Dynamic Kripke Structures

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Abstract

The paper is devoted to dynamic and context-dependent aspects of knowledge and reasoning. The main attention is focused on a relationship between belief revision and inference.
Dynamic Kripke Structures (DKS) are introduced in the paper. Intuitively, DKS consists of a dynamic part and a static part (a usual Kripke Structure). The elements of the dynamic part map a set of possible worlds on itself. A unified view on context-dependent and non-monotonic reasoning – based on Dynamic Kripke Structures – is discussed.
In order to motivate DKS we introduce a set of postulates (a Dynamic Belief Model, DBM) in a style similar to Gärdenfors' approach – propositions are treated as modifications of belief sets. Non-commutativity of conjunction and non-monotonicity of inference are the main results concerning DBM. A semantics of DBM is studied and generalized to DKS.

1 INTRODUCTION

Understanding dynamic and context-dependent aspects of knowledge and reasoning is of crucial importance for artificial intelligence research.

A substantial feature of intelligence is an ability to change beliefs, and to reason correctly with the changed beliefs (in various contexts).

Belief revision and non-monotony of reasoning are examples of some fundamental problems connected with the dynamics and context-sensitivity of reasoning.

The main idea of the paper is a concept of Dynamic Kripke Structure. We hope that the concept is an useful tool for a (unifying) semantic characterization of dynamic and context-dependent aspects of knowledge and (non-monotonic, hypothetical) reasoning.

A Dynamic Kripke Structure (DKS) consists of two components – a static one (a Kripke Structure) and a dynamic one (a set of transformations). Intuitively, DKS enables to specify a change (or a stability) of belief when transitions from one context to another are caused by some events. The contexts are represented by Kripke structures, the events by the transformations.

A Kripke Structure consists of two components. The first is a set of states of the world (some alternative intuitions: possible worlds, contexts, sets of belief sets). The second are some accessibility relations between possible worlds. The accessiblity provides, in a sense, a relaxation of the intuitive idea of the state of the world: the state may be described in an incomplete way and the accessible worlds (epistemic alternatives) are consistent with the description.

Transformations between possible worlds we can imagine as changes in believed hypotheses or as transitions from one (set of) context(s) to another (set of) context(s). The accessibility relation between possible worlds may be changed by the transformation significantly.

Before we introduce the concept of DKS in a rigorous way we will investigate a sequence of constructions. The main reason is to motivate well DKS.

First, in Sections 2 – 4 we try to use and explore an idea of Gärdenfors, see [2]. Propositions are considered as functions from belief states to belief states. An application of a proposition to a belief state is conceived in [2] as an addition of new evidence into the resulting belief state. Conjunction of propositions is defined as composition of functions and it is obvious that the conjunction (as composition of insertions) is commutative.

We extend this view: if a proposition maps an original belief state into a resulting belief state, we assume that the resulting belief state may contain some new information and that some information from the original belief state may be deleted.

We investigate the consequences of this decision: conjunction of propositions is non-commutative and a consequence relation (defined in the same way as in the [2]) is non-monotonic. Let us note that this non-monotony is an obvious consequence of some basic properties [1] of belief revisions and context-dependent reasoning. A set of postulates is proposed in order to formalize the idea.

In Section 5 we provide a semantics appropriate for our postulates. We assign interpretations to belief sets and some mappings (from interpreta-

[1] "…non-monotonic behaviour …is a *symptom*, rather than the essence of non-standard inference", see [8].

tions to interpretations) to propositions. We prove that our postulates are satisfied in the semantics. Furthermore, we prove a representation theorem: we can assign (in a unique way) to each proposition a mapping from interpretations to interpretations.

The result is interesting in its own right. But our goal is to motivate the introduction of more general Dynamic Kripke Structures. In the rest of the paper we generalize the semantic constructions in order to introduce DKS. A non-monotonic consequence relation was introduced in a simple model of belief revision (in the DBM). Our goal is to generalize the semantics of DBM and to create a semantic basis (the DKS) for more general treatment of non-monotonic (hypothetical, context-dependent, non-standard) inference.

2 POSTULATES

We will distinguish two kinds of entities, propositions and belief states. Let $\mathcal{P}$ be a set of propositions and $\mathcal{K}$ be a set of belief states. A, B, C, D, I will be used as variables over $\mathcal{P}$, K (with subscripts or superscripts) will be used as variables over $\mathcal{K}$.

Propositions are mappings from belief states to belief states. If we apply a proposition (function) A to a belief state K, then $A(K)$ represents the resulting belief state.

Now, some postulates are presented. [2] First, we accept a part of Gärdenfors' postulates (from [2]).

(P1) For every A and K there is $K^{'}$ such that $A(K) = K^{'}$

(P2) There is a proposition I such that for each K holds $I(K) = K$

(P3) For every A, B exists a proposition $A \wedge B$ such that for every K holds $A \wedge B(K) = A(B(K))$

(P4) For every A and K holds $A \wedge A(K) = A(K)$

Some definitions, before we supplement the set of postulates.

Definition 1 *A* partial order *is a reflexive, antisymmetric and transitive relation.*

Let L be a set and $\leq$ be a partial order on L. $(L, \leq)$ is a lattice*, if for each $a, b \in L$ there is a least upper bound, denoted by $a \sqcup b$, and a greatest lower bound, $a \sqcap b$.* [3]

[2] Intuitions and motivations are discussed in the Section 3. The both sections could be read in parallel.

[3] A relation $\rho \subseteq S \times S$ (ρ is said to be a relation on S) is

- *reflexive* if for each $a \in S$ holds $(a, a) \in \rho$
- *antisymmetric*, if for a, b such that $(a, b) \in \rho$ and $(b, a) \in \rho$ holds $a = b$
- *transitive*, if holds $(a, c) \in \rho$, whenever $(a, b) \in \rho$ and $(b, c) \in \rho$.

An element $x \in L$ is called upper (lower) bound of a, b, if $a \leq x$ and $b \leq x$ ($x \leq a$ and $x \leq b$). If y is an upper (lower) bound of a, b and for each upper (lower) bound x of a, b holds that $y \leq x$ ($x \leq y$) then y is called the least upper (greatest lower) bound of a, b.

If for $a \in L$ and for each $x \in L$ holds $a \leq x$ ($x \leq a$), then a is the least *(*greatest*) element of L.*

Definition 2 (Conflicting functions) *Let L be a lattice with the least element $\perp$ and the greatest element $\top$. Let $A \neq I \neq B$ be functions from L to L, where I is the identity function.*

We say that A, B are conflicting, if the conditions as follows are satisfied:

(1) $A(\perp) \sqcup B(\top) = \top = A(\top) \sqcup B(\perp)$

(2) $A(\perp) \sqcap B(\top) = \perp = A(\top) \sqcap B(\perp)$

Now, the rest of our postulates:

(P0) $\mathcal{K}$ is a lattice with the least ($K_\perp$) and the greatest ($K_\top$) element.

(P5) For every A there is B such that A, B are conflicting propositions.

(P6) For every A and every K holds $A(K) = (K \sqcup A(K_\perp)) \sqcap A(K_\top)$

Proposition 1 *For every A and K, K', if $K \leq K'$ then $A(K) \leq A(K')$*

Proof: Straightforward. [4] $\square$

Proposition 2 *If A, B are conflicting, then $\forall K \neg(A(K_\perp) \leq B(K) \leq A(K_\top))$.*

Proof: Let us assume $A(K_\perp) \leq B(K) \leq A(K_\top)$ for some K. Hence $A(K_\perp) \sqcup B(K_\top) \leq B(K) \sqcup B(K_\top) \leq A(K_\top) \sqcup B(K_\top)$. From the definition of conflicting functions and from the proposition 1 follows $B(K_\top) = K_\top$.

Similarly, $A(K_\perp) \sqcap B(K_\perp) \leq B(K) \sqcap B(K_\perp) \leq A(K_\top) \sqcap B(K_\perp)$. Therefore, $B(K_\perp) = K_\perp$.

For each K: $B(K) = (K \sqcup B(K_\perp)) \sqcap B(K_\top) = (K \sqcup K_\perp) \sqcap K_\top = K$. Therefore, $B = I$ (in contradiction with the definition of conflicting functions). $\square$

Definition 3 (Consequence relation) *Let be $K \in \mathcal{K}$ and $A, B \in \mathcal{P}$. Then $A |\!\sim B$ iff $\forall K (B \wedge A(K) = A(K))$. We say that B is a consequence of A.*

We note that in this paper only a poor language is considered [5]

We will assume that propositions and belief states are sets $\mathcal{P}$ and $\mathcal{K}$ satisfying the conditions $P0$ – $P6$. Each set of propositions and belief states satisfying $P0$ – $P6$ is called Dynamic Belief Model (DBM).

[4] Complete proofs are in the full version of the paper.

[5] Similarly, Makinson in [5] makes no reference to the particular choice of connectives.

3 INTUITIONS

In this section we give some additional explanations, motivations, and intuitions.

Example 1 *We consider a propositional database (a store of propositional symbols) together with the basic update operations - insert and delete.*

Let a countable set of symbols $\mathcal{S}$ *be given (an alphabet of the propositional logic).*

Let $\mathcal{K}$*, the set of all belief states, be the set of all subsets of* $\mathcal{S}$ *(a belief state is an instance of the propositional database). A partial order on* $\mathcal{K}$ *is the subset relation. The greatest lower (least upper) bound of two sets is their intersection (union).*

The set of all propositions $\mathcal{P}$ *is defined as the set of all pairs* $(ins(\Phi), del(\Psi))$ [6] *, where* Φ *and* Ψ *are subsets of* $\mathcal{S}$ *and* $\Phi \cap \Psi = \emptyset$*.*

Let be $I = (ins(\emptyset), del(\emptyset))$*,* $K_\perp = \emptyset$*,* $K_\top = \mathcal{S}$*.*

Application of a proposition $A = (ins(\Phi), del(\Psi))$ *to a belief state* K *is represented as the belief state* $(K \cup \Phi) \setminus \Psi$*.*

For $A = (ins(\Phi), del(\Psi))$ *and* $B = (ins(\Phi^{'}), del(\Psi^{'}))$ *we define* $B \wedge A$ *as* $(ins((\Phi \cup \Phi^{'}) \setminus \Psi^{'}), del((\Psi \cup \Psi^{'}) \setminus \Phi^{'})$*.*

There is no problem to show that the postulates $P0$ – $P6$ *are satisfied in this example.*

We restrict our attention to the postulate $P5$*. (As a consequence, it will be shown that our conjunction is non-commutative.)*

Let A, B *be propositions. Let us assume that* $A = (ins(\{d, e\}), del(\{a\})), B = (ins(\{a\})),$
$del(\{d, e\}))$*. Clearly,* A, B *are conflicting:* $A(K_\perp) \sqcup B(K_\top) = A(\emptyset) \sqcup B(\mathcal{S}) = \{d, e\} \cup (\mathcal{S} \setminus \{d, e\}) = \mathcal{S} = B(\emptyset) \sqcup A(\mathcal{S})$*. Similarly,* $A(K_\perp) \sqcap B(K_\top) = \emptyset = B(\emptyset) \sqcap A(\mathcal{S})$*. It is straightforward to show that for each* A *there is a* B *such that* A, B *are conflicting.*

In the next section we will show that a conjunction of two conflicting propositions is non-commutative, see Proposition 3. In our example, $e \in A \wedge B(K)$ *for each* K *and* $e \notin B \wedge A(K)$ *for each* K*.*

Now some additional comments and explanations to postulates $P0$ – $P6$ follow.

(P0) We assume that there is a hierarchy of belief states:

- a belief state can be extended to a more rich belief state (or restricted to a more poor belief state)
- a useful abstraction is an initial, or empty, belief state; similarly, a final belief state joining all belief states
- the condition that for two given belief states there is the greatest lower (least upper) bound is a more restrictive one (it assumes a kind of completeness of the set of belief sets); in the conclusions of this paper we discuss the need of a more general set of postulates

[6] *ins* is intended to represent the insert operation, *del* delete.

(P1) Propositions are mappings from belief states to belief states (a proposition stated in a belief state results in a new belief state). Nevertheless, propositions and members of belief states are not necessary identical entities.

(P2) There is a proposition which causes no change of each belief state. It is called the *identity* proposition.

(P3) There is a proposition defined by two successive applications of two propositions. The corresponding operation on propositions is denoted by $\wedge$ (and called conjunction). The set of all propositions is closed under conjunction.

(P4) The repeated application of the same proposition gives nothing new.

(P5) For each revision (insert or delete) there is a complementary revision (in a sense). If A, B are conflicting, then insertion represented by one of them and removal represented by the other are complementary.

(P6) For each proposition A and each belief state K the value of $A(K)$ can be computed from the "representative" applications of A (i.e from $A(K_\top)$ and $A(K_\bot)$) and from K.

4 BASIC PROPERTIES

Proposition 3 (Non-commutativity of conjunction) *If A, B are conflicting, then*
$\forall K(A \wedge B(K) \neq B \wedge A(K))$

Proof: Let us assume that $A \wedge B(K_0) = B \wedge A(K_0)$ for some K_0. From the proposition 1 follows $A(K_\bot) \leq A \wedge B(K_0) \leq A(K_\top)$. Hence, $A(K_\bot) \leq B \wedge A(K_0) \leq A(K_\top)$. Contradiction with the proposition 2. $\Box$

Proposition 4 *If A, B are conflicting propositions and $C| \sim A$, then* $\forall K(C \wedge B(K) \neq B \wedge C(K))$.

Proof: Let be $C \wedge B(K_0) = B \wedge C(K_0)$. Hence, $A \wedge B(C(K_0)) = A \wedge C(B(K_0)) = C \wedge B(K_0) = B \wedge C(K_0) = B \wedge A(C(K_0))$. Contradiction – A, B are not commutative according to the proposition 3. $\Box$

The main theorem of this section expresses the non-monotony of the $|\!\sim$-relation.

Theorem 1 (Non-monotony of consequence relation) *If $A| \sim D$, $D \neq I$, and B, D are conflicting, then $\neg(B \wedge A|\!\sim D)$*

Proof: From the Proposition 3 follows that for each K holds $B \wedge D(K) \neq D \wedge B(K)$. Therefore, for some K_0, $D \wedge B(A(K_0)) \neq B \wedge D(A(K_0)) = B(A(K_0))$ $\Box$

Example 2 *A, B are defined as in the Example 1, $C = (ins(\{d, e, g\}), del(\{a\}))$, hence $C| \sim A$. It is straightforward to show that $B \wedge C| \sim A$ does not hold: Let K be $\{a, b, c\}$, then $B \wedge C(K) = \{a, b, c, g\}$ but $A(B \wedge C(K)) = \{b, c, d, e, g\}$.*

Now we will discuss the properties

$(G2u)$ $A|\!\sim x$ whenever $A|\!\sim y$, $A \cup \{y\}|\!\sim x$

$(G3^{'}u)$ $A \cup \{y\}|\!\sim x$ whenever $A|\!\sim y$, $A|\!\sim x$

stated and studied in [5], see also [1]. $G2u$, and $G3^{'}u$ are intended to be (a part of the) conditions which should satisfy any reasonable formalization of a non-monotonic consequence relation.

We present analogies of $G2u$ and $G3^{'}u$.

Theorem 2 *If $A|\!\sim B$ and $B \wedge A|\!\sim C$, then $A|\!\sim C$.*

Theorem 3 *If $A|\!\sim B$ and $A|\!\sim C$, then $B \wedge A|\!\sim C$.*

5 SEMANTICS

We assign interpretations to the belief sets and accessibility relations between the interpretations to the propositions.

Definition 4 (Interpretations) *Let $\mathcal{S}$ be a set of propositional symbols. An* interpretation *is a pair (T, F), where $T, F \subseteq \mathcal{S}$. An interpretation is called* 3-interpretation *whenever it satisfies the condition $T \cap F = \emptyset$. A 3-interpretation such that $T \cup F = \mathcal{S}$ is called* 2-interpretation.

Notation: We denote the set of all interpretations as Int, the set of all 2-interpretations as Int_2, the set of all 3-interpretations as Int_3. We introduce projections of interpretations $true$ and $false$: $true((T, F)) = T$ and $false((T, F)) = F$. For 2-interpretations it holds that $false((T, F)) = \mathcal{S} \setminus true((T, F))$. Therefore, each $(T, F) \in Int_2$ may be represented as T and we may consider the operations on sets ($\cap$, $\cup$) as defined on Int_2.

Now we define a mapping from belief sets to 2-interpretations.

Definition 5 (Assignment of interpretations to belief sets) *Let $\mathcal{I} : \mathcal{K} \longrightarrow Int_2$ be a mapping that for each K_1, K_2 satisfies the conditions:*

- $K_1 \neq K_2 \Rightarrow \mathcal{I}(K_1) \neq \mathcal{I}(K_2)$
- $\mathcal{I}(K_1 \sqcup K_2) = \mathcal{I}(K_1) \cup \mathcal{I}(K_2)$
- $\mathcal{I}(K_1 \sqcap K_2) = \mathcal{I}(K_1) \cap \mathcal{I}(K_2)$
- $\mathcal{I}(K_\perp) = (\emptyset, \mathcal{S})$ *and* $\mathcal{I}(K_\top) = (\mathcal{S}, \emptyset)$

Note: $\mathcal{I}$ is definable on $\mathcal{K}$ and $\mathcal{S}$, if $card(\mathcal{K}) \leq card(2^{\mathcal{S}})$, where $card(X)$ denotes the cardinality of the set X.

Definition 6 (Possible worlds) $\mathcal{W}_\mathcal{I} = \{(T, F) \in Int_2 : \exists K(\mathcal{I}(K) = (T, F))\}$ *is called $\mathcal{I}$-image of $\mathcal{K}$.*

Note: Elements of $\mathcal{W}_\mathcal{I}$ may be called possible worlds. $\mathcal{W}_\mathcal{I}$ is a lattice with the least and the greatest element. $\mathcal{I}$ is an isomorphism of $\mathcal{K}$ and $\mathcal{W}_\mathcal{I}$.

Definition 7 (Accessibility relations) **(a)** *Let be* $(T, F) \in Int_3$ *and* $(T_1, F_1), (T_2, F_2) \in Int_2$. *We say that* (T_1, F_1) *is* (T, F)-accessible *from* (T_2, F_2) *iff*

(a1) $T_1 = (T_2 \cup T) \setminus F$

(a2) $F_1 = (F_2 \cup F) \setminus T$

(b) *Let be* $A \in \mathcal{P}, K \in \mathcal{K}, \mathcal{I}(K) = (T_2, F_2)$. (T_1, F_1) *is called* A-accessible *from* (T_2, F_2) *iff* $\mathcal{I}(A(K)) = (T_1, F_1)$.

Note: We use 3-interpretations as a tool for a representation of changes. Intuitively, members of T change to true, members of F change to false and things change only when they are forced to (there is no change specified for symbols not in T or F).

Observation 1 *If* $(T_2, F_2) \in Int_2$, $(T, F) \in Int_3$ *and* $T_1 = (T_2 \cup T) \setminus F, F_1 = (F_2 \cup F) \setminus T$, *then* $T_1 \cap F_1 = \emptyset$ *and* $T_1 \cup F_1 = \mathcal{S}$. *(The definition 7 is correct.)*

Observation 2 *For each* $(T_1, F_1), (T_2, F_2) \in Int_2$ *there is a* $(T, F) \in Int_3$ *such that* (T_1, F_1) *is* (T, F)*-accessible from* (T_2, F_2).

Proposition 5 *Let* $(T_1, F_1), (T_2, F_2)$ *be given and* $T = T_1 \setminus T_2, F = T_2 \setminus T_1$. *If* $(T^{'}, F^{'})$ *satisfies both conditions* $(a1), (a2)$, *then* $T \subseteq T^{'}, F \subseteq F^{'}$.

Proof: Let $x \in T$, i.e. $x \in T_1 \wedge x \notin T_2$. From $x \in T_1$ follows $x \notin F^{'}$. Therefore, $x \in T^{'}$.

Similarly, for $x \in F$ holds $x \in T_2 \wedge x \notin T_1$. If $x \notin F^{'}$, then $x \in T_1$ (because of $x \in T_2$) – contradiction. □

Note: $T = T_1 \setminus T_2$ and $F = T_2 \setminus T_1$ are called the *minimal solutions* of the conditions $(a1), (a2)$ from the definition 7.

Theorem 4 (A representation theorem) *For each* K, K' *and each* A *there is exactly one minimal* $(T, F) \in Int_3$ *such that*

$\mathcal{I}(K)$ *is* (T, F)*-accessible from* $\mathcal{I}(K^{'})$ *iff* $K = A(K^{'})$.

Proof: Existence:

$\Leftarrow$

We assume that $K = A(K^{'})$. Hence, $\mathcal{I}(K)$ is A-accessible from $\mathcal{I}(K^{'})$. From the postulate $P6$ and the definition of interpretation follows $\mathcal{I}(K) = \mathcal{I}(A(K^{'})) = \mathcal{I}((K^{'} \sqcup A(K_{\perp})) \sqcap A(K_{\top})) = (\mathcal{I}(K^{'}) \cup \mathcal{I}(A(K_{\perp}))) \cap \mathcal{I}(A(K_{\top})$. Let us denote $\mathcal{I}(A(K_{\perp}))$ as $(T, \mathcal{S} \setminus T)$ and $\mathcal{I}(A(K_{\top}))$ as $(\mathcal{S} \setminus F, F)$. Therefore, $true(\mathcal{I}(K)) = (true(\mathcal{I}(K^{'})) \cup T) \setminus F$. Similarly for $false(\mathcal{I}(K^{'}))$, i.e. $\mathcal{I}(K)$ is (T, F)-accessible from $\mathcal{I}(K^{'})$

Minimality follows from the Proposition 5: $true(\mathcal{I}(A(K_{\perp})) = true(\mathcal{I}(A(K^{'}))\setminus true(\mathcal{I}(K^{'})$ and $false(\mathcal{I}(A(K_{\top})) = true(\mathcal{I}(K^{'}) \setminus true(\mathcal{I}(A(K^{'}))$.

$\Rightarrow$

If $A(K^{'}) \neq K$, then $\mathcal{I}(K) \neq (\mathcal{I}(K^{'}) \cup \mathcal{I}(A(K_{\perp}))) \cap \mathcal{I}(A(K_{\top}))$. Therefore, $true(\mathcal{I}(K)) \neq (true(\mathcal{I}(K^{'})) \cup T) \setminus F$ or $false(\mathcal{I}(K)) \neq (false(\mathcal{I}(K^{'})) \cup F) \setminus T$. Hence $\mathcal{I}(K)$ is not (T, F)-accessible from $\mathcal{I}(K^{'})$. □

Note: The representation theorem shows that each proposition A of a DBM can be represented by a pair $(T, F) \in Int_3$ and this representation is unique if we consider minimal interpretations only.

Definition 8 *If $A \in \mathcal{P}$ and $(T, F) \in Int_3$ satisfy the representation theorem, then we say that (T, F) is* assigned *to A.*

Note: (T, F) is determined uniquely by the Minimality condition (according to the representation theorem).

Consequence 1 *Let (T_A, F_A) be assigned to A. If K_1 is A-accessible from $K_\perp$, then $\mathcal{I}(K_1) = (T_A, \mathcal{S} \setminus T_A)$. If K_2 is A-accessible from $K_\top$, then $\mathcal{I}(K_2) = (\mathcal{S} \setminus F_A, F_A)$.*

Let us investigate the relation between DBM and 3-interpretations closer.

Definition 9 $Acc = \{((w, w_1), w_2) : w_1, w_2 \in \mathcal{W_I}, w \in Int_3$ *is minimal interpretation such that w_2 is w-accessible from w_1*$\}$

Lemma 1 *Each $\phi \in Acc$ is a function (of the type $Int_3 \times \mathcal{W_I} \longrightarrow \mathcal{W_I}$): for each $w_1 \in \mathcal{W_I}$ and $w \in Int_3$ there is exactly one $w_2 \in \mathcal{W_I}$ such that w_2 is w-accessible from w_1.*

Theorem 5 *If $\mathcal{K} = \mathcal{W_I}$ and $\mathcal{P} = Acc$, then postulates $P0 - P6$ of DBM are satisfied.*

Definition 10 *Let be $(T, F), (T^{'}, F^{'}) \in Int_3$. (T, F) is* stronger *than $(T^{'}, F^{'})$ iff $T^{'} \subseteq T, F^{'} \subseteq F$.*

Observation 3 $T \cap F^{'} = \emptyset$, $F \cap T^{'} = \emptyset$.

Proposition 6 *If $A, B \in \mathcal{P}$ and $(T_A, F_A), (T_B, F_B) \in Int_3$ are assigned to A, B, respectively, then $A | \!\sim B$ iff (T_A, F_A) is stronger than (T_B, F_B).*

6 SEMANTICS – REFINEMENTS

We have closed the first part of the paper. In order to progress, let us discuss the basic intuitions, summarize the results and the current state of our exposition and outline a perspective.

Our main ambition is to provide a semantics suitable for formalizations of dynamic aspects of knowledge and reasoning. The static picture of knowledge and reasoning was in the preceding sections identified with belief sets. A complete list of atoms (qualified as true or false) was assigned to a belief set and it determines in a unique way the set of true formulae (for the given belief set).

A dynamics is introduced if we consider changes of belief sets (represented here by some transformations, mappings). Intuitively, the dynamics corresponds to a generation of new sentences (hypotheses) or to a retraction of some hypotheses. Of course, the new knowledge has usually a preliminary status, it is open to some future revisions.

The model of DBM is very simple and a rudimentary one. We have a uniform and rich structured universe of belief sets, to each of them is assigned a complete 2-interpretation and transformations between belief sets correspond to simple insertions and removals (we are not worried about the permissibility of these operations in a given state).

The semantics from the Section 5 is very similar to the formal semantics of Dynamic Logic Programming, see [6]. The dynamic meaning function of [6] assigns to each dynamic predicate symbol a function from ground terms to the binary relations on a set of states (or, equivantely, a function that assigns a binary relation on a set of states to each ground proposition). A-accessiblity of the Section 5 assigns to propositions functions from belief states to belief states.

Our semantics can be characterized by the pair $(\mathcal{W}_{\mathcal{I}}, \{R_A : A \in \mathcal{P}\})$, where $\{R_A : A \in \mathcal{P}\}$ is the set of accessibility relations determined by $A \in \mathcal{P}$. Therefore, we have a kind of Kripke structure as an appropriate semantics for the DBM.

Now, our aim is to outline how to complicate (generalize, refine) this picture. We can accept more weak or more complicated structure of the universe of belief sets. Interpretations of belief sets can be more general – three or four valued – therefore we can account of unknown and inconsistent sentences. Last but not least, we can investigate a variety of transformations between belief sets: transformations violating existing accessibility relations, transformations satisfying some integrity constraints, transformations controlled by some rules or default rules.

Let us start with a sketch of one of the possible refinements of our semantics.

First, we introduce 3-interpretations of belief sets and (an example of) accessibility relation.

Definition 11 *Let be* $\mathcal{I}_3 : \mathcal{K} \longrightarrow Int_3$ *such that* $\forall K_1, K_2 (K_1 \neq K_2 \Rightarrow \mathcal{I}_3(K_1) \neq \mathcal{I}_3(K_2))$. $\mathcal{W}^3_{\mathcal{I}} = \{w \in Int_3 : \exists K \in \mathcal{K}(\mathcal{I}_3(K) = w)\}$

Definition 12 (Accessibility relation) *Let be* $R = \{(w_1, w_2) \in \mathcal{W}^3_{\mathcal{I}} \times \mathcal{W}^3_{\mathcal{I}} : true(w_1) = true(w_2) \wedge false(w_1) \subset false(w_2)\}$

Note: We stress that the symbol $\subset$ denotes the strict inclusion, i.e. $M \not\subset M$. The accessibility relation R from the definition 12 may not be identified with a (T, F)-accessibility, but it is definable in terms of (T, F)-accessibilities. Let $R_{(T,F)}$ be the (T, F)-accessibility relation, i.e. $w_1 R_{(T,F)} w_2$ iff w_2 is (T, F)-accessible from w_1. Then $R = \bigcup_{F \subseteq \mathcal{S}} R_{(\emptyset, F)}$, i.e. $w_1 R w_2$ iff $\exists F \subseteq \mathcal{S}(w_2$ is $(\emptyset, F)$-accessible from $w_1)$.

Example 3 (Closed World Assumption) *Let us fix a set* T. *Let be* $W_T = \{w \in Int_3 : true(w) = T\}$. *For each* T *there is a unique element* $w^T_{max} \in Int_3$ *such that for each* $w \in W_T$ *holds* $false(w) \subset false(w^T_{max})$: *clearly,* $w^T_{max} = (T, F)$, *where* $F = \mathcal{S} \setminus T$.

The set $\{W_T : T \subseteq \mathcal{S}\}$ *is a partition of* $\mathcal{W}^3_{\mathcal{I}}$, *for each* $w \in \mathcal{W}^3_{\mathcal{I}}$ *there is exactly one* W_T *such that* $w \in W_T$.

Now, we can define: $CWA_3(w) = w^T_{max}$, *if* $true(w) = T$.

We turn to the non-monotony of CWA. Let us consider $A \in \mathcal{P}$ *and* $K \in \mathcal{K}$ *such that* $true(\mathcal{I}(K)) = T$*, but* $true(\mathcal{I}(A(K))) \neq T$*. Therefore,* $CWA_3(\mathcal{I}(K)) = w^T_{max}$*, but*
$CWA_3(\mathcal{I}(A(K))) \neq w^T_{max}$*. Revisions result in retracting some CWA-inferences.*

We have defined CWA_3 *function on interpretations. Now we do it for propositions. We can use the correspondence between propositions and interpretations (or between* (T,F)*-accessibility and* A*-accessibility). Let* (T_A, F_A) *be assigned to* A*, where* $T_A = \mathcal{I}(A(K_\perp))$ *and* $F_A = \mathcal{S} \setminus \mathcal{I}(A(K_\top))$*.* $CWA_3(A)$ *is the proposition determined by the assigned interpretation* $(T_{CWA_3(A)}, F_{CWA_3(A)})$*, where* $T_{CWA_3(A)} = T_A$ *and* $F_{CWA_3(A)} = \mathcal{S} \setminus T_A$*. From the definition follows: if for* A, B *holds* $A |\!\sim B$*, then* $CWA_3(A) |\!\sim CWA_3(B)$ *does not hold. Proof:* $T_B \subseteq T_A$*, but* $\mathcal{S} \setminus T_B \not\subseteq \mathcal{S} \setminus T_A$ *and we apply the proposition 6.*

So, we have an example of a Kripke structure, where besides the accessibility relations $R_{(T,F)}$ (or, equivalently, R_A) corresponding to the propositions (or to the 3-interpretations) we have yet another accessibility relation R. The relation is definable in terms of (T,F)-accessibility relations and, consequently, holds: if $w_1 R w_2$, then $R_{(T,F)}(w_1)\ R\ R_{(T,F)}(w_2)$.

The following example shows that there are some accessibility relations ρ such that for some transformation G holds: $w_1\ \rho\ w_2$ but not $G(w_1)\ \rho\ G(w_2)$ and vice versa.

Example 4 *Let us have a theory* $T = \{p \leftarrow q\}$ *and worlds (2-interpretations); each world* $w = (T,F)$ *we represent by* $true(w) = T$*:*

$w_1 = \{p,q,r\}$	$w_4 = \{p,r\}$
$w_2 = \{p,q\}$	$w_5 = \{q,r\}$
$w_3 = \{p\}$	$w_6 = \{q\}$
$w_7 = \{r\}$	$w_8 = \emptyset$

T *is satisfied in* $M = \{w_1, w_2, w_5, w_6, w_7, w_8\}$ *and is not satisfied in* $N = \{w_3, w_4\}$*.*

(a) *Let all worlds from* M *are mutually accessible (and they are the only accessible worlds). We define* ρ *as* $M \times M$*. The* $(\emptyset, \{q\})$*-transformation (let us denote it* G*) violates the accessibility relation:* $G(w_1) = w_4$ *and* $G(w_2) = w_3$*, therefore – if holds* $w\ \rho\ w_1$*, then* $G(w)\ \rho\ G(w_1)$ *does not hold.*

(b) *Similarly, if* $\rho = (M \times M) \cup (N \times N)$*. Then* $\neg(w_1\ \rho\ w_3)$ *but* $G(w_1)\ \rho\ G(w_3)$*.*

We find useful and conceptually clear to separate static and dynamic parts in our semantics. The following definitions do the separation.

Definition 13 *Kripke structure is a pair* (W,R)*, where* W *is a set of interpretations,* R *is a set of accessibility relations (*$R = \{\rho : \rho \subseteq W \times W\}$*).* [7]

[7] Usually only one accessibility relation is considered and instead of the set of interpretations is into the definition introduced a set of possible worlds together with an interpretation assigned to each world.

Definition 14 *A* monoid *is a triple* $(M, \circ, e)$*, where*

- M *is a set,*
- $\circ : M \times M \longrightarrow M$ *is an operation, which is associative, i.e. for every* $x, y, z \in M$ *holds* $x \circ (y \circ z) = (x \circ y) \circ z$
- $e \in M$ *and for every* $x \in M$ *holds* $e \circ x = x \circ e$.

Now we give a definition of DKS. The structure consists of a monoid-part and a Kripke-structure-part. The main idea is a transformation of possible worlds on itself. The transformation is specified by monoid elements.

Definition 15 Dynamic Kripke Structure *is a pair* $(\mathcal{M}, \mathcal{W})$*, where* $\mathcal{M}$ *is a monoid and* $\mathcal{W}$ *is a Kripke structure, and for every* $x \in M$ *there is a function* $f_x : W \longrightarrow W$ *such that*

- f_e *is an identity mapping*
- *for every* $x, y \in M$*, for every* $w \in W$ *holds* $f_x(f_y(w)) = f_{x \circ y}(w)$

Remark 1 *It would be welcomed if the monoid concept in the Definition 15 could be replaced by the group concept (because of nice algebraic properties, based on the existence of inverse elements) and a group action on the set* W *would be considered.* [8]

Unfortunately, the group concept is not in general applicable to our problem. We use notation from the Example 1: let us assume that $A = (ins(\{x, y\}), del(\emptyset))$*,* $B = (ins(\{x\}), del(\emptyset))$*,* $C = (ins(\emptyset), del(\{x\}))$*. If we put* $B \circ A = A$ *and* $B^{-1} = C$ *(both represent a natural choice), we get* $B^{-1} \circ B \circ A \neq A$.

There are more deep grounds for considering monoids instead of groups when defining DKS. The kernel of the problem is in the idempotency. An idempotent group operation is a contradictory notion (with the exception of the group containing only one element $\{e\}$*): for every* a *holds* $a = e$ *(it is a simple consequence of* $a \circ a = a = e \circ a$*).*

Nevertheless, the DKS is a more general concept (than the DBM) and if we find a reasonable motivation, we can reject the idempotency condition. [9]

Example 5 (CWA in terms of DKS) *: Let* $(\mathcal{W}^3_{\mathcal{I}}, \{R\})$ *be a Kripke structure, where* R *is defined by the definition 12. Let* $(\Phi, \circ, e)$ *be a monoid, where* Φ *is a set of functions from* $\mathcal{W}^3_{\mathcal{I}}$ *to* $\mathcal{W}^3_{\mathcal{I}}$.

We define $f_{CWA}(w) = w^T_{max}$ *for* $T = true(w)$.

Let $g : W \longrightarrow W$ *be a function such that* $true(g(w)) \neq T$ *and* $true(g(w)) \cup false(g(w)) \neq \mathcal{S}$*. There is a function* f *such that for every* w *is* $f \circ f_{CWA}(w) = f_{CWA}(w)$ [10] *but* $f \circ (g \circ f_{CWA})(w_0) \neq g \circ f_{CWA}(w_0)$

[8] When discussed an abstract notion of dynamic information structure a perspective from Group Theory was proposed by Van Bentham, see [8].

[9] A group is used in an ongoing research of DKS-based semantics for a logic of action and change.

[10] This condition is satisfied by all functions in the class $\{f : W \longrightarrow W |$ if $w = (T, F)$ and $f(w) = (T', F')$, then $T \subseteq T'$ and $F \subseteq F'\}$. Intuitively, functions from this class move only "undefined" values.

for w_0 such that $true(w_0) = T$. If we express this result in terms of DBM, we have the non-monotonicity property.

The following interpretation of the CWA-example serves as a hint for expressing known types of non-monotonic reasoning in terms of DKS. The function f_{CWA} is a hypotheses generator. [11]*. The function g represents a transition to the falsifying conditions for this hypothesis (it means: what is true in $f_{CWA}(w)$ may be false in $f_{CWA}(g(w))$). We propose to use these pairs of functions (or pairs of classes of functions) – hypotheses generators and revisers (falsifiers) – in applications of DKS to context-dependent reasoning.*

7 CONCLUSIONS

In this paper we have used a dynamic point of view to give an abstract characterization of context-dependent, hypothetical and non-monotonic reasoning. We have proposed a set of postulates for revisions of belief states and we have shown that the associated inference is non-monotonic. The semantics for the postulates we have generalized to the concept of Dynamic Kripke Structures, consisting of two parts – a dynamic part acting on the second, passive part.

We believe that DKS provide a general, uniform and unifying view on various known types of context-dependent, hypothetical and non-monotonic reasoning. When compared to DBM, the concepts of interpretation and Kripke structure make a link to traditional formalizations more easy. An additional level of analysis – a language and its interpretation – seems to be an advantage.

We note again that the dynamics introduced into Kripke structures is a substantial one. The expressive power of Dynamic Kripke Structures is strictly greater than the expressive power of Kripke Structures: We can express changes in accessibility. If f is a function assigned to a monoid member and R is an accessibility relation, then it may hold $f(w_1)\ R\ f(w_2)$ along with $\neg w_1\ R\ w_2$. Therefore, two possible worlds become accessible, if a dynamics is introduced. Similarly, $(w_1\ R\ w_2) \wedge \neg(f(w_1)\ R\ f(w_2))$ may hold.

A close correspondence between belief revision and non-monotonic reasoning is thoroughly studied and well established, see [3]. If we compare our postulates with the AGM-style postulates, the main difference is in our attempt to express by one set of postulates both insertions and deletions.

Our future research should be devoted to a generalization of the present set of postulates in order to catch such issues as integrity constraints, consistency maintaining, revisions (and hypotheses generation) controlled by some rules or default rules. The underlying lattice-structure of the set of belief sets will be relaxed and descriptive possibilities of more gen-

[11] A hypothesis characteristic for CWA we can formulate as "what is not explicitly true is considered as false".

eral structures will be studied. An important part of the future research should be devoted to the applications of DKS-semantics to a characterization of default reasoning, counterfactuals, reasoning about action and change, abduction, context-dependent reasoning, minimal entailment. In the Example 5 was presented an idea of functions expressing an hypothesis characteristic for a type of non-monotonic reasoning and of functions representing transitions to the falsifying conditions for the hypothesis. A goal of the future work is to explore this idea.

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