end{aligned}
$$

Proof

Lemma

Let I and I' be interpretations and Π be a normal disjunctive logic program. Then $I' \subseteq I \Rightarrow \Pi^{I} \subseteq \Pi^{I'}$.

Lemma

Let I be an interpretation and Π be a logic program. Then I is a model of Π iff I is a model of Π^{I}.

Proof of Theorem.

A stable model I of a normal disjunctive logic program Π is a minimal model of Π^{I}.

Let $I' \subset I$ be a model of Π. Then I' is a model of $\Pi^{I'}$. Because $\Pi^{I} \subseteq \Pi^{I'}$, I' is a model of $\Pi^{I'}$. Because I is a minimal model of $\Pi^{I'}$, $I' \not\subset I$. □

Logic Programs without Disjunction

Definition (Support)

Let I be an interpretation and Π be a generalized logic program. A rule $r \in \Pi$ *supports* an atom $A \in I$ if

- $A = head(r)$
- $I \models body(r)$

An interpretation I is *supported* if for all atoms $A \in I$ there exists a rule $r \in \Pi$ such that r supports A.

Example (Supported Model, but Not Stable Model)

```
a :- a.
```

$$M = \{a\}$$

Logic Programs without Disjunction

Theorem

Let I be a stable model of a normal logic program Π. Then I is a supported model of Π.

Proof.

Stable model I of a normal logic program Π is the least model of Π^I. Let $A \in I$ be a non-supported atom. We show that $I' = I \setminus \{A\}$ is a model of Π^I, i.e. I is not the least model of Π.
Let $r \in \Pi^I$. If $A \in body(r)$ then $I' \models r$ because $I' \not\models body(r)$. Let $A \notin body(r)$. If $I' \not\models body(r)$ then $I' \models r$. If $I' \models body(r)$ then $I \models body(r)$. Because I is a model of Π^I, $I \models head(r)$, and because A is not supported, $A \notin head(r)$. Then $I' \models head(r)$. □

Well-Support

Definition (Well-Support)

Let I be an interpretation, Π be a normal logic program and ℓ be a level mapping. A rule $r \in \Pi$ *well-supports* an atom $A \in I$ if r supports A and

- $\ell(A) > \ell(B)$ for all atoms $B \in body(r)$

An interpretation I of Π is *well-supported* if for all atoms $A \in I$ there exists a rule well-supporting A.

Theorem

An interpretation I is a stable model of a normal logic program Π iff there exists a level mapping such that I is well-supported model of Π.

Example

Example

```
 p :- a.
 p :- b.
 a :- not -a.
-a :- not a.
 b :- not -b.
-b :- not b.
 p :- not p.
```

$$\begin{array}{ll} \ell(A) = n+1 & \text{if } A \in I, A \notin T_{P^I} \uparrow n, A \in T_{P^I} \uparrow (n+1) \\ \ell(A) = 0 & \text{if } A \notin I \end{array}$$

$$\ell(A) = n \quad \Rightarrow \quad A \in T_{P^I} \uparrow n$$

Logic Programs with Disjunction

Definition (Support)

Let I be an interpretation and Π be a generalized disjunctive logic program. A rule $r \in \Pi$ *weakly supports* an atom $A \in I$ if

- $A \in head(r)$
- $I \models body(r)$

A rule $r \in \Pi$ *supports* an atom $A \in I$ if in addition

- $\forall A' \in head(r) \colon A' \neq A \Rightarrow I \not\models A'$

An interpretation I is *(weakly) supported* if for all atoms $A \in I$ there exists a rule $r \in \Pi$ such that r (weakly) supports A.

Example (Weakly Supported Model, but Not Supported)

```
a v b.
```

Logic Programs with Disjunction

Theorem

Let I be a stable model of a normal disjunctive logic program Π. Then I is a supported model of Π.

Proof.

Stable model I of a normal disjunctive logic program Π is a minimal model of Π^I. Let $A \in I$ be a non-supported atom. We show that $I' = I \setminus \{A\}$ is a model of Π^I, i.e. I is not a minimal model of Π. Let $r \in \Pi^I$. If $A \in body(r)$ then $I' \models r$ because $I' \not\models body(r)$. Let $A \notin body(r)$. If $I' \not\models body(r)$ then $I' \models r$. If $I' \models body(r)$ then $I \models body(r)$. Because I is a model of Π^I, $I \models head(r)$, and because A is not supported, there exists $B \in I$, $I \models B$. Then $I' \models head(r)$. □

Example

Example

```
a v b.
a :- b.
b :- a.
```

$$M = \{a, b\}$$

Even all atoms are supported, the model M can not be iteratively computed from the empty interpretation only by using strongly supporting rules.