Contextualized Knowledge Repositories for the Semantic Web

Luciano Serafini[a], Martin Homola[a,b]

[a] *Fondazione Bruno Kessler, Via Sommarive 18, 38123 Trento, Italy*
[b] *Comenius University, Faculty of Mathematics, Physics and Informatics, Mlynská dolina, 84248 Bratislava, Slovakia*

Abstract

We propose *Contextualized Knowledge Repository* (CKR): an adaptation of the well studied theories of context for the Semantic Web. A CKR is composed of a set of OWL 2 knowledge bases, which are embedded in a *context* by a set of qualifying attributes (time, space, topic, etc.) specifying the boundaries within which the knowledge base is assumed to be true. Contexts of a CKR are organized by a hierarchical *coverage* relation, which enables an effective representation of knowledge and a flexible method for its reuse between the contexts. The paper defines the syntax and the semantics of CKR; shows that concept satisfiability and subsumption are decidable with the complexity upper bound of 2NExpTime, and it also provides a sound and complete natural deduction calculus that serves to characterize the propagation of knowledge between contexts.

Keywords: Semantic Web, knowledge representation, description logics, context

1. Introduction

More and more ontologies and data-sets are being published using the Semantic Web languages such as RDF and OWL. Especially under the recent Linked Open Data initiative, large knowledge sources such as DBpedia and Freebase, but also many others were conceived and populated. It is also becoming increasingly apparent that large portions of the knowledge available via these sources are not absolute, but instead they are assumed to hold under certain circumstances. Knowledge may be relative to certain time period, a geo-political or geo-cultural region, or certain specific topics, etc. Despite these facts, there is a lack of a widely accepted mechanism to qualify knowledge with the context in which it is supposed to hold. Instead, contextual information is often crafted in the ontology identifier or in attributes like rdfs:comment, owl:AnnotationProperty which do not affect reasoning.

Extensions of the Semantic Web languages with specific mechanisms that allow to qualify knowledge, e.g., w.r.t. its provenance [1] or w.r.t. time and events [2], were proposed. Among other works that offer possible solutions [3, 4, 5], the most interesting are $\mathcal{ALC}_{\mathcal{ALC}}$ [6] and Metaview [7], however, a widely accepted approach has not yet been reached.

On the other hand, theories of context have been investigated for years in the fields of artificial intelligence and knowledge representation. In his seminal paper [8] McCarthy suggested to formalize context in terms of first-class objects and utilize it in reasoning. Further research lead to introduction of the *context as a box* metaphor [9, 10] for suitable representations of contexts. In this approach, each context is a set of formulae which hold under the same circumstances and whose boundaries are delimited by a set of dimensional attributes. The two kinds of knowledge involved here are separated, the knowledge itself is inside the "box" and the contextual meta knowledge is outside. In addition, Lenat [11] proposed to organize the contexts into a hierarchical structure called contextual space based on the values of their dimensions.

To clarify the representational requirements for a contextual representation framework for the Semantic Web let us consider the following scenario. Suppose we want to represent knowledge about Football (FB), FIFA world cups (FWC), national football leagues (NFL), world news (WN), and national news (NN). Suppose also that all the information about FWC and NFL should be included in FB, and that for each nation, all the facts about its NFL should be included in its NN, and also all the information about FWC should be included in WN. On the other hand, only a part of information about NFL should be included in WN (only that of worldwide interest). A well designed contextual representation formalism should support the following requirements:

knowledge about context: knowledge about contexts such as contextual dimensions and relations between contexts as for instance that one context is more specific than some other, should be explicitly represented and reasoned about. For example, we should be able to assert that the context of FWC in 2010 is more specific than the contexts of FB and WN in the same year;

contextually bounded facts: in each context we should be able to state facts with local effect that do not necessarily propagate everywhere. For example, an axiom like "a player is a member of only one team" should be true in some contexts (e.g., FWC, NFL, for each year) but not in more general contexts like FB;

reuse/lifting of facts: to be able to seamlessly reuse the infor-

Email addresses: serafin@fbk.eu (Luciano Serafini), homola@fmph.uniba.sk (Martin Homola)

mation contained in more specific contexts. For example, the facts in FWC should be lifted up into WN and FB. This lifting should be done without spoiling locality of knowledge;

overlapping and varying domains: objects can be present in multiple contexts, but not necessarily in all contexts, e.g., a player can exist in both the FWC context and in the NFL contexts, but many players present in NFL will not be present in FWC;

inconsistency tolerance: two contexts may possibly contain contradicting facts. For instance NN of Italy could assert that "Cassano is the best player of the world", while at the same time the world news report that "Rooney is the best player of the world", without making the whole system inconsistent;

complexity invariance: the qualification of knowledge by context should not increase the complexity.

Based on these requirements, we propose a framework called *Contextualized Knowledge Repository* (CKR), build on top of the expressive description logic $\mathcal{SROIQ}$ [12] that is behind OWL 2 [13]. The CKR framework is tailored for the Semantic Web, but it is rooted in the foundations of contextual knowledge representation laid down by previous research in artificial intelligence. Adopting the context as a box paradigm, a CKR knowledge base is composed of units, called *contexts*, each qualified by a set of dimensional attributes that specify its contextual boundaries. Contexts are organized by a hierarchical *coverage* relation that regulates the propagation of knowledge between them.

After brief preliminaries (Sect. 2) the paper defines the syntax and semantics of CKR (Sect. 3); then it provides a sound and complete natural deduction calculus that serves to characterize the propagation of knowledge between contexts (Sect. 4); and finally it shows that concept satisfiability and subsumption are decidable with the complexity upper bound of 2NExpTime, i.e., same as for $\mathcal{SROIQ}$ (Sect. 5); related work is then discussed and concluding remarks added in Sects. 6, 7. Detailed proofs of all statements are attached in the appendix.

2. Preliminaries

The CKR framework is built on top of the $\mathcal{SROIQ}$ DL [12] which is used as the local language of contexts. This language constitutes the logical foundation of OWL 2 [13] and it is currently the most expressive language relevant to the Semantic Web. In this section, we briefly introduce the necessary DL preliminaries. For more details the reader is referred to the works of Horrocks et al. [12] and Baader et al. [14]. Although semantically CKR is able to handle the full $\mathcal{SROIQ}$ DL, in order to achieve decidability of reasoning we will slightly limit its expressive power as we shall see below.

A DL vocabulary $\Sigma = N_\mathrm{C} \uplus N_\mathrm{R} \uplus N_\mathrm{I}$ is a set of symbols composed of three mutually disjoint countably infinite subsets: the set N_C of atomic concepts including the top concept $\top$ and the bottom concept $\bot$, the set N_R of atomic roles including the universal role U and the identity role I, and the set N_I of individuals.

Concept constructors	Syntax	Semantics
atomic concept	A	$A^\mathcal{I}$
complement	$\neg C$	$\Delta^\mathcal{I} \setminus C^\mathcal{I}$
intersection	$C \sqcap D$	$C^\mathcal{I} \cap D^\mathcal{I}$
existential restriction	$\exists R.C$	$\left\{ x \in \Delta^\mathcal{I} \,\middle|\, \begin{matrix}\exists y \langle x,y\rangle \in R^\mathcal{I} \\ \wedge\, y \in C^\mathcal{I}\end{matrix} \right\}$
self restriction	$\exists R.\mathrm{Self}$	$\left\{ x \in \Delta^\mathcal{I} \,\middle|\, \langle x,x\rangle \in R^\mathcal{I} \right\}$
min. card. restriction	$\geqslant nR.C$	$\left\{ x \in \Delta^\mathcal{I} \,\middle|\, \begin{matrix}\sharp\{y \mid \langle x,y\rangle \in R^\mathcal{I} \\ \wedge\, y \in C^\mathcal{I}\} \geq n\end{matrix} \right\}$
nominal	$\{a\}$	$\{a^\mathcal{I}\}$
Role constructors	Syntax	Semantics
atomic role	R	$R^\mathcal{I}$
inverse role	R^-	$\left\{ \langle y,x\rangle \,\middle|\, \langle x,y\rangle \in R^\mathcal{I} \right\}$
role composition	$S \circ Q$	$S^\mathcal{I} \circ Q^\mathcal{I}$
Axioms	Syntax	Semantics
concept inclusion (GCI)	$C \sqsubseteq D$	$C^\mathcal{I} \subseteq D^\mathcal{I}$
role inclusion (RIA)	$S \sqsubseteq R$	$S^\mathcal{I} \subseteq R^\mathcal{I}$
reflexivity assertion	$\mathrm{Ref}(R)$	$R^\mathcal{I}$ is reflexive
role disjointness	$\mathrm{Dis}(P,R)$	$P^\mathcal{I} \cap R^\mathcal{I} = \emptyset$
concept assertion	$C(a)$	$a^\mathcal{I} \in C^\mathcal{I}$
role assertion	$R(a,b)$	$\langle a^\mathcal{I}, b^\mathcal{I}\rangle \in R^\mathcal{I}$
negated role assertion	$\neg R(a,b)$	$\langle a^\mathcal{I}, b^\mathcal{I}\rangle \notin R^\mathcal{I}$

Table 1: Syntax and Semantics of $\mathcal{SROIQ}$

Complex concepts (complex roles) are recursively defined as the smallest set containing all concepts (roles) that can be inductively constructed using the concept (role) constructors in Table 1, where A is any atomic concept, C and D are any concepts, P and R are any atomic roles, S and Q are any (possibly complex) roles, a and b are any individuals, and n stands for any positive integer.

A $\mathcal{SROIQ}$ knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \rangle$ consists of a TBox $\mathcal{T}$ which contains GCI axioms; an RBox $\mathcal{R}$ which contains RIA axioms, reflexivity and role disjointness axioms; and an ABox $\mathcal{A}$ which contains assertions. The syntax of all axioms is shown at the bottom of Table 1. The closure $\sqsubseteq^*_\mathcal{R}$ of the RBox $\mathcal{R}$ is defined as follows: if $R \sqsubseteq Q \in \mathcal{R}$ and $Q_1 \circ Q \circ Q_2 \sqsubseteq S \in \mathcal{R}$ then $Q_1 \circ R \circ Q_2 \sqsubseteq_\mathcal{R} S$; and $\sqsubseteq^*_R$ is the transitive and reflexive closure on $\sqsubseteq_\mathcal{R}$.

A DL interpretation is a pair $\mathcal{I} = \left\langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \right\rangle$ where $\Delta^\mathcal{I}$ is a set called interpretation domain and $\cdot^\mathcal{I}$ is the interpretation function which provides denotations for individuals, concepts and roles. In $\mathcal{SROIQ}$, as much as in any classical DL, $\Delta^\mathcal{I}$ is required to be non-empty. We will see later on that in CKR we will relax from this requirement. The interpretation function $\cdot^\mathcal{I}$ assigns an element $a^\mathcal{I}$ of $\Delta^\mathcal{I}$ to each individual a, a subset $C^\mathcal{I}$ of $\Delta^\mathcal{I}$ to each concept C, and a subset $R^\mathcal{I}$ of the product $\Delta^\mathcal{I} \times \Delta^\mathcal{I}$ to each role R. In addition for any complex concept and role the respective semantic constraint listed in Table 1 must be satisfied by $\cdot^\mathcal{I}$, plus

$\top^{\mathcal{I}} = \Delta^{\mathcal{I}}$, $\bot^{\mathcal{I}} = \emptyset$, $U^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, and $I^{\mathcal{I}} = \{\langle x, x\rangle \mid x \in \Delta^{\mathcal{I}}\}$[1]. An axiom ϕ is satisfied by an interpretation $\mathcal{I}$ (denoted $\mathcal{I} \models_{\mathrm{DL}} \phi$) if $\mathcal{I}$ satisfies the respective semantic constraint listed in Table 1. An interpretation $\mathcal{I}$ is a model of $\mathcal{K}$ (denoted $\mathcal{I} \models_{\mathrm{DL}} \mathcal{K}$) if it satisfies all axioms of $\mathcal{K}$.

Only simple roles are allowed in the min cardinality restriction, in the self restriction constructors, as well as in the reflexivity and the disjointness axioms. Simple roles are defined recursively as follows: a) atomic role is simple if it does not occur on the right-hand side of a RIA in $\mathcal{R}$; b) an inverse role R^- is simple if R is simple; c) if R occurs on the right-hand side of a RIA in $\mathcal{R}$ and each such RIA is of the from $S \sqsubseteq R$ where S is a simple role, than R is also simple. Also the universal role U is not allowed on the left-hand side of RIA axioms.

There are additional $\mathcal{SROIQ}$ constructors and axioms [12]. Specifically, concept constructors $C \sqcup D$, $\forall R.C$, $\leqslant nR.C$ and $=nR.C$, and RBox axioms $\mathrm{Sym}(R)$, $\mathrm{Tra}(R)$, $\mathrm{Irr}(R)$. Although we occasionally use some of them to simplify the notation, they are all fully reducible[2] into the core constructs listed in Table 1 which allows us to leave them out when laying out the theoretical foundation of CKR. Note that also $\mathrm{Ref}(R)$ is reducible, but only in cases when R is simple (i.e., by replacing it with $\top \sqsubseteq \exists R.\mathrm{Self}$).

Two basic reasoning tasks for $\mathcal{SROIQ}$ and for any DL are concept satisfiability, the task to decide for a given (possibly complex) concept C whether there is a model $\mathcal{I}$ of $\mathcal{K}$ such that $C^{\mathcal{I}}$ is nonempty; and entailment, the task to decide for some axiom ϕ whether $\mathcal{I} \models_{\mathrm{DL}} \phi$ for all models $\mathcal{I}$ of $\mathcal{K}$ (which is then denoted by $\mathcal{K} \models_{\mathrm{DL}} \phi$). These two tasks are known to be inter-reducible [14, 12]. A tableaux algorithm that decides these two tasks for $\mathcal{SROIQ}$ was given by Horrocks et al. [12]. The algorithm, however, requires further syntactic restrictions on the language, as discussed below.

It is required that the RBox is regular, which is defined as follows [12]. A regular order on roles $\prec$ is a strict partial order (i.e., a transitive and irreflexive binary relation) on roles such that $S \prec R$ iff $S^- \prec R$ for any roles S, R. Given a regular order on roles $\prec$, a RIA is $\prec$-regular if it has one of the following forms: a) $R \circ R \sqsubseteq R$; b) $R^- \sqsubseteq R$; c) $S_1 \circ \cdots \circ S_n \sqsubseteq R$ and $S_i \prec R$ for $1 \leq i \leq n$; d) $R \circ S_1 \circ \cdots \circ S_n \sqsubseteq R$ and $S_i \prec R$ for $1 \leq i \leq n$; e) $S_1 \circ \cdots \circ S_n \circ R \sqsubseteq R$ and $S_i \prec R$ for $1 \leq i \leq n$. An RBox $\mathcal{R}$ is regular if there exists a regular order $\prec$ on its roles such that all RIA of $\mathcal{R}$ are $\prec$-regular.

$\mathcal{SROIQ}$ is decidable for knowledge bases with regular RBox. This is because the restriction assures the existence of a regular automaton corresponding to the language generated by the RBox, which is then used by the tableaux algorithm [12]. More recently an alternative condition called RBox stratification was introduced by Kazakov [15]. Given a preorder (a transitive and reflexive binary relation) $\precsim$ on roles, let $R \approx S$ if $R \precsim S$ and $S \precsim R$, and let $R \prec S$ if $R \precsim S$ and not $R \approx S$. A RIA $R_1 \circ \cdots \circ R_n \sqsubseteq R$ is $\precsim$-admissible if $R_i \precsim R$ for $1 \leq i \leq n$. Given a set of $\precsim$-admissible RIA $\mathcal{R}$ a RIA $Q \sqsubseteq S$ is $\precsim$-stratified in $\mathcal{R}$, if for every R such that $R \approx S$ and $Q = Q_1 \circ R \circ Q_2$ there exists P such that $Q_1 \circ R \sqsubseteq^*_{\mathcal{R}} P$ and $P \circ Q_2 \sqsubseteq^*_{\mathcal{R}} S$. Finally, $\mathcal{R}$ is stratified if there is some preorder $\precsim$ on $\mathcal{R}$ such that every RIA of $\mathcal{R}$ is $\precsim$-admissible and every RIA $R \sqsubseteq S$ such that $R \sqsubseteq^*_{\mathcal{R}} S$ is $\precsim$-stratified in $\mathcal{R}$.

Stratification is less restrictive then regularity, i.e., every regular RBox $\mathcal{R}$ can be extended into a stratified RBox $\mathcal{R}'$ modulo-$\mathcal{R}$-equivalent to $\mathcal{R}$, but not vice versa [15]. As also showed by Kazakov [15], for deciding satisfiability and entailment with respect to a $\mathcal{SROIQ}$ knowledge base with stratified RBox it is possible to use the same tableaux algorithm introduced by Horrocks et al. [12]. This is because the existence of the regular automaton required by the algorithm is also assured for stratified RBoxes. In order to show decidability of reasoning in CKR we will rely on this result. In addition, we will slightly limit the expressive power of $\mathcal{SROIQ}$ as the local language: disjointness axioms $\mathrm{Dis}(R, S)$ will be excluded from CKR and role reflexivity axioms $\mathrm{Ref}(R)$ will only be allowed if R is simple. Therefore the DL on which CKR is built can be described as almost full $\mathcal{SROIQ}$. We deem this to be a reasonable sacrifice in the expressivity of the local language that allows us to achieve the contextualized representation of knowledge that is enabled by CKR.

3. Contextualized Knowledge Repository

A CKR is composed of a set of *contexts*. Following the "context as a box" metaphor [9], a context contains a set of logical statements and it is qualified by a set of contextual attributes, also called *dimensions*. An example of this type of representation is shown in Fig. 1 where an excerpt from a context representing the Italian national football league in 2010 is depicted.

time = 2010, location = Italy, topic = NFL

Team $\sqsubseteq$ =22has_player.Player
Player $\sqsubseteq$ $\leqslant$1plays_for.Team
Team(Milan)
plays_for(Cassano, Milan)
...

Figure 1: Italian national football league under the context as a box metaphor

It is apparent from this simple example, that there are two layers in this kind of representation, the knowledge itself (we will call this layer the object knowledge) and the data about the knowledge (which we will call meta knowledge). McCarthy [8] proposed to use an unique language for both types of knowledge, namely quantified modal logic. While this is very powerful from the representational perspective (e.g., the structure of context may be inferred out of the object knowledge, etc.), it easily leads to undecidability. At the opposite extreme there are approaches such as multi-context systems [16], distributed

[1] The identity role I is not originally part of $\mathcal{SROIQ}$ [12], however it can be easily introduced as syntactic sugar by adding the axioms $\top \sqsubseteq \exists I.\mathrm{Self}$ and $\top \sqsubseteq \neg{\geqslant}2I.\top$.

[2] $C \sqcup D$ reduces into $\neg(\neg C \sqcap \neg D)$; $\forall R.C$ reduces into $\neg\exists R.\neg C$; $\leqslant nR.C$ reduces into $\neg{\geqslant}n{+}1R.C$; $=nR.C$ reduces into ${\geqslant}nR.C \sqcap \neg{\geqslant}n{+}1R.C$; $\mathrm{Sym}(R)$ reduces into $R^- \sqsubseteq R$; $\mathrm{Tra}(R)$ reduces into $R \circ R \sqsubseteq R$; $\mathrm{Irr}(R)$ reduces into $\exists R.\mathrm{Self} \sqsubseteq \bot$.

[17] or package-based description logics [18], where the context structure is fixed and it is not possible to specify knowledge about contexts, which limits their practical applicability. We therefore propose an intermediate approach, by allowing to specify the context structure and properties in a (simple) logical *meta language*, but avoiding to mix it with the *object language* used within each context in order to maintain good computational properties. In our approach, the meta knowledge influences the object knowledge (in terms of logical consequence) but not vice versa.

3.1. Language of Contextual Representation

In CKR, contextual attributes are specified in the meta language, with vocabulary called the meta vocabulary. The content of each context is specified in the object language, with vocabulary called the object vocabulary. Both of these languages will be DL. The meta vocabulary contains a specific set of symbols in order to identify contexts and to assign dimensional values to contexts. It contains a distinguished set of individuals that will be used as context identifiers. Each dimension is represented by a dedicated role A that will be used to assign dimensional values to the contexts, a set of admissible dimensional values D_A which are individuals, and a role $\prec_A$ which will be used to model the cover relation between dimensional values.

Definition 1 (Meta vocabulary)**.** *A meta vocabulary* Γ *is a DL vocabulary that contains:*

1. *a set of individuals called* context identifiers*;*
2. *a finite set of roles* $\mathbf{A}$ *called* dimensions*;*
3. *a set of individuals* D_A *called* dimensional values*, for every dimension* $A \in \mathbf{A}$*;*
4. *a role* $\prec_A$*, called* coverage relation*, for every dimension* $A \in \mathbf{A}$.

The number of dimensions $k = |\mathbf{A}|$ is assumed to be a fixed constant. This will be important in order not to introduce an additional complexity blow up. Also, relevant research on contextual dimensions suggests that their number is usually very limited [11]. The meta assertions of the form $A(C, d)$ for a context identifier C and some $d \in D_A$ (e.g., $\mathsf{time}(\mathsf{c0}, \mathsf{2010})$), state that the value of the dimension A of the context C is d. The meta assertions of the form $d \prec_A e$ (e.g., $\mathsf{Italy} \prec_{\mathsf{space}} \mathsf{Europe}$)[3] state that the value d of the dimension A is covered by the value e. Depending on the dimension, the coverage relation has different intuitive meanings, e.g., if A is space then the coverage relation is topological containment, if A is topic then it is topic specificity. It is of course up to the modeller to pick and represent the dimensional coverage appropriately.

The meta vocabulary allows us to construct dimensional vectors of the form $\{A_{i_1}{:=}d_1, \ldots, A_{i_n}{:=}d_n\}$ which are composed of attribute-value declarations such that each A_{i_k} is a dimension of Γ and each d_k is a value from $D_{A_{i_k}}$. In accordance with the context as a box paradigm, dimensional vectors will be used to identify each context by a specific set of dimensional values. They are either full, if a value for each dimension in $\mathbf{A}$ is given, or partial, if some dimensions are missing. The set of all full dimensional vectors of Γ forms the dimensional space in which contexts will be located.

Definition 2 (Dimensional space)**.** *Given a meta vocabulary* Γ *with n dimensions* $\mathbf{A} = \{A_1, \ldots, A_n\}$*, let us define:*

1. *a full dimensional vector in* Γ *is a set of attribute-value declarations* $\mathbf{d} = \{A_1{:=}d_1, \ldots, A_n{:=}d_n\}$ *such that* $d_k \in D_{A_k}$ *for every k with* $1 \le k \le n$*;*
2. *a partial dimensional vector in* Γ *is a set of attribute-value declarations* $\mathbf{d_B} = \{A_{i_1}{:=}d_1, \ldots, A_{i_m}{:=}d_m\}$ *such that* $0 \le m \le n$*,* $d_k \in D_{A_{i_k}}$ *for every k with* $1 \le k \le m$*, and* $\mathbf{B} = \{A_{i_1}, \ldots, A_{i_m}\} \subset \mathbf{A}$*;*
3. $\mathfrak{D}_\Gamma$*, the dimensional space respective to* Γ*, is the set of all full dimensional vectors in* Γ*;*
4. $\mathbf{d_B}{+}\mathbf{e_C}$*, the completion of* $\mathbf{d_B}$ *w.r.t.* $\mathbf{e_C}$*, given two partial dimensional vectors* $\mathbf{d_B}$ *and* $\mathbf{e_C}$*, is equal to* $\mathbf{d_B} \cup \{(A_{i_k}{:=}d_k) \in \mathbf{e_C} \mid A_{i_k} \notin \mathbf{B}\}$.

We use bold Latin letters $\mathbf{d}, \mathbf{e}, \mathbf{f}$, etc. to denote dimensional vectors. Given a dimensional vector $\mathbf{d}$ and $A \in \mathbf{A}$, we denote by d_A the value assigned to A in $\mathbf{d}$ (i.e., such that $(A{:=}d_A) \in \mathbf{d}$). If $\mathbf{d}$ is partial and it does not contain a value for A, then d_A is undefined. Analogously for vectors denoted by $\mathbf{e}$, $\mathbf{f}$, etc. Observe that in fact for any full dimensional vector $\mathbf{d}$, and any subset of dimensions $\mathbf{B} \subseteq \mathbf{A}$, a partial dimensional vector $\mathbf{d_B}$ is obtained by projection of $\mathbf{d}$ with respect to the dimensions in $\mathbf{B}$ (i.e., $\mathbf{d_B} = \{B{:=}d_B \mid B \in \mathbf{B}\}$). By definition $\mathbf{d_A} = \mathbf{d}$. Note that the empty dimensional vector $\{\}$ is also a partial dimensional vector in Γ.

One may object that for encoding a set of attributes with structured value-sets in the meta knowledge, such a powerful formalism as DL may be unnecessary. Several examples using simply ordered sets are found in the literature [19, 5, 20]. We have however intentionally chosen DL, as this brings the option to employ reasoning also at the meta level. As the dimensional values are normal DL individuals, one may assign them into classes and express constraints on them. Consider for instance the location dimension corresponding to geographical regions. One may sort the values into classes such as Country, City, CapitalCity, etc., and require e.g. $\mathsf{Country} \sqsubseteq \exists{\prec_{\mathsf{location}}}^-.\mathsf{CapitalCity}$, that is that every country has a capital city as its subregion. One may further notice that for those contexts which have an instance of City assigned as the value of the location dimension, there may possibly be a large part of shared knowledge, especially axioms at the TBox level. In such a case it might be useful to group all these axioms into some kind of a *context class* (we have investigated this in our previous work [21] and implemented context classes in our prototype implementation). This line of research is beyond the scope of this paper, but it justifies DL as our choice of meta language. In addition, in Sect. 5 we show that the complexity of reasoning with CKR is in the same class as for the object language (i.e., 2NExpTime in case of $\mathcal{SROIQ}$).

[3] To improve legibility we will use infix notation for the coverage relations, e.g., we will equivalently use $d \prec_A e$ instead of $\prec_A(d, e)$.

Inside the contexts, knowledge is encoded using the object vocabulary. This is again a DL-vocabulary. While the object vocabulary is shared between all contexts in a CKR knowledge base, the symbols may have different interpretation in different contexts. This is very natural when modeling contextualized information. For instance, in the context of FIFA WC 2010 the concept Finalist represents the finalist teams of the FIFA WC 2010, while in the context of FIFA WC 2006, the same concept represents the finalists of the 2006 edition of FIFA WC. Locality, however, does not imply opacity. When information propagates across contexts, we need a way to refer to the specific interpretation of a symbol in a remote context. To be able to do this, we introduce so called *qualified symbols* into the object vocabulary. These are symbols with a dimensional vector in subscript which indicates with respect to which context the symbol should be interpreted.

Definition 3 (Object vocabulary)**.** *Let* Γ *be a meta vocabulary. Given any DL-vocabulary* Σ^{B}*, an object vocabulary* Σ *is an extension of* Σ^{B} *such that for every concept/role symbol* X *in* Σ^{B} *(including* $\top$*,* U*,* I*, but excluding* $\bot$*), and for every dimensional vector* $\mathbf{d}$ *(full or partial),* Σ *contains the concept/role symbol* $X_{\mathbf{d}}$.

An object vocabulary Σ is possibly constructed on top of any DL-vocabulary Σ^{B}. In such a case, Σ^{B} is called the base-vocabulary of Σ. Any symbol from $\Sigma \setminus \Sigma^{\mathrm{B}}$ is called qualified symbol. Concepts and roles of Σ^{B} are non-qualified, but they can also be perceived as qualified with respect to the empty dimensional vector $\{\}$. If no ambiguity arises, we skip the brackets and the attribute names, so instead of e.g. $X_{\{\mathsf{location}:=\mathsf{Italy},\mathsf{time}:=\mathsf{2010}\}}$ we will write $X_{\mathsf{Italy},\mathsf{2010}}$, etc. Qualified symbols will be given a special interpretation by the CKR semantics, but they are used just like any other concept/role symbols. For instance, in the context of football 2005–2010 one would like to define the concept TopTeam as the set of teams that reached the final phase in at least one edition of the FIFA WC in the last 5 years (note that FIFA WC is run every 4 years). Given the dimensional value FWC for the topic FIFA World Cup and 2006, 2010, etc. for years, the concept TopTeam can be defined with the following axiom:

$$\mathsf{TopTeam} \equiv \mathsf{Finalist}_{\mathsf{FWC},\mathsf{2010}} \sqcup \mathsf{Finalist}_{\mathsf{FWC},\mathsf{2006}}$$

Such an approach is reminiscent of the knowledge qualification and unqualification operations (also context *push* and *pop*) as known from the literature [9]. These operations allow for a statement to be popped out of the context, preserving its meaning, by modifying it to make the contextual parameters explicit. Or in the opposite direction, a qualified statement can be pushed inside a context and some of its qualifying parameters stripped. Later on we will formalize these operations in the CKR framework using a special operator called the @ operator.

3.2. Syntax of CKR

A context is a unit of knowledge, from which a CKR knowledge base is composed. Each context has an identifier, a set of dimensional attributes, one for each dimension, which are respective to some meta vocabulary Γ, and it features a DL knowledge base over some object vocabulary Σ.

Definition 4 (Context)**.** *Given a meta vocabulary* Γ *and an object vocabulary* Σ*, a* context *on* $\langle\Gamma,\Sigma\rangle$ *is a triple* $\langle C, \mathrm{dim}(C), \mathrm{K}(C)\rangle$ *where:*

1. C *is a* context identifier *of* Γ*;*
2. $\mathrm{dim}(C)$ *is a full dimensional vector of* $\mathfrak{D}_\Gamma$*;*
3. $\mathrm{K}(C)$ *is a* $\mathcal{SROIQ}$ *knowledge base over* Σ*.*

Note that while symbols appearing inside contexts can possibly be qualified with partial dimensional vectors, $\mathrm{dim}(C)$, the dimensional vector on which the context C resides, is always a full dimensional vector in $\mathfrak{D}_\Gamma$. We use the notation $C_{\mathbf{d}}$ to denote a context with $\mathrm{dim}(C) = \mathbf{d}$.

Finally, a CKR knowledge base is composed of a collection of contexts and an additional DL knowledge base over the meta vocabulary which will be called meta knowledge. The meta knowledge assigns dimensional values to each context and it also asserts a hierarchical organization of contexts, which will be called context coverage. This hierarchy is recorded by asserting a strict partial order on the dimensional values of each dimensional attribute $A \in \mathbf{A}$ using the role $\prec_A$.

Definition 5 (Contextualized Knowledge Repository)**.** *Let* Γ *be a meta vocabulary and let* Σ *be an object vocabulary. A* Contextualized Knowledge Repository (CKR) *on* $\langle\Gamma,\Sigma\rangle$ *is a pair* $\mathfrak{K} = \langle\mathfrak{M},\mathfrak{C}\rangle$ *such that:*

1. $\mathfrak{C}$ *is a set of contexts on* $\langle\Gamma,\Sigma\rangle$*;*
2. $\mathfrak{M}$*, called meta knowledge, is a DL knowledge base on* Γ *such that:*
   (a) *every* $A \in \mathbf{A}$ *is declared a functional role;*
   (b) *for every* $C \in \mathfrak{C}$ *with* $\mathrm{dim}(C) = \mathbf{d}$ *and for every* $A \in \mathbf{A}$ *we have* $\mathfrak{M} \models A(C, d_A)$*;*
   (c) *for every* $A \in \mathbf{A}$*, the relation* $\{d \prec_A d' \mid \mathfrak{M} \models \prec_A(d,d')\}$ *is a strict partial order on* D_A*.*

To indicate that a formula ϕ belongs to a context $C_{\mathbf{d}}$ of a CKR $\mathfrak{K}$ we will often write $\mathbf{d} : \phi$ instead of just ϕ. Similarly, by the notation $\mathbf{d} : \phi \in \mathfrak{K}$ we mean that $\phi \in C_{\mathbf{d}}$, where $C_{\mathbf{d}}$ is a context of $\mathfrak{K}$; and by $\mathfrak{K} \cup \{\mathbf{d} : \phi\}$ ($\mathfrak{K} \setminus \{\mathbf{d} : \phi\}$) we denote a new CKR constructed by adding ϕ to $C_{\mathbf{d}}$ (respectively subtracting it).

Note that functional roles required by the definition above are quite common in DL ($\mathcal{SHIF}$ and all more expressive logics). Although this is not so common, in $\mathcal{SROIQ}$ (and therefore in OWL 2) it is possible to implement also strict partial order of the dimensional coverage (particularly, by asserting $\mathrm{Tra}(\prec_A)$ and $\mathrm{Irr}(\prec_A)$). On the other hand, in simpler logics it is possible to assure appropriate structure of the dimensions simply by enumerating the coverage in the ABox. The order is required to ensure a reasonable hierarchical organization of the contexts in a CKR knowledge base. In expressive logics we are able to verify the order automatically. If the logic is not expressive enough, we can still organize the knowledge base and verify the order extra-logically (e.g., we can use some other programmatic means for that). Therefore CKR may indeed be built with simpler, more tractable logics: an RDFS-based version has already been developed [22].

The coverage relation between the dimensional values of each dimension encoded in the meta knowledge provides the base for the coverage between dimensional vectors and contexts. One dimensional vector covers another, if its dimensional values cover the values of the other, one by one. That is, the coverage between dimensional vectors ($\prec$) is a product of the dimensional order relations $\prec_A$. One context covers another if the same holds for their associated dimensional vectors. We will also introduce a handy notation for coverage with respect to a subset of dimensions only ($\prec_{\mathbf{B}}$).

Definition 6 (Coverage)**.** *Given a CKR $\mathfrak{K}$ on $\langle \Gamma, \Sigma \rangle$ with dimensions $\mathbf{A}$, given any dimension $A \in \mathbf{A}$ and any two dimensional values $d, d' \in D_A$, given any two dimensional vectors $\mathbf{d}$ and $\mathbf{e}$ (full or partial), any subset $\mathbf{B} \subseteq \mathbf{A}$, and given any two contexts C and C' we say that:*

1. *d covers d' w.r.t. A (denoted $d \prec_A d'$) if $\mathfrak{M} \models \prec_A(d, d')$;*
2. *$\mathbf{e}$ covers $\mathbf{d}$ w.r.t. $\mathbf{B}$ (denoted $\mathbf{d} \prec_{\mathbf{B}} \mathbf{e}$) if $d_B \prec_B e_B$ for all $B \in \mathbf{B}$;*
3. *$\mathbf{e}$ covers $\mathbf{d}$ (denoted $\mathbf{d} \prec \mathbf{e}$) if $\mathbf{d} \prec_{\mathbf{A}} \mathbf{e}$;*
4. *C' covers C (denoted $C \prec C'$) if $\dim(C) \prec \dim(C')$.*

Note that $\mathbf{d} \prec \mathbf{e}$ implies that $\mathbf{d}$ and $\mathbf{e}$ are defined on the same set of dimensions. If $\mathbf{d} \prec_{\mathbf{B}} \mathbf{e}$, then $\mathbf{d}$ and $\mathbf{e}$ may be defined on a different set of dimensions but both must be defined on all dimensions of $\mathbf{B} \subseteq \mathbf{A}$.

Intuitively, if one context covers another, its perspective is broader. For instance, the context concerned with football in general would cover the contexts of FIFA World Cup and contexts concerned with national football leagues. To give an example, let us now formally model the coverage relation for the contexts described in the introduction. We will have the topic dimension with the following values in D_{topic}: FB (football), FWC (FIFA World Cup), NFL (National football league), WN (world news), NN (national news). The space dimension will have the values world, africa and italy in D_{space}. The time dimension will have only one value 2010. The following coverage between the dimensional values will be asserted in the ABox of $\mathfrak{M}$:

$$\begin{array}{lll} \mathsf{FWC} \prec_{\mathsf{topic}} \mathsf{WN} & \mathsf{NFL} \prec_{\mathsf{topic}} \mathsf{FB} & \mathsf{africa} \prec_{\mathsf{space}} \mathsf{world} \\ \mathsf{FWC} \prec_{\mathsf{topic}} \mathsf{FB} & \mathsf{NFL} \prec_{\mathsf{topic}} \mathsf{NN} & \mathsf{italy} \prec_{\mathsf{space}} \mathsf{world} \end{array}$$

The fact that the FWC is covered by WN in this example is due to world news report on the World Cup together with other topics, therefore this context is broader. Similarly for NFL and NN. The context coverage relation generated from this coverage between dimensional values is shown in Fig. 2.

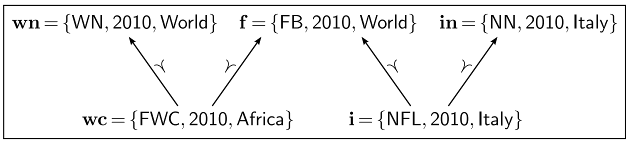

Figure 2: Coverage relation between contexts

3.3. Semantics of CKR

The semantics of CKR relies on the DL semantics inside each context (local semantics), while the relations between the contexts are handled by some additional semantic conditions. Local interpretation and local models are like standard DL-interpretations and models with two notable exceptions: empty domains are allowed; and, while all contexts in a CKR share a common object vocabulary Σ, not every symbol of Σ needs to be interpreted by each local interpretation. This will be especially true in case of individuals which may but also may not be meaningful in a given context.

Definition 7 (Local Interpretation)**.** *Given a CKR $\mathfrak{K}$ over $\langle \Gamma, \Sigma \rangle$ with $\Sigma = N_C \uplus N_R \uplus N_I$, and a context $C_{\mathbf{d}}$ of $\mathfrak{K}$, a pair $\mathcal{I}_{\mathbf{d}} = \left\langle \Delta_{\mathbf{d}}, \cdot^{\mathcal{I}_{\mathbf{d}}} \right\rangle$ is a local interpretation of $C_{\mathbf{d}}$ if:*

1. *either $\Delta_{\mathbf{d}} = \emptyset$;*
2. *or there exists $N'_I \subseteq N_I$ s.t. $\mathcal{I}_{\mathbf{d}}$ is a DL-interpretation over $\Sigma' = N_C \uplus N_R \uplus N'_I$.*

Note that for any complex concept or role X, $X^{\mathcal{I}_{\mathbf{d}}}$ is defined only if it is defined also for every individual occurring in X. In the following, whenever we write $X^{\mathcal{I}_{\mathbf{d}}}$ then we also mean that $\mathcal{I}_{\mathbf{d}}$ is defined for X. Observe in the definition below, that in a local model $\mathcal{I}_{\mathbf{d}}$ of $C_{\mathbf{d}}$, $\mathcal{I}_{\mathbf{d}}$ is necessarily defined on every individual actually occurring in $C_{\mathbf{d}}$. It may be defined on some individuals in addition due to the semantic relations between contexts.

Definition 8 (Local Model)**.** *Given a CKR $\mathfrak{K}$, a context $C_{\mathbf{d}}$ of $\mathfrak{K}$, a local interpretation $\mathcal{I}_{\mathbf{d}}$ is a* local model *of $C_{\mathbf{d}}$ (denoted $\mathcal{I}_{\mathbf{d}} \models_{\mathrm{DL}} C_{\mathbf{d}}$) if $\mathcal{I}_{\mathbf{d}} \models_{\mathrm{DL}} \phi$ for every axiom $\phi \in \mathrm{K}(C_{\mathbf{d}})$.*

Note that the local interpretation with empty domain trivially satisfies any TBox or RBox axiom (e.g., $C \sqsubseteq D$ is satisfied because both $C^{\mathcal{I}_{\mathbf{d}}} = \emptyset$ and $D^{\mathcal{I}_{\mathbf{d}}} = \emptyset$ if $\Delta_{\mathbf{d}} = \emptyset$). Therefore such an interpretation is always a model of any context that does not explicitly contain individuals. On the other hand, if at least one individual is contained inside a context then such an interpretation is no longer a model.

A model of a CKR knowledge base is a collection of local models, one for each context, which are bound together by further semantic conditions in order to take into account relations between contexts. In a CKR model, local domains may possibly overlap, reflecting the fact that the contexts may possibly describe same things from a different perspective. Local domains will be organized in accordance with the coverage hierarchy. In addition special attention is given to individuals, which are interpreted equally if they occur in two contexts that share a common super-context, and the meaning for the qualified concepts and roles is provided.

Definition 9 (CKR Model)**.** *A model of a CKR $\mathfrak{K}$ is a collection $\mathfrak{I} = \{\mathcal{I}_{\mathbf{d}}\}_{\mathbf{d} \in \mathfrak{D}_\Gamma}$ of local models such that for all $\mathbf{d}$, $\mathbf{e}$, and $\mathbf{f}$, for every atomic concept A, atomic role R, atomic concept/role X and individual a:*

1. $(\top_{\mathbf{d}})^{\mathcal{I}_{\mathbf{f}}} \subseteq (\top_{\mathbf{e}})^{\mathcal{I}_{\mathbf{f}}}$ *if* $\mathbf{d} \prec \mathbf{e}$
2. $(A_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} \subseteq (\top_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}}$
3. $(R_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} \subseteq (\top_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} \times (\top_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}}$
4. $a^{\mathcal{I}_{\mathbf{e}}} = a^{\mathcal{I}_{\mathbf{d}}}$, *given* $\mathbf{d} \prec \mathbf{e}$, *either if* $a^{\mathcal{I}_{\mathbf{d}}}$ *is defined,*
   or $a^{\mathcal{I}_{\mathbf{e}}}$ *is defined and* $a^{\mathcal{I}_{\mathbf{e}}} \in \Delta_{\mathbf{d}}$
5. $(X_{\mathbf{d}_{\mathbf{B}}})^{\mathcal{I}_{\mathbf{e}}} = (X_{\mathbf{d}_{\mathbf{B}+\mathbf{e}}})^{\mathcal{I}_{\mathbf{e}}}$

6. $(X_\mathbf{d})^{\mathcal{I}_\mathbf{e}} = (X_\mathbf{d})^{\mathcal{I}_\mathbf{d}}$ *if* $\mathbf{d} \prec \mathbf{e}$
7. $(A_\mathbf{f})^{\mathcal{I}_\mathbf{d}} = (A_\mathbf{f})^{\mathcal{I}_\mathbf{e}} \cap \Delta_\mathbf{d}$ *if* $\mathbf{d} \prec \mathbf{e}$
8. $(R_\mathbf{f})^{\mathcal{I}_\mathbf{d}} = (R_\mathbf{f})^{\mathcal{I}_\mathbf{e}} \cap (\Delta_\mathbf{d} \times \Delta_\mathbf{d})$ *if* $\mathbf{d} \prec \mathbf{e}$
9. $\mathcal{I}_\mathbf{d} \models_{\mathrm{DL}} C_\mathbf{d}$

Let us now explain the semantic constraints imposed in CKR models passing through the conditions of the definition one by one.

Condition 1 Given $C_\mathbf{d} \prec C_\mathbf{e}$, the perspective of $C_\mathbf{d}$ is narrower than of $C_\mathbf{e}$ and vice versa the perspective of $C_\mathbf{e}$ is broader than of $C_\mathbf{d}$. Condition 1 implements this in the semantics. It has two practical consequences: Firstly, together with Conditions 5 and 6 it implies that $\Delta_\mathbf{d}$ is required to be a subset of $\Delta_\mathbf{e}$ in any CKR model if $\mathbf{d} \prec \mathbf{e}$. This follows as indicated above the = and ⊆ relations:

$$\Delta_\mathbf{d} = \top^{\mathcal{I}_\mathbf{d}} \overset{(5)}{=} \top_\mathbf{d}^{\mathcal{I}_\mathbf{d}} \overset{(6)}{=} \top_\mathbf{d}^{\mathcal{I}_\mathbf{e}} \overset{(1)}{\subseteq} \top_\mathbf{e}^{\mathcal{I}_\mathbf{e}} \overset{(5)}{=} \top^{\mathcal{I}_\mathbf{e}} = \Delta_\mathbf{e} \qquad (1)$$

This is a basic premise in order to make the knowledge of $C_\mathbf{d}$ accessible to $C_\mathbf{e}$ by the latter constraints.

The second consequence of Condition 1 is that if $C_\mathbf{d} \prec C_\mathbf{e}$, then any other context $C_\mathbf{f}$ is aware of this in the sense that $\top_\mathbf{d}^{\mathcal{I}_\mathbf{f}} \subseteq \top_\mathbf{e}^{\mathcal{I}_\mathbf{f}}$. This allows for some basic desirable properties of reasoning about the knowledge of other contexts. For instance, in any context it is entailed that $\top_{\mathsf{FWC,2010,Africa}} \sqsubseteq \top_{\mathsf{FB,2010,World}}$ (given that $\mathsf{FWC} \prec \mathsf{FB}$ and $\mathsf{Africa} \prec \mathsf{World}$). Hence if an individual is known to belong to the context of FIFA WC 2010, we always know that it also belongs to the context of football of the same year.

Conditions 2 and 3 take care that in every context $C_\mathbf{d}$ the interpretations of symbols qualified with some $\mathbf{f}$ are roofed under the concept $\top_\mathbf{f}$ which thus represents the domain of $C_\mathbf{f}$ as viewed inside $C_\mathbf{d}$ and in a CKR model $\top_\mathbf{f}^{\mathcal{I}_\mathbf{d}}$ represents the image that $\mathcal{I}_\mathbf{d}$ keeps of $\Delta_\mathbf{f}$. That is for instance $\mathsf{Team}_{\mathsf{FWC,2010,Africa}} \sqsubseteq \top_{\mathsf{FWC,2010,Africa}}$ holds in any context. Or if a qualified role such as $\mathsf{hasPlayer}_{\mathsf{FWC,2010,Africa}}$ occurs in some context, it is assured by the semantics that all its possible values are always instances of $\top_{\mathsf{FWC,2010,Africa}}$, i.e., individuals that are known to occur in $C_{\mathsf{FWC,2010,Africa}}$.

As implied by further conditions the image of $\Delta_\mathbf{f}$ in $C_\mathbf{d}$ (i.e., the set $\top_\mathbf{f}^{\mathcal{I}_\mathbf{d}}$) may be but as well may not be entirely precise, depending on how $C_\mathbf{d}$ and $C_\mathbf{f}$ are related by the coverage. As we learned from equation (1), this image is necessarily precise in cases when $\mathbf{d} \prec \mathbf{f}$. In the opposite case, the image of the domain of a super-context is a narrowing of the original domain (i.e., $\top_\mathbf{f}^{\mathcal{I}_\mathbf{d}} \subseteq \Delta_\mathbf{f}$ if $\mathbf{d} \prec \mathbf{f}$). If neither $\mathbf{d} \prec \mathbf{f}$ nor $\mathbf{f} \prec \mathbf{d}$ then the image is even less precise and there can be elements in $\top_\mathbf{f}^{\mathcal{I}_\mathbf{d}}$ which do not belong to $\Delta_\mathbf{f}$ at all.

Condition 4 is responsible for the semantic treatment of individuals in CKR. If a narrower context $C_\mathbf{d}$ is covered by a broader context $C_\mathbf{e}$, and an individual a is defined in both of these contexts, then the interpretation of a must be equal in both of these contexts. This is assured by propagating the semantics of a from $C_\mathbf{d}$ into $C_\mathbf{e}$, but not necessarily the other way around: if $a^{\mathcal{I}_\mathbf{d}}$ is defined then $a^{\mathcal{I}_\mathbf{e}}$ must be defined and must be equal to $\mathcal{I}_\mathbf{e}$; on the other hand, if $a^{\mathcal{I}_\mathbf{e}}$ is defined then $a^{\mathcal{I}_\mathbf{d}}$ must be defined to the same value only if $a^{\mathcal{I}_\mathbf{e}}$ is part of the $C_\mathbf{e}$'s image of $\top_\mathbf{d}$.

One practical consequence of this treatment is that if the same individual a occurs in two context which share at least one common super-context, it has the same interpretation. Another consequence is that it allows to predicate about (non)existence of objects in a context from a broader context. For instance if $\top_{\mathsf{FWC,2010,Africa}}(\mathsf{England})$ and $\neg\top_{\mathsf{FWC,2010,Africa}}(\mathsf{Egypt})$ are stated in a context broader than $C_{\mathsf{FWC,2010,Africa}}$ (e.g., in $C_{\mathsf{FB,2010,World}}$), as a consequence it is implied that England participates in the last FIFA WC while Egypt does not. That is, on the semantic level the individual England is always defined in this context while the individual Egypt is always undefined in it.

Condition 5 provides meaning for partially qualified symbols. It assures that the values for attributes which are not specified are always taken from the current context in which the expression appears. Therefore in the end all symbols even those partially qualified are treated as fully qualified by the semantics. It is important to understand that also symbols with no qualifying vectors are viewed as qualified symbols, they are qualified with the empty dimensional vector {}. Their qualification is taken from the context in which they appear and they are thenceforth treated as fully qualified by the semantics.

Due to this kind of treatment, partially qualified symbols are in fact some syntactic sugar added to the framework, For instance, instead of $\mathsf{Coach}_{\mathsf{FB,World}}$ we can equivalently use $\mathsf{Coach}_{\mathsf{FB,2010,World}}$ inside $C_{\mathsf{FWC,2010,Africa}}$ and instead of playsFor we can equivalently use $\mathsf{playsFor}_{\mathsf{FWC,2010,Africa}}$ in the very same context. On the other hand, we consider partially qualified symbols necessary in order to achieve practical usability of the framework.

Condition 6, 7, and 8 provide semantics for qualified symbols. It is ensured that the meaning of a symbol $X_\mathbf{d}$ is based on its interpretation in $C_\mathbf{d}$ as much as the partially overlapping domains allow. Therefore the propagation of knowledge is respective to the hierarchy of contexts as follows.

Condition 6 states that the interpretation of $X_\mathbf{d}$ is strictly bound to $X^{\mathcal{I}_\mathbf{d}}$ in all contexts that cover $C_\mathbf{d}$. This is indeed possible due to the fact that $\Delta_\mathbf{d}$ is totally contained the interpretation domains of all such contexts, which is assured by Condition 1. For example, the interpretation of $\mathsf{Team}_{\mathsf{FWC,2010,Africa}}$ in $C_{\langle\mathsf{FB,2010,World}\rangle}$ is the same as the interpretation of Team in the context $C_{\langle\mathsf{FWC,2010,Africa}\rangle}$.

Conditions 7 and 8 assure that given $C_\mathbf{d} \prec C_\mathbf{e}$ and a symbol $X_\mathbf{f}$, where $\mathbf{f}$ is not necessarily related to $\mathbf{d}$ or $\mathbf{e}$, the two interpretations of $X_\mathbf{f}$ in $\mathcal{I}_\mathbf{d}$ and $\mathcal{I}_\mathbf{e}$ are equal modulo the interpretation domain of the narrower context $\Delta_\mathbf{d}$. This especially implies that if a particular individual (or pair of individuals if X is a role) occurs in both contexts $C_{\mathbf{d}}$ and $C_{\mathbf{e}}$, then it ether belongs to both $X_{\mathbf{f}}^{\mathcal{I}_{\mathbf{d}}}$ and $X_{\mathbf{f}}^{\mathcal{I}_{\mathbf{e}}}$ or it belongs to none of them. Consider our example CKR from Fig. 2. In this CKR the contexts $C_{\mathbf{wn}}$ (the context of world news 2010) and $C_{\mathbf{f}}$ (the context of football in 2010) are unrelated. Therefore if a qualified concept $\mathsf{Player}_{\mathbf{f}}$ occurs inside $C_{\mathbf{wn}}$ we cannot be sure that all its instances belong to the domain of $C_{\mathbf{f}}$. Due to Condition 7 however, the interpretations assigned to $\mathsf{Player}_{\mathbf{f}}$ by $C_{\mathbf{wn}}$ and $C_{\mathbf{wc}}$ must agree as much as the partially overlapping domains permit (because $\mathbf{wc} \prec \mathbf{wn}$). Similarly, as $\mathbf{wc} \prec \mathbf{f}$, the interpretations assigned to $\mathsf{Player}_{\mathbf{f}}$ by $C_{\mathbf{wc}}$ and $C_{\mathbf{f}}$ must agree as much as the domains permit. Hence if one of the instances of $\mathsf{Player}_{\mathbf{f}}$ in $C_{\mathbf{wn}}$ is a constant Rooney, which also appears in $C_{\mathbf{wc}}$, then due to Condition 7 we have that $\mathsf{Rooney}^{\mathcal{I}_{\mathbf{e}}} \in \Delta_{\mathbf{wc}}$ and we already showed that $\Delta_{\mathbf{wc}} \subseteq \Delta_{\mathbf{f}}$ in equation (1). For further details see Examples 1–3 where we study some basic properties of such a semantics.

Condition 9 states that in a CKR model, the local interpretation $\mathcal{I}_{\mathbf{d}}$ of each context $C_{\mathbf{d}}$ is also a model of $C_{\mathbf{d}}$ according to the local semantics, i.e., that of DL.

The two classic reasoning tasks for DL are satisfiability of concepts and entailment (especially of subsumption formulae) with respect to a knowledge base. In a CKR model, a formula may be satisfied in one context but in another it may be unsatisfied. In addition, for some contexts in a knowledge base, all admissible CKR models may have a local model with empty domain whereas for other contexts there may be CKR models with non-empty local domain. Therefore given a CKR $\mathfrak{K}$ one has to specify with respect to which context the reasoning task in question is to be evaluated. Such reasoning tasks will be called **d**-satisfiability of concepts and **d**-entailment.

Definition 10 (**d**-satisfiability of concepts)**.** *Given a CKR knowledge base $\mathfrak{K}$ over $\langle\Gamma,\Sigma\rangle$ with $\mathbf{d} \in \mathfrak{D}_\Gamma$ and a concept C over Σ, we say that C is **d**-satisfiable w.r.t. $\mathfrak{K}$ if there exists a CKR model $\mathfrak{I} = \{\mathcal{I}_{\mathbf{e}}\}_{\mathbf{e}\in\mathfrak{D}_\Gamma}$ of $\mathfrak{K}$ such that $C^{\mathcal{I}_{\mathbf{d}}} \neq \emptyset$.*

Definition 11 (**d**-entailment)**.** *Given a CKR knowledge base $\mathfrak{K}$ over $\langle\Gamma,\Sigma\rangle$ with $\mathbf{d} \in \mathfrak{D}_\Gamma$ and any formula ϕ over Σ with syntax listed in Table 1 under Axioms, we say that ϕ is **d**-entailed by $\mathfrak{K}$ (denoted by $\mathfrak{K} \models \mathbf{d} : \phi$) if for every CKR model $\mathfrak{I} = \{\mathcal{I}_{\mathbf{e}}\}_{\mathbf{e}\in\mathfrak{D}_\Gamma}$ of $\mathfrak{K}$ we have $\mathcal{I}_{\mathbf{d}} \models_{\mathrm{DL}} \phi$.*

In addition, we consider satisfiability of a CKR knowledge base as a decision task. In this case it makes sense to define **d**-satisfiability as well as global satisfiability.

Definition 12 (**d**-satisfiability)**.** *A CKR knowledge base $\mathfrak{K}$ over $\langle\Gamma,\Sigma\rangle$ with $\mathbf{d} \in \mathfrak{D}_\Gamma$ is said to be **d**-satisfiable if there exists a CKR model $\mathfrak{I} = \{\mathcal{I}_{\mathbf{e}}\}_{\mathbf{e}\in\mathfrak{D}_\Gamma}$ of $\mathfrak{K}$ such that $\Delta_{\mathbf{d}} \neq \emptyset$.*

Definition 13 (Global satisfiability)**.** *A CKR knowledge base $\mathfrak{K}$ over $\langle\Gamma,\Sigma\rangle$ is said to globally satisfiable if there exists a CKR model $\mathfrak{I} = \{\mathcal{I}_{\mathbf{d}}\}_{\mathbf{d}\in\mathfrak{D}_\Gamma}$ of $\mathfrak{K}$ such that for every $\mathbf{d} \in \mathfrak{D}_\Gamma$ we have $\Delta_{\mathbf{d}} \neq \emptyset$.*

As usual with DL, the **d**-entailment (of concept subsumption) and **d**-(un)satisfiability are inter-reducible: $\mathfrak{K} \models \mathbf{d} : C \sqsubseteq D$ iff $C \sqcap \neg D$ is not **d**-satisfiable w.r.t. $\mathfrak{K}$; on the other hand, C is **d**-satisfiable w.r.t. $\mathfrak{K}$ iff $\mathfrak{K} \not\models \mathbf{d} : C \sqsubseteq \bot$. In addition, C is **d**-satisfiable w.r.t. $\mathfrak{K}$ iff $\mathfrak{K} \cup \{\mathbf{d} : C(a)\}$ is **d**-satisfiable where a is a new constant previously unused in $\mathfrak{K}$. This follows from the fact that local models are all valid $\mathcal{SROIQ}$ models and from the fact that the reduction holds in $\mathcal{SROIQ}$. For details on these reductions see for instance the DL Handbook by Baader et al. [14].

4. Reasoning in CKR

In this section we provide a characterization of entailment in CKR in terms of a natural deduction (ND) calculus [23]. CKR entailment is the product of two orthogonal semantic entailments: *local entailment* and *cross-context entailment*, The former is induced by the local semantics and coincides with entailment in $\mathcal{SROIQ}$; the latter is induced by the constraints that Definition 9 imposes on each pair of contexts related via coverage. $\mathcal{SROIQ}$ entailment, i.e., $\phi_1, \ldots, \phi_n \models_{\mathrm{DL}} \phi$, is known to be decidable.[4] We therefore assume a black box decision procedure that checks if $\phi_1, \ldots, \phi_n \models_{\mathrm{DL}} \phi$.

Reasoning rules in the ND calculus allow to deduce conclusions in one of the contexts based on evidence from other contexts, they are therefore a kind of *bridge rules* [16]. As an example consider the following simple bridge rule:

$$\frac{\mathbf{d} : A \sqsubseteq B \quad \mathbf{d} \prec \mathbf{e}}{\mathbf{e} : A_{\mathbf{d}} \sqsubseteq B_{\mathbf{d}}} \tag{2}$$

The rule implies that whenever $A \sqsubseteq B$ is true in a context $C_{\mathbf{d}}$ such that $\mathbf{d} \prec \mathbf{e}$, then $A_{\mathbf{d}} \sqsubseteq B_{\mathbf{d}}$ should be true in $C_{\mathbf{e}}$. This is indeed sound thanks to conditions 5 and 6 of Definition 9 which together impose that in any CKR model $\mathfrak{I}$ the interpretation of A and B in $\mathcal{I}_{\mathbf{d}}$ coincide respectively with the interpretations of $A_{\mathbf{d}}$ and $B_{\mathbf{d}}$ in $\mathcal{I}_{\mathbf{e}}$.

The rationale of rule (2) is that a statement in a narrower context, namely $C_{\mathbf{d}}$, can be embedded into a larger context, namely $C_{\mathbf{e}}$, by applying a transformation that preserves semantics. We generalize this idea by introducing the notion of embedding between DL knowledge bases and by showing that in CKR such embedding preserves the meaning of $\mathcal{SROIQ}$ expressions.

4.1. Embedding of DL knowledge bases

A DL embedding is a mapping that embeds a DL knowledge base with a narrower perspective into another one with a broader perspective. The vocabulary of the context being embedded splits in two parts: Σ_c that contains symbols which are completely specified with respect to the embedded context, and Σ_e that contains the remaining symbols which are called external. For instance, the symbol $\mathsf{Player}_{\mathsf{sports}}$ is external in the context $C_{\mathsf{FB,2010,World}}$, This is because $\mathsf{FB} \prec \mathsf{sports}$. If we state

[4] Decidability is guaranteed under the assumption that the knowledge base is stratified (see [15] for more details), and we have to impose this condition also for CKR. This point is discussed later.

the axiom $\mathsf{Player}_{\mathsf{sports}} \sqsubseteq \exists.\mathsf{playsFor.Team}$ here, it is only valid in this context where all the players are football players and football is a team sport. In other sports such as tennis players need not have to play for a team. Therefore, when embedding the axiom into the broader context of sports we need to take care to embed the proper meaning of the axiom it has in FB and so we need to pay attention to external symbols.

Definition 14 (DL embedding)**.** *Let Σ and Σ' be two DL alphabets, and let Σ be partitioned into two disjoint sets Σ_c and Σ_e with $\top \in \Sigma_c$. A* DL embedding *is a total function $f : \Sigma \to \Sigma'$ that maps individuals, atomic concepts, and atomic roles of Σ to individuals, atomic concepts, and atomic roles of Σ' respectively. The extension f^* of f that maps complex expressions and axioms over Σ into complex expressions and axioms over Σ' is defined as given in Table 2.*

The DL embedding is done on the syntactic level. On the semantic level, if one knowledge base is embedded into another, we should be able to embed models of the former knowledge base to the models of the latter. A pair of such models is said to be complying with the embedding.

Definition 15 (Embedding-complying interpretations)**.** *Two DL-interpretations $\mathcal{I}$ and $\mathcal{I}'$ of Σ and Σ' respectively comply with the DL embedding f if:*

1. *$a^{\mathcal{I}} = f(a)^{\mathcal{I}'}$, for each individual a of Σ such that $a^{\mathcal{I}}$ is defined;*
2. *$X^{\mathcal{I}} = f(X)^{\mathcal{I}'}$, for each concept/role $X \in \Sigma_c$;*
3. *$A^{\mathcal{I}} = f(A)^{\mathcal{I}'} \cap f(\top)^{\mathcal{I}'}$, for each concept $A \in \Sigma_e$;*
4. *$R^{\mathcal{I}} = f(R)^{\mathcal{I}'} \cap f(\top)^{\mathcal{I}'} \times f(\top)^{\mathcal{I}'}$, for each role $R \in \Sigma_e$.*

If two interpretations $\mathcal{I}$ and $\mathcal{I}'$ comply with the embedding f then $\mathcal{I}'$ is apparently an extension of $\mathcal{I}$. It contains the domain Δ of $\mathcal{I}$ as $\Delta = f(\top)^{\mathcal{I}'} \subseteq \Delta'$ and the f-images of all internal symbols of Σ_c are interpreted inside $f(\top)^{\mathcal{I}'}$. The images of external symbols of Σ_e can possibly exceed $f(\top)^{\mathcal{I}'}$ when interpreted in $\mathcal{I}'$ but we can always obtain the corresponding interpretations of their pre-images by restriction to $f(\top)^{\mathcal{I}'}$. This corresponds to the fact that the symbols external to Σ are not completely specified in $\mathcal{I}$.

An important point is that the meaning of any symbol, internal or external, with respect to $\mathcal{I}$ can always be retained from $\mathcal{I}'$. The following lemma shows that this is also true for complex descriptions composed of a mixture of internal and external symbols and as a consequence also the meaning of axioms is preserved.

Lemma 1. *If two DL-interpretations $\mathcal{I}$ and $\mathcal{I}'$ comply with the embedding $f : \Sigma \to \Sigma'$, then, for every concept C, $C^{\mathcal{I}} = (f^*(C))^{\mathcal{I}'}$, for every role R, $R^{\mathcal{I}} = (f^*(R))^{\mathcal{I}'}$, and for every axiom ϕ, $\mathcal{I} \models \phi$ iff $\mathcal{I}' \models f^*(\phi)$.*

Proof. (Sketch.) Full proof is listed in Appendix A.1. The first claim of the lemma, concerned with concepts and roles, is proved by structural induction. The base case (i.e., for atomic concepts and roles) follows from the fact that the interpretations $\mathcal{I}$ and $\mathcal{I}'$ comply with the embedding f (Definition 15). For every type of complex concept and role we then have to argue the claim from the induction hypothesis, from the construction of f^* (Table 2), and from basic properties of DL-interpretations. The second claim of the lemma that is concerned with axioms is then proved for each type of axioms, mostly as a consequence of the first claim. □

Armed with this result we will now show that given any CKR $\mathfrak{K}$ and any two contexts $C_{\mathbf{d}}$ and $C_{\mathbf{e}}$ such that $\mathbf{d} \prec \mathbf{e}$ it is possible to construct a DL-embedding between $C_{\mathbf{d}}$ and $C_{\mathbf{e}}$. For convenience we will call this embedding the $@\mathbf{d}$ operator. For any construct ϕ the embedded value will be $\phi@\mathbf{d}$. The operator will allow us to characterize the knowledge propagation in CKR along the two basic axes, from narrower to broader context and vice versa. Later on in Sect. 5 we will find another use for embeddings, when showing how CKR can be reduced into a regular DL knowledge base.

Definition 16 ($@\mathbf{d}$ operator)**.** *For every full dimensional vector $\mathbf{d}$, the operator $(\cdot)@\mathbf{d}$ is defined as $f^*_{\mathbf{d}}(\cdot)$, where $f_{\mathbf{d}}$ is an embedding from Σ into itself defined as follows:*

- *$f_{\mathbf{d}}(a) = a$ for every individual a;*
- *$f_{\mathbf{d}}(X_{\mathbf{d}'_{\mathbf{B}}}) = X_{\mathbf{d}'_{\mathbf{B}}+\mathbf{d}}$ for every concept/role X;*
- *$\Sigma_c = \{X_{\mathbf{d}'_{\mathbf{B}}} \in \Sigma \mid \mathbf{d}'_{\mathbf{B}} \preceq \mathbf{d}_{\mathbf{B}}\}$; $\Sigma_e = \Sigma \setminus \Sigma_c$.*

For instance if the concept Team occurs in $C_{\mathbf{d}}$ with $\mathbf{d} = \{\mathsf{FWC}, \mathsf{2010}, \mathsf{Africa}\}$, it belongs to Σ_c as $\mathbf{d}'_{\mathbf{B}} \preceq \mathbf{d}_{\mathbf{B}}$ for $\mathbf{B} = \emptyset$. Hence $\mathsf{Team}@\mathbf{d} = \mathsf{Team}_{\mathsf{FWC,2010,Africa}}$. This is natural, as in a context wider than $C_{\mathbf{d}}$ the concept $\mathsf{Team}_{\mathsf{FWC,2010,Africa}}$ is fully defined by Team in $C_{\mathbf{d}}$. But $\mathsf{NationalTeam}_{\mathsf{FB}} \notin \Sigma_c$ as $\mathsf{FB} \not\preceq \mathsf{FWC}$. Hence we have $\mathsf{NationalTeam}_{\mathsf{FB}}@\mathbf{d} = \mathsf{NationalTeam}_{\mathsf{FB,2010,Africa}} \sqcap \top_{\mathsf{FWC,2010,Africa}}$. Intuitively, in order to embed $\mathsf{NationalTeam}_{\mathsf{FB}}$ from $C_{\mathbf{d}}$ into a broader context one must restrict it to $\top_{\mathsf{FWC,2010,Africa}}$ because its interpretation in the broader context may be broader.

Lemma 2. *Given a CKR $\mathfrak{K}$ with two contexts $C_{\mathbf{d}}$ and $C_{\mathbf{e}}$ such that $\mathbf{d} \preceq \mathbf{e}$, and given any model $\mathfrak{I}$ of $\mathfrak{K}$, the pair of local interpretations $\mathcal{I}_{\mathbf{d}}$ and $\mathcal{I}_{\mathbf{e}}$ complies with the embedding $f_{\mathbf{d}}$ respective to the operator $@\mathbf{d}$.*

Proof. Let $\mathfrak{K}$ be a CKR over $\langle \Gamma, \Sigma \rangle$. Let $C_{\mathbf{d}}, C_{\mathbf{e}}$ be two contexts of $\mathfrak{K}$ such that $\mathbf{d} \preceq \mathbf{e}$. Let the $@\mathbf{d}$ operator be defined on the embedding $f_{\mathbf{d}} : \Sigma \to \Sigma$ as given in Definition 16, that is, we have $\Sigma_c = \{X_{\mathbf{d}'_{\mathbf{B}}} \in \Sigma \mid \mathbf{d}'_{\mathbf{B}} \preceq \mathbf{d}_{\mathbf{B}}\}$ and $\Sigma_e = \Sigma \setminus \Sigma_c$. We need to show that all four conditions of Definition 15 are satisfied:

1. $a^{\mathcal{I}_{\mathbf{d}}} = f_{\mathbf{d}}(a)^{\mathcal{I}_{\mathbf{e}}}$, for each individual a of Σ such that $a^{\mathcal{I}_{\mathbf{d}}}$ is defined: this follows from the Condition 4 of Definition 9;
2. $X^{\mathcal{I}_{\mathbf{d}}} = f_{\mathbf{d}}(X)^{\mathcal{I}_{\mathbf{e}}}$, for each concept/role $X \in \Sigma_c$. In this case $X = Y_{\mathbf{d}'_{\mathbf{B}}}$ for some $\mathbf{d}'_{\mathbf{B}} \preceq \mathbf{d}_{\mathbf{B}}$. If $\mathbf{d}'_{\mathbf{B}} = \mathbf{d}_{\mathbf{B}}$ then $Y_{\mathbf{d}'_{\mathbf{B}}+\mathbf{d}} = Y_{\mathbf{d}}$ and the proposition follows either trivially if $\mathbf{d} = \mathbf{e}$ or directly from Condition 6 if $\mathbf{d} \prec \mathbf{e}$. Now assume that $\mathbf{d}'_{\mathbf{B}} \prec \mathbf{d}_{\mathbf{B}}$. From Definition 9 we have $X^{\mathcal{I}_{\mathbf{d}}} = Y^{\mathcal{I}_{\mathbf{d}}}_{\mathbf{d}'_{\mathbf{B}}} = Y^{\mathcal{I}_{\mathbf{d}}}_{\mathbf{d}'_{\mathbf{B}}+\mathbf{d}}$ (Condition 5), and $Y^{\mathcal{I}_{\mathbf{d}}}_{\mathbf{d}'_{\mathbf{B}}+\mathbf{d}} = Y^{\mathcal{I}_{\mathbf{e}}}_{\mathbf{d}'_{\mathbf{B}}+\mathbf{d}}$ because $\mathbf{d}'_{\mathbf{B}}+\mathbf{d} \prec \mathbf{d} \preceq \mathbf{e}$ (Condition 6). Finally, from the construction of $f_{\mathbf{d}}$, $Y^{\mathcal{I}_{\mathbf{e}}}_{\mathbf{d}'_{\mathbf{B}}+\mathbf{d}} = f_{\mathbf{d}}(Y_{\mathbf{d}'_{\mathbf{B}}})^{\mathcal{I}_{\mathbf{e}}} = f_{\mathbf{d}}(X)^{\mathcal{I}_{\mathbf{e}}}$;

$f^*(A) = \begin{cases} f(A) & \text{if } A \in \Sigma_c \\ f(\top) \sqcap f(A) & \text{if } A \in \Sigma_e \end{cases}$	$f^*(C \sqcap D) = f^*(C) \sqcap f^*(D)$	$f^*(C(a)) = f^*(C)(f(a))$
$f^*(R) = \begin{cases} f(R) & \text{if } R \in \Sigma_c \\ f(I) \circ f(R) \circ f(I) & \text{if } R \in \Sigma_e \end{cases}$	$f^*(\exists R.C) = \begin{cases} \exists f(R).f^*(C) & \text{if } R \in \Sigma_c \\ f(\top) \sqcap \exists f(R).f^*(C) & \text{if } R \in \Sigma_e \end{cases}$	$f^*(R(a,b)) = f(R)(f(a), f(b))$
$f^*(\neg C) = f(\top) \sqcap \neg f^*(C)$	$f^*(\exists R.\mathrm{Self}) = \begin{cases} \exists f(R).\mathrm{Self} & \text{if } R \in \Sigma_c \\ f(\top) \sqcap \exists f(R).\mathrm{Self} & \text{if } R \in \Sigma_e \end{cases}$	$f^*(\neg R(a,b)) = \neg f(R)(f(a), f(b))$
$f^*(\bot) = \bot$	$f^*(\geqslant nR.C) = \begin{cases} \geqslant n f(R).f^*(C) & \text{if } R \in \Sigma_c \\ f(\top) \sqcap \geqslant n f(R).f^*(C) & \text{if } R \in \Sigma_e \end{cases}$	$f^*(C \sqsubseteq D) = f^*(C) \sqsubseteq f^*(D)$
$f^*(R^-) = (f(R))^-$	$f^*(\{a\}) = \{f(a)\}$	$f^*(R \sqsubseteq S) = f^*(R) \sqsubseteq f(S)$
$f^*(R \circ S) = f^*(R) \circ f^*(S)$		$f^*(a = b) = f(a) = f(b)$
		$f^*(a \neq b) = f(a) \neq f(b)$

Table 2: DL-embedding on complex expressions and axioms

$$\frac{\mathbf{d}:\phi_1 \;\ldots\; \mathbf{d}:\phi_n \quad \{\phi_1 \ldots \phi_n\} \models \phi}{\mathbf{d}:\phi}\,\mathsf{LReas} \qquad \frac{\mathbf{d}:\bot(a)}{\mathbf{e}:\top \sqsubseteq \bot}\,\mathsf{Bot} \qquad \frac{\mathbf{d} \leq \mathbf{e}}{\mathbf{f}:A_\mathbf{d} \sqsubseteq \top_\mathbf{e}} \quad \frac{-}{\mathbf{f}:\exists R_\mathbf{d}\top \sqsubseteq \top_\mathbf{d}} \quad \frac{-}{\mathbf{f}:\top \sqsubseteq \forall R_\mathbf{d}\top_\mathbf{d}}\,\mathsf{Top}$$

$$\frac{\mathbf{e}:\phi@\mathbf{d} \quad \mathbf{e}:\top_\mathbf{d}(a_1) \;\cdots\; \mathbf{e}:\top_\mathbf{d}(a_n) \quad \mathbf{d} \leq \mathbf{e}}{\mathbf{d}:\phi}\,\mathsf{Push} \qquad \frac{\mathbf{d}:\phi \quad \mathbf{d} \leq \mathbf{e}}{\mathbf{e}:\phi@\mathbf{d}}\,\mathsf{Pop}$$

$$\frac{\mathbf{d}:A \sqcup B(x) \quad \begin{array}{c}[\mathbf{d}:A(x)] \\ \mathbf{e}:\phi\end{array} \quad \begin{array}{c}[\mathbf{d}:B(x)] \\ \mathbf{e}:\phi\end{array}}{\mathbf{e}:\phi}\,\sqcup\mathsf{E} \qquad \frac{\mathbf{d}:\exists R.A(x) \quad \begin{array}{c}[\mathbf{d}:R(x,y),\ \mathbf{d}:A(y)] \\ \mathbf{e}:\phi\end{array}}{\mathbf{e}:\phi}\,\exists\mathsf{E} \qquad \frac{\begin{array}{c}[\mathbf{d}:\top(a)] \\ \mathbf{d}:\top \sqsubseteq \bot\end{array}}{\mathbf{d}:\top \sqsubseteq \bot}\,a\mathsf{E}$$

$$\frac{\mathbf{d}:\geqslant nR.A(x) \quad \begin{array}{c}[\mathbf{d}:y_i \neq y_j,\ \mathbf{d}:R(x,y_i),\ \mathbf{d}:A(y_i)]_{1 \leq i \neq j \leq n} \\ \mathbf{e}:\phi\end{array}}{\mathbf{e}:\phi}\,(\geqslant n)\mathsf{E}$$

Restrictions: **1)** LReas can be applied if every individual occurring in ϕ occurs in a ϕ_i for some $1 \leq i \leq n$; **2)** in the Push rule $a_1, \ldots, a_n$ are assumed to be all individuals occuring in ϕ; **3)** the individuals a, y, and y_i, $1 \leq i \leq n$, occuring in aE, $\exists$E, and $(\geqslant n)$E are new, not occuring elsewhere in $\mathfrak{K}$ and the proof apart from the assumptions discharded by these rules.

Table 3: CKR inference rules

3. $A^{\mathcal{I}_\mathbf{d}} = f_\mathbf{d}(A)^{\mathcal{I}_\mathbf{e}} \cap f_\mathbf{d}(\top)^{\mathcal{I}_\mathbf{e}}$, for each concept $A \in \Sigma_e$; In this case $A = B_{\mathbf{d}'_\mathbf{B}}$ with $\mathbf{d}'_\mathbf{B} \not\leq \mathbf{d}_\mathbf{B}$. From Definition 9 we have $A^{\mathcal{I}_\mathbf{d}} = B^{\mathcal{I}_\mathbf{d}}_{\mathbf{d}'_\mathbf{B}} = B^{\mathcal{I}_\mathbf{d}}_{\mathbf{d}'_\mathbf{B}+\mathbf{d}}$ (Condition 5), and $B^{\mathcal{I}_\mathbf{d}}_{\mathbf{d}'_\mathbf{B}+\mathbf{d}} = B^{\mathcal{I}_\mathbf{e}}_{\mathbf{d}'_\mathbf{B}+\mathbf{d}} \cap \Delta_\mathbf{d}$ (Condition 7). As $\Delta_\mathbf{d} = \top^{\mathcal{I}_\mathbf{d}} = \top^{\mathcal{I}_\mathbf{d}}_\mathbf{d} = \top^{\mathcal{I}_\mathbf{e}}_\mathbf{d}$ (Condition 5, then Condition 6), we finally get $B^{\mathcal{I}_\mathbf{e}}_{\mathbf{d}'_\mathbf{B}+\mathbf{d}} \cap \Delta_\mathbf{d} = B^{\mathcal{I}_\mathbf{e}}_{\mathbf{d}'_\mathbf{B}+\mathbf{d}} \cap \top^{\mathcal{I}_\mathbf{e}}_\mathbf{d} = f_\mathbf{d}(B_{\mathbf{d}_\mathbf{B}})^{\mathcal{I}_\mathbf{e}} \cap f_\mathbf{d}(\top)^{\mathcal{I}_\mathbf{e}} = f_\mathbf{d}(A)^{\mathcal{I}_\mathbf{e}} \cap f_\mathbf{d}(\top)^{\mathcal{I}_\mathbf{e}}$ from the construction of $f_\mathbf{d}$;
4. $R^{\mathcal{I}_\mathbf{d}} = f_\mathbf{d}(R)^{\mathcal{I}_\mathbf{e}} \cap f_\mathbf{d}(\top)^{\mathcal{I}_\mathbf{e}} \times f_\mathbf{d}(\top)^{\mathcal{I}_\mathbf{e}}$, for each role $R \in \Sigma_e$: this case is exactly analogous to the previous one, only we need to use Condition 8 instead of Condition 7 of Definition 9 which is concerned with roles.

□

4.2. ND calculus for CKR

We now briefly introduce natural deduction (ND), for more details see the work of Prawitz [23]. An ND calculus is a set of inference rules of the form:

$$\frac{\alpha_1 \quad \cdots \quad \alpha_n \quad \begin{array}{c}[B_{n+1}] \\ \alpha_{n+1}\end{array} \quad \cdots \quad \begin{array}{c}[B_{n+m}] \\ \alpha_{n+m}\end{array}}{\alpha}\,\rho \qquad (3)$$

with $n, m \geq 0$, where for all i, α_i and α are formulae, B_i are sets of formulae. The formulae α_i are the *premises* of ρ, α is the conclusion, and B_i are the *assumptions discharged* by ρ. A *deduction* of α depending on a set of formulae Φ is a tree rooted in α inductively constructed starting from a set of assumptions included in Φ by applying the inference rules. Formally deduction is defined by induction:

1. a formula α is a deduction of α depending on $\{\alpha\}$;
2. if for each $1 \leq i \leq n+m$, Π_i is a deduction of α_i depending on Φ_i and the calculus contains the rule (3), then

$$\frac{\Pi_1 \cdots \Pi_{n+m}}{\alpha}\rho$$

is a deduction of α depending on $\left(\bigcup_{i=1}^{n} \Phi_i\right) \cup \left(\bigcup_{i=n+1}^{n+m} (\Phi_i \setminus B_i)\right)$.

A formula α is derivable from Φ if there is a deduction of α depending on a subset of Φ. α is provable if it is derivable form the empty set.

A ND system for a CKR $\mathfrak{K} = \langle \mathfrak{M}, \mathfrak{C} \rangle$ over $\langle \Gamma, \Sigma \rangle$ is shown in Table 3. The premises of the ND rules of our calculus are either object formulae of the form $\mathbf{d} : \phi$ where $\mathbf{d} \in \mathfrak{D}_\Gamma$ and ϕ is a DL formula over Σ, or meta formulae μ over Γ. Conclusions and discharged assumptions are always object formulae.

Definition 17 (Derivability in CKR). *Given a CKR $\mathfrak{K} = \langle \mathfrak{M}, \mathfrak{C} \rangle$ over $\langle \Gamma, \Sigma \rangle$ and a set of object formulae Φ, an object formula $\mathbf{d} : \phi$ is derivable from $\mathfrak{K}$ and Φ (denoted by $\mathfrak{K}, \Phi \vdash \mathbf{d} : \phi$) if it is derivable in the calculus given in Table 3 from the set Ψ which contains the following formulae:*

1. $\mathbf{e} : \chi$*, for every* $\chi \in C_\mathbf{e}$ *and for every* $\mathbf{e} \in \mathfrak{D}_\Gamma$*;*
2. μ*, for every meta formula such that* $\mathfrak{M} \models_{\mathrm{DL}} \mu$*;*
3. ψ*, for all* $\psi \in \Phi$.

Instead of $\mathfrak{K}, \emptyset \vdash \mathbf{d} : \phi$ we simply write $\mathfrak{K} \vdash \mathbf{d} : \phi$. Even if ND derivations are formally defined as trees, we will often present them as a sequence of derivation steps. This can be naturally achieved, we only have to track the set of premises from which the resulting formula in each step is derived. Hereafter $\vdash$ always denotes this calculus as formally defined by Definition 17, and whenever we say proof calculus or just calculus, we refer exactly to this calculus. The proof calculus allows us to define the syntactic notion of **d**-consistence of a CKR knowledge base.

Definition 18 (**d**-consistence). *A CKR $\mathfrak{K}$ is **d**-consistent if it is not possible to prove $\mathbf{d} : \top \sqsubseteq \bot$ by the calculus, i.e., if $\mathfrak{K} \nvdash \mathbf{d} : \top \sqsubseteq \bot$. Otherwise $\mathfrak{K}$ is **d**-inconsistent.*

The first main result of this work is presented in Theorem 1 where the calculus is showed to be a sound and complete characterization of logical consequence in CKR. In other words, the calculus rules show us how logical consequence is propagated between contexts in a CKR knowledge base.

Theorem 1 (Soundness and Completeness). *For every CKR $\mathfrak{K}$ over $\langle \Gamma, \Sigma \rangle$, for every $\mathbf{d} \in \mathfrak{D}_\Gamma$, and for every formula ϕ over Σ, $\mathfrak{K} \vdash \mathbf{d} : \phi$ if and only if $\mathfrak{K} \models \mathbf{d} : \phi$.*

Proof. (Sketch.) The full proof is attached in Appendix A.2. The soundness is proved by showing for each rule that it is sound, i.e., that starting from valid premises it only derives valid conclusions.

In order to prove the completeness we make use of the reductions between reasoning tasks. For any formula $\mathbf{d} : \phi$, a CKR $\mathfrak{K}'$ can be constructed such that $\mathfrak{K} \models \mathbf{d} : \phi$ if and only if $\mathfrak{K}'$ is **d**-unsatisfiable. We therefore prove, that if $\mathfrak{K}'$ is **d**-unsatisfiable then a proof exists in the CKR calculus that $\mathfrak{K}'$ is **d**-inconsistent. This guarantees, that whenever $\mathfrak{K} \models \mathbf{d} : \phi$ then there exists a calculus proof that supports this. The implication – if $\mathfrak{K}'$ is **d**-unsatisfiable then the calculus proves that $\mathfrak{K}'$ is **d**-inconsistent – is proven by contraposition, i.e., we prove that if there is no calculus proof concluding that $\mathfrak{K}'$ is **d**-inconsistent then it is **d**-satisfiable. This is proved by a variant of the Henkin construction of a model based on constants (see e.g. [24]). For details see the appendix. □

Let us show some examples of deductions in CKR. Consider the CKR with structure depicted in Fig. 2. Example 1 shows how knowledge is propagated from $C_\mathbf{wc}$ into $C_\mathbf{i}$ via the common super-context $C_\mathbf{f}$, and Example 2 shows how knowledge is propagated from $C_\mathbf{wn}$ into $C_\mathbf{f}$ via the common sub-context $C_\mathbf{wc}$. Finally Example 3 shows how contradicting knowledge can coexist in different separated context.

Example 1. *The following deduction shows how the subsumption $\mathbf{wc} : \mathsf{WChamp} \sqsubseteq \mathsf{Player}$ propagates from the FIFA WC context $C_\mathbf{wc}$ to the Italian National League context $C_\mathbf{i}$. Notice that the result of this deduction, i.e., $\mathbf{i} : \mathsf{WChamp}_\mathbf{wc} \sqsubseteq \mathsf{Player}_\mathbf{wc}$, in the context $C_\mathbf{i}$ is weaker than the premise as it holds only on the set of players of the Italian National League. In other words, the knowledge shifting from $C_\mathbf{wc}$ to $C_\mathbf{i}$ is limited by the domain of interpretation of $C_\mathbf{i}$.*

(1)	$\mathbf{wc} : \mathsf{WChamp} \sqsubseteq \mathsf{Player}$	premise
(2)	$\mathbf{f} : (\mathsf{WChamp} \sqsubseteq \mathsf{Player})@\mathbf{wc}$	Pop, $\mathbf{wc} \preceq \mathbf{f}$
(3)	$\mathbf{f} : \mathsf{WChamp}_\mathbf{wc} \sqsubseteq \mathsf{Player}_\mathbf{wc}$	by @
(4)	$\mathbf{f} : \mathsf{WChamp}_\mathbf{wc} \sqcap \top_\mathbf{i} \sqsubseteq \mathsf{Player}_\mathbf{wc} \sqcap \top_\mathbf{i}$	LReas
(5)	$\mathbf{f} : (\mathsf{WChamp}_\mathbf{wc} \sqsubseteq \mathsf{Player}_\mathbf{wc})@\mathbf{i}$	by @
(6)	$\mathbf{i} : \mathsf{WChamp}_\mathbf{wc} \sqsubseteq \mathsf{Player}_\mathbf{wc}$	Push, $\mathbf{i} \preceq \mathbf{f}$

Example 2. *The following deduction shows how $\mathbf{wn} : \mathsf{Player}_\mathbf{f} \sqsubseteq \mathsf{Pro}$ (i.e., every football player mentioned in the world news is a professional) propagates from $C_\mathbf{wn}$ to $C_\mathbf{f}$, trough the common sub-context $C_\mathbf{wc}$.*

(1)	$\mathbf{wn} : \mathsf{Player}_\mathbf{f} \sqsubseteq \mathsf{Pro}$	premise
(2)	$\mathbf{wn} : (\mathsf{Player}_\mathbf{f} \sqsubseteq \mathsf{Pro})@\mathbf{wn}$	Pop, $\mathbf{wn} \preceq \mathbf{wn}$
(3)	$\mathbf{wn} : \mathsf{Player}_\mathbf{f} \sqsubseteq \mathsf{Pro}_\mathbf{wn}$	by @
(4)	$\mathbf{wn} : \mathsf{Player}_\mathbf{f} \sqcap \top_\mathbf{wc} \sqsubseteq \mathsf{Pro}_\mathbf{wn} \sqcap \top_\mathbf{wc}$	by LReas
(5)	$\mathbf{wc} : \mathsf{Player}_\mathbf{f} \sqsubseteq \mathsf{Pro}_\mathbf{wn}$	Push, $\mathbf{wc} \preceq \mathbf{wn}$
(6)	$\mathbf{f} : \mathsf{Player}_\mathbf{f} \sqcap \top_\mathbf{wc} \sqsubseteq \mathsf{Pro}_\mathbf{wn} \sqcap \top_\mathbf{wc}$	Pop, $\mathbf{wc} \preceq \mathbf{f}$
(7)	$\mathbf{f} : \mathsf{Player}_\mathbf{f} \sqcap \top_\mathbf{wc} \sqsubseteq \mathsf{Pro}_\mathbf{wn}$	LReas

Notice that we did not infer $\mathbf{f} : \mathsf{Player}_\mathbf{f} \sqsubseteq \mathsf{Pro}_\mathbf{wn}$, i.e., that every single player of football is understood as a professional player in the world news, but the fact that this subsumption holds only for the players of the FIFA world cup domain.

Example 3. *Suppose that the Italian News context $C_\mathbf{in}$ contains the facts that Rooney does not take part to the Italian league in 2010, i.e., $\neg\top_\mathbf{i}(\mathsf{Rooney})$, and that he is not considered a good football player, i.e., $\neg\mathsf{GoodPlayer}_\mathbf{f}(\mathsf{Rooney})$. Suppose also that the world news context $C_\mathbf{wn}$ contains the opposite evaluation, i.e., $\mathsf{GoodPlayer}_\mathbf{f}(\mathsf{Rooney})$. In the CKR of Fig. 2, these two contradicting statements do not necessarily lead to inconsistency. Indeed, to derive inconsistency one has to find a context where to combine the two contradicting facts. However, to transfer the facts $\mathbf{wn} : \mathsf{GoodPlayer}_\mathbf{f}(\mathsf{Rooney})$ and $\mathbf{in} : \neg\mathsf{GoodPlayer}_\mathbf{f}(\mathsf{Rooney})$ into a common context, one has to pass through $C_\mathbf{i}$. But the fact that* Rooney *is not an individual of $C_\mathbf{i}$ disables any inference about* Rooney *in $C_\mathbf{i}$. Model-theoretically we admit CKR models where $\mathsf{Rooney}^{\mathcal{I}_\mathbf{wn}} \neq \mathsf{Rooney}^{\mathcal{I}_\mathbf{in}}$.*

4.3. Properties of Reasoning

With help of the ND calculus we will be able to formulate and prove some interesting properties of reasoning with CKR knowledge bases. We will first examine the propagation of

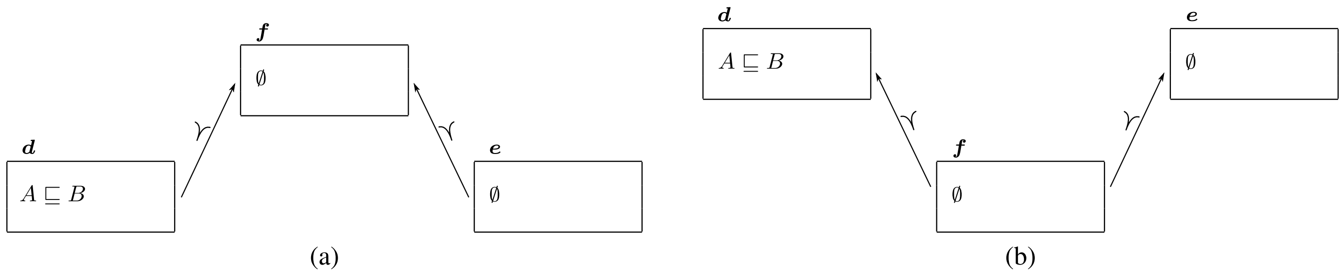

Figure 3: Basic knowledge propagation: (a) common super-context scheme; (b) common sub-context scheme

knowledge in CKR that is enabled by the qualified symbols. We will show that it occurs only in contexts that are connected by the coverage hierarchy. We will start by considering the basic cases of connected contexts.

Let us generalize the situation discussed in Example 1. Consider the CKR $\mathfrak{K}$ represented in Fig. 3 (a), composed of three contexts $C_{\mathbf{d}}$, $C_{\mathbf{e}}$ and $C_{\mathbf{f}}$ such that $\mathbf{d} \preceq \mathbf{f}$ and $\mathbf{e} \preceq \mathbf{f}$. We will call $C_{\mathbf{f}}$ a *common super-context* of $C_{\mathbf{d}}$ and $C_{\mathbf{e}}$. It shows that the mere existence of a common super-context enables communication from $C_{\mathbf{d}}$ to $C_{\mathbf{e}}$.

Property 1 (Communication via a common super-context)**.** *In every CKR $\mathfrak{K}$ with two contexts $C_{\mathbf{d}}$, $C_{\mathbf{e}}$ that share a common super-context $C_{\mathbf{f}}$, we have:*

$$\mathfrak{K} \vdash \mathbf{d} : A \sqsubseteq B \implies \mathfrak{K} \vdash \mathbf{e} : A_{\mathbf{d}} \sqsubseteq B_{\mathbf{d}}$$

Proof. Proof in the ND calculus:

(1)	$\mathbf{d} : A \sqsubseteq B$	Premise
(2)	$\mathbf{f} : A_{\mathbf{d}} \sqsubseteq B_{\mathbf{d}} \quad (= (A \sqsubseteq B)@\mathbf{d})$	Pop, $\mathbf{d} \preceq \mathbf{f}$
(3)	$\mathbf{f} : A_{\mathbf{d}} \sqcap \top_{\mathbf{e}} \sqsubseteq B_{\mathbf{d}} \sqcap \top_{\mathbf{e}} \quad (= (A_{\mathbf{d}} \sqsubseteq B_{\mathbf{d}})@\mathbf{e})$	LReas
(4)	$\mathbf{e} : A_{\mathbf{d}} \sqsubseteq B_{\mathbf{d}}$	Push, $\mathbf{e} \preceq \mathbf{f}$

□

Notice that in order to enable the communication between $C_{\mathbf{d}}$ and $C_{\mathbf{e}}$, the mere presence of a common super-context is required. This will work even if, as in Fig. 3 (a), the super-context is empty. One may ask whether the common super-context $C_{\mathbf{f}}$ is a "symmetric channel" which induces also the inverse communication, i.e., does the subsumption $A_{\mathbf{d}} \sqsubseteq B_{\mathbf{d}}$ if entailed in $C_{\mathbf{e}}$ propagate back to $C_{\mathbf{d}}$ in the form $A \sqsubseteq B$. This is not the case, it is proven by the counter model $\mathfrak{I}$, in which: $\Delta_{\mathbf{d}} \cap \Delta_{\mathbf{e}} = \emptyset$, $\Delta_{\mathbf{f}} = \Delta_{\mathbf{d}} \cup \Delta_{\mathbf{e}}$, $A^{\mathcal{I}_{\mathbf{d}}} = \Delta_{\mathbf{d}}$ and $B^{\mathcal{I}_{\mathbf{d}}} = \emptyset$. By the conditions of Definition 9 we have that $A_{\mathbf{d}}^{\mathcal{I}_{\mathbf{e}}} = B_{\mathbf{d}}^{\mathcal{I}_{\mathbf{e}}} = \emptyset$. Hence we have found a model that satisfies $\mathbf{e} : A_{\mathbf{d}} \sqsubseteq B_{\mathbf{d}}$ but not $\mathbf{d} : A \sqsubseteq B$.

If in a CKR $\mathfrak{K}$, $C_{\mathbf{f}} \preceq C_{\mathbf{d}}$ and $C_{\mathbf{f}} \preceq C_{\mathbf{e}}$ then we call $C_{\mathbf{f}}$ a common sub-context of $C_{\mathbf{d}}$ and $C_{\mathbf{e}}$. This situation is dual to the previous case. We have already outlined this situation in Example 2, and in general it is depicted in Fig. 3 (b). We will see that to some extent communication between two contexts is also induced by a common sub-context. In this case, however, the domain of the common sub-context comes into play and effectively restricts the amount of information that can be communicated.

Property 2 (Communication via a common sub-context)**.** *In every CKR $\mathfrak{K}$ with two contexts $C_{\mathbf{d}}$, $C_{\mathbf{e}}$ that share a common sub-context $C_{\mathbf{f}}$, we have:*

$$\mathfrak{K} \vdash \mathbf{d} : A \sqsubseteq B \implies \mathfrak{K} \vdash \mathbf{e} : A_{\mathbf{d}} \sqcap \top_{\mathbf{f}} \sqsubseteq B_{\mathbf{d}} \sqcap \top_{\mathbf{f}}$$

Proof. Again we prove the claim by the ND calculus:

(1)	$\mathbf{d} : A \sqsubseteq B$	Premise
(2)	$\mathbf{d} : A_{\mathbf{d}} \sqsubseteq B_{\mathbf{d}} \quad (= (C \sqsubseteq D)@\mathbf{d})$	Pop, $\mathbf{d} \preceq \mathbf{d}$
(3)	$\mathbf{d} : A_{\mathbf{d}} \sqcap \top_{\mathbf{f}} \sqsubseteq B_{\mathbf{d}} \sqcap \top_{\mathbf{f}} \quad (= (A_{\mathbf{d}} \sqsubseteq B_{\mathbf{d}})@\mathbf{f})$	LReas
(4)	$\mathbf{f} : A_{\mathbf{d}} \sqsubseteq B_{\mathbf{d}}$	Push, $\mathbf{f} \preceq \mathbf{d}$
(5)	$\mathbf{e} : A_{\mathbf{d}} \sqcap \top_{\mathbf{f}} \sqsubseteq B_{\mathbf{d}} \sqcap \top_{\mathbf{f}} \quad (= (A_{\mathbf{d}} \sqsubseteq B_{\mathbf{d}})@\mathbf{f})$	Pop, $\mathbf{f} \preceq \mathbf{e}$

□

Therefore we see that the amount of communication is effectively constrained by the domain of the common sub-context $C_{\mathbf{f}}$. We now show that the full amount of communication, as much as in the case of common super-context, is not possible, i.e., the fact that $A \sqsubseteq B$ is entailed in $C_{\mathbf{d}}$ does not imply $A_{\mathbf{d}} \sqsubseteq B_{\mathbf{d}}$ in $C_{\mathbf{e}}$. Of course this is under the assumption that there is no common super-context shared by $C_{\mathbf{d}}$ and $C_{\mathbf{e}}$. The claim is proved by a counterexample given by the following interpretation $\mathfrak{I}$ of the CKR $\mathfrak{K}$:

- $\Delta_{\mathbf{d}} = \{x, y\}$,
  $\mathcal{I}_{\mathbf{d}}(A) = \mathcal{I}_{\mathbf{d}}(A_{\mathbf{d}}) = \{x\}$, $\mathcal{I}_{\mathbf{d}}(B) = \mathcal{I}_{\mathbf{d}}(B_{\mathbf{d}}) = \{x, y\}$,
  $\mathcal{I}_{\mathbf{d}}(\top_{\mathbf{f}}) = \{x\}$, $\mathcal{I}_{\mathbf{d}}(\top_{\mathbf{e}}) = \{x, y\}$;
- $\Delta_{\mathbf{f}} = \{x\}$,
  $\mathcal{I}_{\mathbf{f}}(A_{\mathbf{d}}) = \mathcal{I}_{\mathbf{f}}(B_{\mathbf{d}}) = \{x\}$,
  $\mathcal{I}_{\mathbf{f}}(\top_{\mathbf{d}}) = \mathcal{I}_{\mathbf{f}}(\top_{\mathbf{e}}) = \{x\}$;
- $\Delta_{\mathbf{e}} = \{x, z\}$,
  $\mathcal{I}_{\mathbf{e}}(A_{\mathbf{d}}) = \{x, z\}$, $\mathcal{I}_{\mathbf{e}}(B_{\mathbf{d}}) = \{x\}$,
  $\mathcal{I}_{\mathbf{e}}(\top_{\mathbf{f}}) = \{x\}$, $\mathcal{I}_{\mathbf{e}}(\top_{\mathbf{d}}) = \{x, z\}$.

Clearly, $\mathfrak{I}$ satisfies Definition 9 and hence it is a model of $\mathfrak{K}$. But on the other hand, $A_{\mathbf{d}}^{\mathcal{I}_{\mathbf{e}}} \not\subseteq B_{\mathbf{d}}^{\mathcal{I}_{\mathbf{e}}}$, that is to say, $\mathfrak{K} \not\models \mathbf{e} : A_{\mathbf{d}} \sqsubseteq B_{\mathbf{d}}$.

Analogously to the previous case, $C_{\mathbf{f}}$ can be seen as a communication channel, in this case however $C_{\mathbf{f}}$ constitutes the "intersection" of $C_{\mathbf{d}}$ and $C_{\mathbf{e}}$ and allows to pass only knowledge within its domain, which is contained in both domains of $C_{\mathbf{d}}$ and $C_{\mathbf{e}}$.

We now proceed by showing how the two propagation patterns described above, can be composed to define a general propagation pattern between any pair of contexts in a CKR knowledge base that are connected. Two context with dimensions $\mathbf{d}$ and $\mathbf{e}$ are connected in $\mathfrak{K}$, if there is a sequence $\mathbf{d} = \mathbf{d}_1, \ldots, \mathbf{d}_n = \mathbf{e}$, with $n \geq 1$, and for all $1 \leq i < n$ either $\mathbf{d}_i \prec \mathbf{d}_{i+1}$ or $\mathbf{d}_i \succ \mathbf{d}_{i+1}$. The sequence $\mathbf{d}_1, \ldots, \mathbf{d}_n$ is called a path connecting $C_\mathbf{d}$ and $C_\mathbf{e}$. For any $1 \leq i \leq n$, $\mathbf{d}_i$ is a minimum of $\mathbf{d}_1, \ldots, \mathbf{d}_n$ if one of the following conditions holds:

1. $i = 1$, and $\mathbf{d}_1 \prec \mathbf{d}_2$
2. $i = n$, and $\mathbf{d}_{n-1} \succ \mathbf{d}_n$
3. $1 < i < n$ and $\mathbf{d}_{i-1} \succ \mathbf{d}_i \prec \mathbf{d}_{i+1}$.

If for two contexts $C_\mathbf{d}$ and $C_\mathbf{e}$ of $\mathfrak{K}$ there exists no path that connects them, they are called isolated contexts.

The following property shows that if two contexts $C_\mathbf{d}$ and $C_\mathbf{e}$ are not directly covered one by another but instead connected by a path of multiple contexts, then still subsumptions entailed in $C_\mathbf{d}$ at least partially propagate into $C_\mathbf{e}$ (and vice versa). The amount of information that is propagated is effectively constrained by the domains of those contexts which are minima on the path that connects $C_\mathbf{d}$ and $C_\mathbf{e}$.

Property 3. *Given a CKR $\mathfrak{K}$ with $\mathbf{d}_{i_1}, \ldots, \mathbf{d}_{i_k}$ being all the minima of the path $\mathbf{d}_1, \ldots, \mathbf{d}_n$ connecting $C_\mathbf{d}$ with $C_\mathbf{e}$. Then*

$$\mathbf{d} : A \sqsubseteq B \vdash \mathbf{e} : A_\mathbf{d} \sqcap \bigsqcap_{1 \leq j \leq k} \top_{\mathbf{d}_{i_j}} \sqsubseteq B_\mathbf{d} \sqcap \bigsqcap_{1 \leq j \leq k} \top_{\mathbf{d}_{i_j}}$$

Proof. (Sketch.) The proof can be obtained by an iterative application of the two Properties 1 and 2. □

For instance, consider our example CKR depicted in Fig. 2. If in the context of Italian news $C_\mathbf{in}$ it is asserted that all players of Inter Milan are considered good players, e.g., by the axiom $\exists \mathsf{playsFor}_\mathbf{f}.\{\mathsf{Inter\_Milan}\} \sqsubseteq \mathsf{GoodPlayer}$, this subsumption propagates into $C_\mathbf{wn}$ as $(\exists \mathsf{playsFor}_\mathbf{f}.\{\mathsf{Inter\_Milan}\}) \sqcap \top_\mathbf{i} \sqcap \top_\mathbf{wc} \sqsubseteq \mathsf{GoodPlayer}_\mathbf{in} \sqcap \top_\mathbf{i} \sqcap \top_\mathbf{wc}$, as the contexts $C_\mathbf{i}$ (Italian league) and $C_\mathbf{wc}$ (FIFA WC) are the two minima on the path connecting $C_\mathbf{wn}$ to $C_\in$. As a consequence this subsumption will certainly apply in $C_\mathbf{wn}$ on all players that participate to both FIFA WC and the Italian league.

On the other hand, if two contexts are not connected by any path, they are totally independent. This is a simple consequence of the following property.

Property 4. *Let $\mathfrak{K} = \langle \mathfrak{M}, \mathfrak{C} \rangle$ be a CKR over $\langle \Gamma, \Sigma \rangle$ that is $\mathbf{d}$-satisfiable and for the context $C_\mathbf{d} \in \mathfrak{C}$ there is no other $C_\mathbf{e} \in \mathfrak{C}$ with $\mathbf{d} \prec \mathbf{e}$ or $\mathbf{e} \prec \mathbf{d}$. Let us construct $\mathfrak{K}' = \langle \mathfrak{M}, \mathfrak{C} \setminus \{C_\mathbf{d}\} \rangle$. Then for any $\mathbf{f} \in \mathfrak{D}_\Gamma$, $\mathbf{f} \neq \mathbf{d}$, and for any DL formula ϕ over Σ we have:*

$$\mathfrak{K} \models \mathbf{f} : \phi \iff \mathfrak{K}' \models \mathbf{f} : \phi$$

Proof. (Sketch.) The property is proven by establishing a one-to-one correspondence between the models of $\mathfrak{K}$ and $\mathfrak{K}'$ (disregarding $C_\mathbf{d}$):

- given a model $\mathfrak{I}$ of $\mathfrak{K}$, a model $\mathfrak{I}'$ of $\mathfrak{K}'$ is constructed simply by taking $\mathfrak{I}$ and deleting $\mathcal{I}_\mathbf{d}$ from it;
- given a model $\mathfrak{I}'$ of $\mathfrak{K}'$, a model $\mathfrak{I}$ of $\mathfrak{K}$ is constructed by taking $\mathfrak{I}'$ and extending it with $\mathcal{I}_\mathbf{d} = \langle \emptyset, \emptyset \rangle$ (i.e., the interpretation with empty domain), and by setting $X_\mathbf{d}^{\mathcal{I}_\mathbf{e}} = \emptyset$ for every concept/role X and for every $\mathbf{e} \in \mathfrak{D}_\Gamma$.

Therefore if $\mathcal{I}_\mathbf{f} \models \phi$ in every model of $\mathfrak{K}$, also $\mathcal{I}'_\mathbf{f} \models \phi$ in every model of $\mathfrak{K}'$ and vice versa. □

We can find another justification of Property 4 by inspecting the calculus rules in Table 3. Apart from the Bot rule, no other rule enables transfer of consequence between context which are not directly related in terms of coverage. The Bot rule is only applicable if one context is inconsistent, therefore in fact no transfer of knowledge may possibly occur between two unconnected contexts that are consistent.

As an important consequence of Condition 4 of Definition 9 we will show that equality among individuals propagates across contexts if at least one of the individuals involved is defined in a common sub-context.

Property 5 (Propagation of equality)**.** *Given a CKR $\mathfrak{K}$ over $\langle \Gamma, \Sigma \rangle$ with two contexts $C_\mathbf{d}$, $C_\mathbf{e}$ sharing a common sub-context $C_\mathbf{f}$, and given two individuals $a, b \in \Sigma$, the following holds:*

$$\mathfrak{K} \vdash \mathbf{d} : a = b \;\wedge\; \mathfrak{K} \vdash \mathbf{f} : \top(a) \implies \mathfrak{K} \vdash \mathbf{e} : a = b$$

Proof. The proof again in the ND calculus

(1)	$\mathbf{d} : a = b$	Premise
(2)	$\mathbf{f} : \top(a)$	Premise
(3)	$\mathbf{d} : \top_\mathbf{f}(a)$	Pop, $\mathbf{f} \preceq \mathbf{d}$
(4)	$\mathbf{d} : \top(a)$	LReas
(5)	$\mathbf{d} : \top_\mathbf{d}(a)$	Pop, $\mathbf{d} \preceq \mathbf{d}$
(6)	$\mathbf{d} : \top_\mathbf{d}(b)$	From (1) and (5) by LReas
(7)	$\mathbf{f} : a = b$	From (1), (5), (6) and $\mathbf{f} \preceq \mathbf{d}$ by Push
(8)	$\mathbf{e} : a = b$	Pop, $\mathbf{f} \preceq \mathbf{e}$

□

One of our desiderata in contextualized knowledge representation is certain inconsistency tolerance by the system. Inconsistency should not necessarily pollute whole CKR if it occurs in one of the contexts. Let us analyze the propagation of inconsistency in CKR. As a direct consequence of Property 4, we have that inconsistency does not propagate to contexts which are not connected with the inconsistent part of the system.

Property 6. *Let $C_\mathbf{d}$ and $C_\mathbf{e}$ be any two isolated context of a CKR $\mathfrak{K}$ such that $\mathfrak{K}$ is $\mathbf{e}$-consistent, and for no context $C_\mathbf{f}$ and for no individual $a \in \Sigma$ we have $\mathfrak{K} \vdash \mathbf{f} : \bot(a)$. Then:*

$$\mathfrak{K}, \mathbf{d} : \top \sqsubseteq \bot \nvdash \mathbf{e} : \top \sqsubseteq \bot$$

Also, inconsistency does not necessarily propagate from a narrower context into a broader one, at least not until individuals come into play.

Property 7. *Let $\mathfrak{K}$ be CKR such that for no context $C_{\mathbf{f}}$ we have $\mathfrak{K} \vdash \mathbf{f} : \bot(a)$ for any individual $a \in \Sigma$. Then for any two contexts $C_{\mathbf{d}}$ and $C_{\mathbf{e}}$ of $\mathfrak{K}$ such that $\mathbf{d} \prec \mathbf{e}$ we have:*

$$\mathfrak{K}, \mathbf{d} : \top \sqsubseteq \bot \nvdash \mathbf{e} : \top \sqsubseteq \bot$$

Proof. Consider a CKR $\mathfrak{K}$ with two contexts $C_{\mathbf{d}}, C_{\mathbf{e}}, \mathbf{d} \prec \mathbf{e}$, with the only axiom $\top \sqsubseteq \bot$ in $C_{\mathbf{d}}$, and with $C_{\mathbf{e}}$ empty. The model $\mathfrak{I}$ in which $\mathcal{I}_{\mathbf{d}} = \langle \emptyset, \emptyset \rangle$ and $\mathcal{I}_{\mathbf{e}} = \langle \{x\}, \{\top \mapsto \{x\}, \bot \mapsto \emptyset, \top_{\mathbf{d}} \mapsto \emptyset\} \rangle$ is indeed a model of $\mathfrak{K}$ in which $\mathbf{e} : \top \sqsubseteq \bot$ is not entailed. The property is now a direct consequence of the soundness of the calculus. □

Thus we see that allowing local models with empty domains is an important preposition in order to minimize inconsistency propagation. On the other hand, if the inconsistency appears in a broader context $C_{\mathbf{e}}$, it is pushed also into all contexts covered by $C_{\mathbf{e}}$. This is due to the fact that the empty domain of $\mathcal{I}_{\mathbf{e}}$ makes all domains under $C_{\mathbf{e}}$ necessarily empty. We can also directly prove this by the CKR calculus.

Property 8. *Given a CKR $\mathfrak{K}$ with two contexts $C_{\mathbf{d}}$ and $C_{\mathbf{e}}$ such that $\mathbf{d} \prec \mathbf{e}$, we have:*

$$\mathfrak{K}, \mathbf{e} : \top \sqsubseteq \bot \vdash \mathbf{d} : \top \sqsubseteq \bot$$

Proof. First, we have $\mathfrak{K}, \mathbf{e} : \top \sqsubseteq \bot \vdash \mathbf{e} : \top_{\mathbf{d}} \sqsubseteq \bot$ by LReas, and consecutively $\mathfrak{K}, \mathbf{e} : \top \sqsubseteq \bot \vdash \mathbf{d} : \top \sqsubseteq \bot$ by Push. □

The inconsistency tolerance of a CKR knowledge base reaches its limit as soon as individuals appear in one of the inconsistent contexts. In such a case whole CKR becomes inconsistent.

Property 9. *Let $C_{\mathbf{d}}$ and $C_{\mathbf{e}}$ be any contexts in a CKR $\mathfrak{K}$, we have that:*

$$\mathfrak{K}, \mathbf{d} : \top(a), \mathbf{d} : \top \sqsubseteq \bot \vdash \mathbf{e} : \top \sqsubseteq \bot$$

Proof. By LReas we obtain $\mathfrak{K}, \mathbf{d} : \top(a), \mathbf{d} : \top \sqsubseteq \bot \vdash \mathbf{d} : \bot(a)$. Consequently we have $\mathfrak{K}, \mathbf{d} : \top(a), \mathbf{d} : \top \sqsubseteq \bot \vdash \mathbf{e} : \top \sqsubseteq \bot$ by the Bot rule. □

Model-theoretically speaking, if an individual occurs in an inconsistent context, not even the local interpretation with empty domain qualifies for a local model. Hence there is no model for the CKR, because it requires a local model for every context. In the calculus this situation is handled by the Bot rule, which is the only rule applicable if there is an inconsistent context with an individual present, and also it is the only rule which propagates its conclusion arbitrarily, even to isolated contexts.

Finally, an inconsistent context with empty domain blocks the communication. If $C_{\mathbf{d}}$ and $C_{\mathbf{e}}$ are connected through a path that contains an inconsistent context $C_{\mathbf{f}}$, then such a path does not contribute to the transfer of knowledge between $C_{\mathbf{d}}$ and $C_{\mathbf{e}}$.

Property 10. *Given a CKR $\mathfrak{K}$ with two contexts $C_{\mathbf{d}}$ and $C_{\mathbf{e}}$ such that for each path $\mathbf{d}_1, \ldots, \mathbf{d}_n$ connecting $C_{\mathbf{d}}$ and $C_{\mathbf{e}}$ there is $1 < k < n$ such that $\mathfrak{K} \vdash \mathbf{d}_k : \top \sqsubseteq \bot$, then for any two formulae ϕ, ψ over Σ such that $\mathfrak{K} \cup \{\mathbf{d} : \phi\}$ is $\mathbf{d}$-satisfiable, we have:*

$$\mathfrak{K}, \mathbf{d} : \phi \vdash \mathbf{e} : \psi \iff \mathfrak{K} \vdash \mathbf{e} : \psi$$

Proof. (Sketch.) Let Π be a proof of $\mathbf{e} : \phi$ from $\mathfrak{K} \cup \{\mathbf{d} : \phi\}$. Let $\pi = \mathbf{d}_1 : \phi, \ldots, \mathbf{d}_n : \psi$, where $\mathbf{d}_1 = \mathbf{d}$, $\mathbf{d}_n = \mathbf{e}$ be the path in Π that starts in the assumption node $\mathbf{d} : \phi$, it follows the conclusions consecutively derived from $\mathbf{d} : \phi$, and finally reaches $\mathbf{e} : \psi$.

Observe first that the Bot rule was not applied anywhere on the path π, because this would allow us to derive inconsistency in $\mathbf{d}$ but we assumed that the contrary, i.e., that $\mathfrak{K} \cup \{\mathbf{d} : \phi\}$ is $\mathbf{d}$-satisfiable.

From the fact that the path π was constructed by rule applications, and all calculus rules apart from Bot only allow to derive a consequence in a context directly related by the coverage, it follows that $\mathbf{d}_1, \ldots, \mathbf{d}_n$ is a path in the CKR $\mathfrak{K}$. From the assumptions, at least for one $1 < k < n$ such that $\mathfrak{K} \vdash \mathbf{d}_k : \top \sqsubseteq \bot$. Therefore the subtree of Π rooted in $\mathbf{d}_1 : \phi$ can be replaced by $\mathbf{d}_k : \phi_k$ since $\mathfrak{K} \vdash \mathbf{d}_k : \phi_k$ by LReas. □

5. Decidability and Complexity

Decidability of CKR entailment is proved indirectly by embedding CKR into a single DL knowledge base. To do this we reuse the notion of embedding between DL knowledge bases as previously defined in Sect. 4.1. First we need a vocabulary that is robust enough to keep track of all semantic relations inside a CKR knowledge base. Since each qualified symbol $X_{\mathbf{d}}$ may have different meanings in different contexts, we need to introduce one version $X^{\mathbf{e}}_{\mathbf{d}}$ of the symbol per each context $C_{\mathbf{e}}$. We know from the semantics that non-qualified concept and role symbols have the same meaning as if qualified with respect to the context where they appear. In addition, also constants may possibly have different meaning in different contexts, therefore for each constant a we introduce a version $a^{\mathbf{e}}$ for each context $C_{\mathbf{e}}$.

Definition 19 (Transformed vocabulary $\#(\Gamma, \Sigma)$)**.** *Given a pair of meta/object vocabularies $\langle \Gamma, \Sigma \rangle$, let $\Sigma^{\mathrm{B}} = N^{\mathrm{B}}_{\mathrm{C}} \uplus N^{\mathrm{B}}_{\mathrm{R}} \uplus N^{\mathrm{B}}_{\mathrm{I}}$ be the base-vocabulary of Σ. Let us define a DL-vocabulary $\#(\Gamma, \Sigma) = \#N_{\mathrm{C}} \uplus \#N_{\mathrm{R}} \uplus \#N_{\mathrm{I}}$ such that:*

1. $\#N_{\mathrm{C}} = \{A^{\mathbf{e}}_{\mathbf{d}} \mid A \in N^{\mathrm{B}}_{\mathrm{C}} \wedge \mathbf{d}, \mathbf{e} \in \mathfrak{D}_\Gamma\}$*;*
2. $\#N_{\mathrm{R}} = \{R^{\mathbf{e}}_{\mathbf{d}} \mid R \in N^{\mathrm{B}}_{\mathrm{R}} \wedge \mathbf{d}, \mathbf{e} \in \mathfrak{D}_\Gamma\} \cup \{S^{\mathbf{d},\mathbf{e},\mathbf{f}}_R \mid R \in N^{\mathrm{B}}_{\mathrm{R}} \wedge \mathbf{d}, \mathbf{e}, \mathbf{f} \in \mathfrak{D}_\Gamma\}$*, where S is some new symbol not appearing in Σ;*
3. $\#N_{\mathrm{I}} = \{a^{\mathbf{e}} \mid a \in N^{\mathrm{B}}_{\mathrm{I}} \wedge \mathbf{e} \in \mathfrak{D}_\Gamma\} \cup \{\mathsf{undef}\}$*, where* undef *is a new symbol not appearing in Σ.*

The role symbols of the form $S^{\mathbf{d},\mathbf{e},\mathbf{f}}_R$ which we also added to $\#N_{\mathrm{R}}$ are auxiliary and will be later used to maintain decidability of the transformed knowledge base. In addition we have introduced a new constant undef. This constant is needed because some of the constants in CKR need not to be necessarily defined in all contexts. This will be simulated by allowing some of the constants in $\#N_{\mathrm{I}}$ to be equal to undef.

For each full dimensional vector $\mathbf{d} \in \mathfrak{D}_\Gamma$, we now define an operator $(\cdot)\#\mathbf{d}$ which will be based on an embedding $g_{\mathbf{d}}$ of Σ into $\#(\Gamma, \Sigma)$.

Definition 20 ($\#\mathbf{d}$ operator)**.** *Given a pair of meta/object vocabularies $\langle \Gamma, \Sigma \rangle$, for every full dimensional vector $\mathbf{d} \in \mathfrak{D}_\Gamma$, $(\cdot)\#\mathbf{d}$*

is defined as $g^*_{\mathbf{d}}(\cdot)$, *where* $g_{\mathbf{d}}$ *is an embedding from* Σ *to* $\#(\Gamma,\Sigma)$ *defined as follows:*

- $g_{\mathbf{d}}(a) = a^{\mathbf{d}}$ *for every individual* a;
- $g_{\mathbf{d}}(X_{\mathbf{f_B}}) = X^{\mathbf{d}}_{\mathbf{f_B+d}}$ *for every concept/role* $X_{\mathbf{f_B}}$;
- $\Sigma_c = \Sigma$; $\Sigma_e = \emptyset$.

Observe that in this case the split of Σ into the internal part Σ_c and the external part Σ_e is different: $\Sigma_c = \Sigma$ and $\Sigma_e = \emptyset$. This is in line with the fact that the single DL knowledge base which is the result of the transformation has complete information about all symbols in every context. That is, in terms of CKR we could see the transformed knowledge base as if placed on top of all contexts with respect to the coverage $\prec$. Using the $(\cdot)\#\mathbf{d}$ operator we now transform a CKR knowledge base $\mathfrak{K}$ into a DL theory $\#(\mathfrak{K})$ over $\#(\Gamma,\Sigma)$.

Definition 21 (Transformed CKR $\#(\mathfrak{K})$)**.** *For every CKR* $\mathfrak{K}$ *over* $\langle\Gamma,\Sigma\rangle$, *let* $\#(\mathfrak{K})$ *be a DL knowledge base over* $\#(\Gamma,\Sigma)$ *such that for every individual a, concept A, role R, concept/role X (all atomic), and for any full dimensional vectors* $\mathbf{d},\mathbf{e},\mathbf{f}$ *it contains the following axioms:*

1. $\top^{\mathbf{f}}_{\mathbf{d}} \sqsubseteq \top^{\mathbf{f}}_{\mathbf{e}}$ *for* $\mathbf{d} \prec \mathbf{e}$;
2. $A^{\mathbf{d}}_{\mathbf{e}} \sqsubseteq \top^{\mathbf{d}}_{\mathbf{e}}$;
3. $\exists R^{\mathbf{d}}_{\mathbf{e}}.\top \sqsubseteq \top^{\mathbf{d}}_{\mathbf{e}}$ *and* $\top \sqsubseteq \forall R^{\mathbf{d}}_{\mathbf{e}}.\top^{\mathbf{d}}_{\mathbf{e}}$;
4. *add the following three axioms*[5] *(if the indicated condition is true):*
   a) $\top^{\mathbf{d}}_{\mathbf{d}} \sqcap \{a^{\mathbf{e}}\} \sqsubseteq \{a^{\mathbf{d}}\}$ *if* $\mathbf{d} \prec \mathbf{e}$;
   b) $\{a^{\mathbf{d}}\} \sqsubseteq \{a^{\mathbf{e}}, \mathsf{undef}\}$ *if* $\mathbf{d} \prec \mathbf{e}$;
   c) $\neg\top^{\mathbf{d}}_{\mathbf{d}}(\mathsf{undef})$;
6. $X^{\mathbf{d}}_{\mathbf{d}} \equiv X^{\mathbf{e}}_{\mathbf{d}}$ *if* $\mathbf{d} \prec \mathbf{e}$;
7. $A^{\mathbf{d}}_{\mathbf{f}} \equiv A^{\mathbf{e}}_{\mathbf{f}} \sqcap \top^{\mathbf{d}}_{\mathbf{d}}$ *if* $\mathbf{d} \prec \mathbf{e}$;
8. *if* $\mathbf{d} \prec \mathbf{e}$, *add the following 4 axioms:*
   a) $I^{\mathbf{d}}_{\mathbf{d}} \circ R^{\mathbf{e}}_{\mathbf{f}} \circ I^{\mathbf{d}}_{\mathbf{d}} \sqsubseteq R^{\mathbf{d}}_{\mathbf{f}}$ *and* $R^{\mathbf{d}}_{\mathbf{f}} \sqsubseteq R^{\mathbf{e}}_{\mathbf{f}}$,
   b) $I^{\mathbf{d}}_{\mathbf{d}} \circ R^{\mathbf{e}}_{\mathbf{f}} \sqsubseteq S^{\mathbf{d,e,f}}_{R}$ *and* $S^{\mathbf{d,e,f}}_{R} \circ I^{\mathbf{d}}_{\mathbf{d}} \sqsubseteq R^{\mathbf{d}}_{\mathbf{f}}$;
9. $\phi\#\mathbf{d}$ *for all* $\phi \in \mathrm{K}(C)$ *and* $\mathbf{d} = \dim(C)$.

The axioms added to $\#(\mathfrak{K})$ in the previous definition correspond step-by-step to the conditions of Definition 9; in each step we add axioms to deal with the respective condition. Step 5 is missing, as we do not deal with incomplete symbols directly; as we previously explained incomplete symbols are a kind of syntactic sugar, and therefore all symbols $X_{\mathbf{d_B}}$ occurring in $C_{\mathbf{e}}$ are represented by $X^{\mathbf{e}}_{\mathbf{d_B+e}}$ in this construction as they have the same meaning. In Step 4, the first two axioms (a) and (b) directly correspond to Condition 4 of Definition 9, axiom (c) is needed to assure that the newly added constant undef will not be identified with any individual of the original CKR. In Step 8, the pair of axioms (a) actually serves to maintain Condition 8 of Definition 9; the second pair (b) added in this step has no influence on the semantics of $\#(\mathfrak{K})$, but it serves to maintain $\#(\mathfrak{K})$ decidable, as further discussed below.

Thanks to the transformation we are now able to check $\mathbf{d}$-satisfiability of a CKR knowledge base $\mathfrak{K}$ by checking for satisfiability/entailment in $\#(\mathfrak{K})$. This is formally established by the following two lemmata.

[5] Please note that in our previous report [25] we mistakenly introduced a simpler version of this step that, most notably, did not involve nominals. This simpler construction is not correct.

Lemma 3. *If* $\mathfrak{K}$ *is* $\mathbf{d}$*-satisfiable then* $\#(\mathfrak{K})$ *is satisfiable.*

Proof. (Sketch.) As $\mathfrak{K}$ is $\mathbf{d}$-satisfiable, there exists a model $\mathfrak{I}$ of $\mathfrak{K}$ with $\Delta_{\mathbf{d}} \neq \emptyset$. Let us construct a DL interpretation $\mathcal{I} = \left\langle \Delta, \cdot^{\mathcal{I}} \right\rangle$ over $\#(\Gamma,\Sigma)$ as follows:

1. $\Delta = \bigcup_{\mathbf{d}\in\mathfrak{D}_\Gamma} \Delta_{\mathbf{d}} \cup \{x_{\mathsf{undef}}\}$ where x_{undef} is a new element not occurring in $\Delta_{\mathbf{d}}$ for all $\mathbf{d} \in \mathfrak{D}_\Gamma$;
2. $(a^{\mathbf{d}})^{\mathcal{I}} = a^{\mathcal{I}_{\mathbf{d}}}$ if $a^{\mathcal{I}_{\mathbf{d}}}$ is defined otherwise $(a^{\mathbf{d}})^{\mathcal{I}} = x_{\mathsf{undef}}$ for every individual a and for every $\mathbf{d} \in \mathfrak{D}_\Gamma$;
   $\mathsf{undef}^{\mathcal{I}} = x_{\mathsf{undef}}$;
3. $(A^{\mathbf{d}}_{\mathbf{e}})^{\mathcal{I}} = C^{\mathcal{I}_{\mathbf{d}}}_{\mathbf{e}}$ for every atomic concept C of Σ and for every $\mathbf{d},\mathbf{e} \in \mathfrak{D}_\Gamma$;
4. $(R^{\mathbf{d}}_{\mathbf{e}})^{\mathcal{I}} = R^{\mathcal{I}_{\mathbf{d}}}_{\mathbf{e}}$ for every atomic role R of Σ and for every $\mathbf{d},\mathbf{e} \in \mathfrak{D}_\Gamma$;
   $(S^{\mathbf{d,e,f}}_{R})^{\mathcal{I}} = (I^{\mathbf{d}}_{\mathbf{d}})^{\mathcal{I}} \circ (R^{\mathbf{e}}_{\mathbf{f}})^{\mathcal{I}}$ for every role R and for all $\mathbf{d},\mathbf{e},\mathbf{f} \in \mathfrak{D}_\Gamma$;

Clearly $\Delta \neq \emptyset$. It remains to prove that all the axioms of $\#(\mathfrak{K})$ are satisfied by $\mathcal{I}$. Depending on the type of axiom, this largely a consequence of Definition 9 or Lemma 1. Full proof is listed in the appendix. ☐

Lemma 4. *If there is* $\mathbf{d}$ *such that* $\#(\mathfrak{K}) \not\models \top^{\mathbf{d}}_{\mathbf{d}} \sqsubseteq \bot$, *then* $\mathfrak{K}$ *is* $\mathbf{d}$*-satisfiable.*

Proof. (Sketch.) Given a CKR $\mathfrak{K}$ let $\mathcal{I}$ be a model of $\#(\mathfrak{K})$ such that $(\top^{\mathbf{d}}_{\mathbf{d}})^{\mathcal{I}}$ is not empty. This model exists since by hypothesis $\#(\mathfrak{K}) \not\models \top^{\mathbf{d}}_{\mathbf{d}} \sqsubseteq \bot$. Let us construct the CKR model $\mathfrak{I} = \{\mathcal{I}_{\mathbf{d}}\}_{\mathbf{d}\in\mathfrak{D}_\Gamma}$, where for every $\mathbf{d} \in \mathfrak{D}_\Gamma$, $\mathcal{I}_{\mathbf{d}} = \langle \Delta_{\mathbf{d}}, \mathcal{I}_{\mathbf{d}} \rangle$ is defined as follows:

1. $\Delta_{\mathbf{d}} = (\top^{\mathbf{d}}_{\mathbf{d}})^{\mathcal{I}}$;
2. $a^{\mathcal{I}_{\mathbf{d}}} = (a^{\mathbf{d}})^{\mathcal{I}}$ if $(a^{\mathbf{d}})^{\mathcal{I}} \neq \mathsf{undef}^{\mathcal{I}}$ otherwise $a^{\mathcal{I}_{\mathbf{d}}}$ is undefined for every individual a;
3. $(X_{\mathbf{f_B}})^{\mathcal{I}_{\mathbf{d}}} = (X^{\mathbf{d}}_{\mathbf{f_B+d}})^{\mathcal{I}}$ for every atomic concept/role $X_{\mathbf{f_B}}$.

It remains to prove that the conditions of Definition 9 are satisfied. This follows from the construction of $\#(\mathfrak{K})$, for the full proof see the appendix. ☐

For any given CKR $\mathfrak{K}$, $\#(\mathfrak{K})$ is a $\mathcal{SROIQ}$ knowledge base. As discussed in Sect. 2, reasoning in $\mathcal{SROIQ}$ is known to be decidable only under certain restrictions. Since the RIA introduced in Step 8 of the construction $\#(\mathfrak{K})$ spoil the regularity of the role hierarchy of $\#(\mathfrak{K})$, we rely on RBox stratification. Particularly, in Step 8, in order to maintain the stratification of the RBox after the pair of axioms (a) is introduced, we add also the pair (b) that is based on the requirements for stratified RBoxes [15]. Thus the stratification of $\#(\mathfrak{K})$ is not broken in Step 8. Since a new role $S^{\mathbf{d,e\,f}}_{R}$ is used in each iteration of this step, this has no influence on the semantics of other symbols. This is an important step in order to avoid introduction of undecidability solely by the reduction of $\mathfrak{K}$ into $\#(\mathfrak{K})$. As undecidability may still be caused by complex dependencies in the role hierarchy of $\mathfrak{K}$, therefore we will use stratification of $\#(\mathfrak{K})$ as a sufficient condition to distinguish cases when this is not true. We summarize the findings of this section in the following theorem.

Theorem 2. *If* $\#(\mathfrak{K})$ *is stratified, then checking if* $\mathfrak{K} \models \mathbf{d} : \phi$ *is decidable with the complexity upper-bound of* 2NExpTime.

Proof. (Sketch.) The decidability follows directly from Lemmata 3, 4 and from the fact that $\#(\mathfrak{K})$ is stratified. The complexity upper-bound follows from the fact that the reduction is polynomial, more precisely, cubic. Given a CKR $\mathfrak{K}$ of size m it produces a $\mathcal{SROIQ}$ knowledge base $\#(\mathfrak{K})$ of size $O(m^3)$. An important fact to establish this result is that the number of dimensions in $\mathfrak{K}$ is assumed to be a fixed constant. This is justified by relevant research on properties of context space (e.g., Lenat [11] suggests that twelve dimensions should be enough). Also that the number of contexts n is always smaller than the size of the knowledge base m, as when a new context is introduced, axioms are added to $\mathfrak{M}$ which is part of $\mathfrak{K}$. Full proof is listed in the appendix. □

6. Related Work

The theoretical foundations of contextualized knowledge representation were laid down by McCarthy [8]. It is based on the idea to represent logically and reason also about the meta knowledge that constraints the validity of the knowledge represented in the first place (object knowledge). McCarthy proposed a unique language for both kinds of knowledge, namely quantified modal logic. This approach allows for great representation power, but easily leads to undecidability. Therefore in CKR we avoid mixing the meta knowledge and the object knowledge arbitrarily; the meta knowledge semantically influences the object knowledge, but not the other way around.

Among the most influential works in contextualized knowledge representation is undoubtedly the one of Lenat [11], who proposed a structured knowledge base organized in units called micro theories, which in our framework correspond to contexts. Lenat also proposed dimensional parameters to be attached to the micro theories, and investigated on the types of dimensions and the structure of the dimensional space. The paper describes the basic set of twelve dimensions that should be satisfactory for most applications. This theoretical framework was implemented in the CYC system [26] which is a successful commercial product. The CKR framework shares notable similarities with this approach, in that the knowledge is organized in contexts, which are arranged in a dimensional space. However, the propagation of knowledge between contexts is implemented on a different basis in CKR, there are no qualified symbols in the Lenat's approach [11]. In addition, CKR is fully compatible with $\mathcal{SROIQ}$ (and therefore OWL 2), so that knowledge available via the Semantic Web can be directly stored and retrieved in a contextualized way.

Perceiving the need for some means of representing context in the Semantic Web, Both aRDF [19] and Context Description Framework [5] extend RDF triples with an n-tuple of qualification attributes with partially ordered domains. Apart from CKR being on top of OWL 2, it differs from these approaches by qualifying whole theories and not each formula separately. This approach is more compact as usually the context is shared a by group of formulae.

Straccia et al. [20] enable RDFS graphs to be annotated with values from a lattice. The semantics of the framework is based on an interpretation structure that is common in multi-valued logics. This effectively restricts the dimensional structure to a complete lattice, as for every two contexts there must be a meet ($\wedge$) and a join ($\vee$) and also global bottom ($\bot$) and top ($\top$) must exist. Contrary to this, the CKR semantics permits any directed acyclic graph, even an unconnected one. This is intentional, as we want to permit certain dimensions to be modeled based on exiting ontologies. The location dimension may for instance be based on the Geonames[6] ontology. Also, if for instance a believer dimension is added, a separated dimensional space may be necessary as no knowledge propagation between believers is desired. Also the top class (rdfs : Resource) has the same semantics in all contexts, and all constants are equally defined in all contexts – in CKR this is controlled by the context hierarchy – and no equivalent of qualified symbols is available in this framework.

Another extension of RDFS to cope with context was proposed by Guha et al. [3] and further developed in Bao et al. [27]. A new predicate $\mathsf{isin}(c, \phi)$ is used to assert that the triple ϕ occurs in the context c. A set of operators to combine contexts ($c_1 \wedge c_1$, $c_1 \vee c_2$, $\neg c$) and to relate contexts ($c \Rightarrow c_2$, $c \to c_2$) is defined, making the approach particularly suited for manipulating contexts. Unfortunately, no sound and complete axiomatization or decision procedure was provided so far.

The contextual DL $\mathcal{ALC}_{\mathcal{ALC}}$ [6] is a multi-modal extension of the $\mathcal{ALC}$ DL with the contextual modal operator $[C]_r A$ representing "all objects of type A in all contexts of type C reachable from the current context via relation r." In both $\mathcal{ALC}_{\mathcal{ALC}}$ and CKR contextual structure is formalized in a meta language separated from the object language used to describe the domain. The main difference between CKR and $\mathcal{ALC}_{\mathcal{ALC}}$ is that CKR is more expressive in the object language ($\mathcal{SROIQ}$ vs. $\mathcal{ALC}$) but less expressive in the contextual assertions, allowing qualification of knowledge only w.r.t. individual contexts rather than context classes as in $\mathcal{ALC}_{\mathcal{ALC}}$. The effect of this choice is that in CKR the complexity of reasoning is the same as in the object language (i.e., 2NExpTime) while in $\mathcal{ALC}_{\mathcal{ALC}}$ the complexity jumps to 2ExpTime compared to ExpTime for $\mathcal{ALC}$. On the other hand, this comparison is only preliminary, and it will be more accurate to compare the two frameworks w.r.t. the same local language. We plan to investigate an $\mathcal{ALC}$-based CKR as future work.

The Metaview approach [7] enriches OWL ontologies with logically treated annotations and it can be used to model contextual meta data similarly to CKR albeit on per-axiom basis. The main difference is that in the Metaview approach the contextual level has no direct implications on ontology reasoning, but it makes possible to reason about the ontology or even data. Also a contextually sensitive query language MQL is provided. The examples presented in this paper concentrate especially on modeling provenance of data and associated confidence, and the framework seems well suited for this purpose. Our research is concerned with other aspects of context, e.g., to break down the data-set into smaller well manageable units, knowledge reuse, etc. A context-aware query language was also designed and im-

[6]http://www.geonames.org/

plemented for CKR [21]. It would be interesting to further compare the frameworks. And also, from the point of view of CKR, to cope with the goals suggested by the Metaview approach, for instance we can try modeling different confidence-levels of data by means of a new dimension.

Related to our approach is also the data tailoring technique that was described by Tanca [28]. Here, the contextual structure is captured by a "context dimension tree", which is in fact a refinement of dimensional vectors into a tree form. Top level dimensions are thus specialized into sub-dimensions, and only leaf configurations represent possible contexts associated with different views of the data. This serves to provide the user with an appropriate view, based on her context. Multiple dimensions relevant for this application are suggested, such as time, topic of interest, but also interface (e.g. human or machine). The dimension tree is combined with a set of constraints to limit the valid combinations which serves to filter out some of the irrelevant configurations. The notable parallel of this approach to ours is that a structured dimensional space is used to break down the data set into relevant portions.

On the semantic level, CKR is also related to approaches such as multi-context systems [16], distributed description logics [17], $\mathcal{E}$-connections [29], but especially approaches concerned with semantic importing such as package-based description logics (P-DL) [18] and semantic imports [30]. In P-DL imports of symbols are implemented by relating the elements of interpretation domains with one-to-one mappings. The work of Pan et al. [30] goes even closer to our approach by assuming that the interpretation domains of distinct ontologies may overlap. In both cases additional semantic constraints are introduced to support various desired properties of the importing paradigm.

In our current work, we use similar techniques, however we use them to meet different goals. Borrowing the viewpoint of the semantic imports paradigm, we may observe that imports are implemented between the contexts of CKR, however, to various extents depending on the relation of the two contexts in question. If the contexts are directly related by the coverage, all information from the narrower context is accessible in the broader context using a technique similar to importing. On the other hand, the narrower of the two contexts may only access part of the other context's information. If two contexts are related indirectly, then the importing is even more limited. Thus we can see that similar techniques are being used in order to characterize a complex scenario of information reuse in accordance with the underlaying ideas of the AI theories of context which is carefully crafted in the semantic conditions asserted in CKR models.

In addition we would like to mention also the work done on ontology versioning and semantic difference in description logics [31, 32]. While there are certain obvious similarities between ontology versioning and contextual representation, most notably in both cases there are multiple knowledge bases that one has to handle and reason with, the motivation and also the problems one faces in these two approaches are different. In ontology versioning we face the problem of knowledge evolution, there are multiple versions of the ontology and we are interested in characterization of the changes (the notion of semantic difference), which would allow to store and reason with the different versions efficiently. That is, at any time one wants to draw conclusions from a particular revision of interest. In contextual representation on the other hand, we are dealing with knowledge chunks which are not evolving versions of one another, but rather one is complementary to each other. When we draw conclusions from one of them, they are possibly influenced by the other chunks, but not in the sense of outdated/updated information. Although implemented differently in each of the approaches, the most interesting similarity is probably the need to break down the information into multiple units and to be able to combine it efficiently when it is relevant.

7. Conclusion

With increasing numbers of ontologies and data-sets being published on the Web under the initiatives such as Semantic Web and Linked Open Data, the need of having a way to consider, process, and take advantage also of the context associated with knowledge becomes more and more apparent. Multiple approaches to deal with context have been proposed; we have reviewed a number of them in the previous section. It is not yet the case that a commonly acknowledged representation framework and methodology to deal with context on the Semantic Web has been found.

Building on the foundations of contextual knowledge representation [8, 16, 10, 11] we have proposed Contextualized Knowledge Repository (CKR) – a context-aware representation framework specifically tailored for the Semantic Web. CKR offers several distinctive features. The knowledge base is structured into units called contexts with contextual attributes explicitly assigned; this allows to group axioms and data that are assumed to hold under similar circumstances, and improves topical organization and maintenance of the knowledge base. Such approach is also well in line with the context as a box paradigm which opens the possibility to exploit the existing body of research on contextual representation [9, 10] in applications of the framework.

Contextual attributes have structured value-sets which results into a hierarchical organization of contexts in a dimensional space. This makes the relations between contexts explicit and allows to maintain contexts with different levels of generality. Part of the knowledge expressed in a context has local validity, but part may as well influence related contexts. CKR allows knowledge to be lifted between contexts with so called qualified concepts and roles. This provides a significant level of control to the knowledge engineer who may exactly specify which symbols have only local meaning and which are reused between contexts. The complexity of the lifting mechanism is hidden from the user as it is automatically implemented by the semantics; there is no need to express lifting axioms directly.

The knowledge inside each context is fully expressed using the standard Semantic Web languages. The CKR framework in this paper uses the $\mathcal{SROIQ}$ DL which corresponds to OWL 2, but an RDFS-based version has been developed as well [22]. The meta knowledge is gathered in a separate knowledge base which is expressed in the same language as well. Therefore adopting the framework by a user familiar with the standard Semantic Web languages should not be difficult.

In this paper we have described syntax and semantics of CKR built on top of the $\mathcal{SROIQ}$ DL. We have provided a sound and complete ND calculus that characterizes logical consequence in CKR, particularly focusing on cross-context entailment (i.e., the transfer of knowledge between contexts). We have also studied basic properties of cross-context entailment in CKR. Finally, we have showed that reasoning with CKR is decidable with computational complexity same as for $\mathcal{SROIQ}$ (i.e., within the class 2NExpTime).

In the future, we plan to investigate the formal properties of CKR based on more tractable fragments of OWL 2, e.g., OWL-Horst [33]. We have already developed a prototype on top of the Sesame 2 triple store which uses RDFS as local language [22]. In the prototype, contexts have been naturally implemented with named graphs [34]. We also plan to study tableaux based reasoning techniques for CKR which would allow to develop a reasoner for the DL-based CKR, and also to investigate on additional meta level constructs (such as for instance context classes) and novel applications of meta level reasoning.

In order to evaluate the practical applicability of CKR we are currently undergoing an experimental study in which we model and populate a CKR knowledge base of non trivial size. We have chosen the domain of football tournaments and more specifically the different editions of FIFA WC. This domain was broken down into a number of contexts, some of them more general such as the generic contexts sports and football, some more specific such as the particular editions of FIFA WC (e.g., $C_{\mathsf{FIFA\_WC,2010,South\_Africa}}$), but also as specific as the stages of each tournament, and further down to single matches (e.g., $C_{\mathsf{FIFA\_WC\_Match\_42,2010,South\_Africa}}$). In every context we put axioms that are relevant to this context. For instance the axioms $\mathsf{GoalKeeper} \sqsubseteq \mathsf{Player}$, $\mathsf{Midfielder} \sqsubseteq \mathsf{Player}$ and $\mathsf{Player} \sqsubseteq \mathsf{Sportsman}_{\mathsf{sports}}$ belong to the generic context of football as they serve to model various player positions in the game and the fact that all football players are sportsmen. On the other hand the axioms $\mathsf{TeamA} \sqsubseteq \neg\mathsf{TeamB}$ and $\mathsf{TeamA} \sqcup \mathsf{TeamB} \sqsubseteq \mathsf{Team}_{\mathsf{FIFA\_WC}}$ belong to the context of a particular match of the FIFA WC, and serve to assure that there are two teams involved in this match, that these two teams are distinct and are both part of the respective edition of FIFA WC. The knowledge base was populated with data available from Freebase, DBPedia, and other sources on the Web, and then stored in our prototype implementation of CKR. In this study, we aim to evaluate the representational aspects, that is, how easy is to model with the CKR framework, whether the resulting modeling is efficient and practical, etc., but also on the computational aspects such as the efficiency of query answering. We plan to publish our results in near future.

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Appendix A. Proofs

Appendix A.1. Proof of Lemma 1

Lemma 1. *If $\mathcal{I}$ and $\mathcal{I}'$ comply with $f : \Sigma \to \Sigma'$, then, for every concept C, $C^{\mathcal{I}} = f^*(C)^{\mathcal{I}'}$, for every role R, $R^{\mathcal{I}} = f^*(R)^{\mathcal{I}'}$, and for every axiom ϕ, $\mathcal{I} \models \phi$ iff $\mathcal{I}' \models f^*(\phi)$.*

Let us have two DL-alphabets Σ and Σ', a DL-embedding $f : \Sigma \to \Sigma'$ and two respective DL-interpretations $\mathcal{I}$ and $\mathcal{I}'$ complying with f. From Definition 15 this implies the following four facts which we denote by (†):

1. $a^{\mathcal{I}} = f(a)^{\mathcal{I}'}$, for all individuals a of Σ such that $a^{\mathcal{I}}$ is defined;
2. $X^{\mathcal{I}} = f(X)^{\mathcal{I}'}$, for all symbols $X \in \Sigma_c$;
3. $A^{\mathcal{I}} = f(A)^{\mathcal{I}'} \cap f(\top)^{\mathcal{I}'}$, for all concepts $A \in \Sigma_e$;
4. $R^{\mathcal{I}} = f(R)^{\mathcal{I}'} \cap f(\top)^{\mathcal{I}'} \times f(\top)^{\mathcal{I}'}$, for all roles $R \in \Sigma_e$.

Let us first realize how the domain $\Delta^{\mathcal{I}}$ of $\mathcal{I}$ is embedded into the domain $\Delta^{\mathcal{I}'}$ of Σ'. Later in the proof we will denote this observation by (‡):

$$\Delta^{\mathcal{I}} = \top^{\mathcal{I}} = f(\top)^{\mathcal{I}'} \subseteq \Delta^{\mathcal{I}'}$$

The second equation is due to $\top \in \Sigma_c$ and from the fact that $\mathcal{I}$ and $\mathcal{I}'$ comply with the embedding f. The other two equations trivially follow from $\mathcal{I}$ and $\mathcal{I}'$ being DL-interpretations.

We will now prove that for every concept or role X it holds that $X^{\mathcal{I}} = f^*(X)^{\mathcal{I}'}$. The proof is by structural induction on X:

if $X = A \in \Sigma_c$, then $f^*(A)^{\mathcal{I}'} =$
$= f(A)^{\mathcal{I}'}$ by definition of f^*
$= A^{\mathcal{I}}$ from (†,2)

if $X = A \in \Sigma_e$, then $f^*(A)^{\mathcal{I}'} =$
$= f(A) \sqcap f(\top)^{\mathcal{I}'}$ by the definition of f^*
$= f(A)^{\mathcal{I}'} \cap f(\top)^{\mathcal{I}'}$ by the interpretation of $\sqcap$
$= (A)^{\mathcal{I}}$ from (†,3)

if $X = \neg C$, then $f^*(\neg C)^{\mathcal{I}'} =$
$= f(\top) \sqcap \neg f^*(C)^{\mathcal{I}'}$ by definition of f^*
$= f(\top)^{\mathcal{I}'} \cap \neg f^*(C)^{\mathcal{I}'}$ by interpretation of $\sqcap$
$= f(\top)^{\mathcal{I}'} \cap (\Delta^{\mathcal{I}'} \setminus f^*(C)^{\mathcal{I}'})$ by interpretation of $\neg$
$= f(\top)^{\mathcal{I}'} \setminus f^*(C)^{\mathcal{I}'}$ due to $f(\top)^{\mathcal{I}'} \subseteq \Delta^{\mathcal{I}'}$
$= \Delta^{\mathcal{I}} \setminus f^*(C)^{\mathcal{I}'}$ from (‡)
$= \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$ by induction
$= \neg C^{\mathcal{I}}$ by interpretation of $\neg$

if $X = C \sqcap D$, then $f^*(C \sqcap D)^{\mathcal{I}'} =$
$= f^*(C) \sqcap f^*(D)^{\mathcal{I}'}$ by definition of f^*
$= f^*(C)^{\mathcal{I}'} \cap f^*(D)^{\mathcal{I}'}$ by interpretation of $\sqcap$
$= C^{\mathcal{I}} \cap D^{\mathcal{I}}$ by induction
$= C \sqcap D^{\mathcal{I}}$ by interpretation of $\sqcap$

if $X = \exists R.C$ and $R \in \Sigma_c$, then $f^*(\exists R.C)^{\mathcal{I}'} =$
$= \exists f(R).f^*(C)^{\mathcal{I}'}$ by definition of f^* and $R \in \Sigma_c$
$= \{x \in \Delta^{\mathcal{I}'} \mid \exists y\ (x,y) \in f(R)^{\mathcal{I}'} \wedge y \in f^*(C)^{\mathcal{I}'}\}$ by interpretation of $\exists$
$= \{x \in \Delta^{\mathcal{I}'} \mid \exists y\ (x,y) \in f(R)^{\mathcal{I}'} \wedge y \in C^{\mathcal{I}}\}$ by induction
$= \{x \in \Delta^{\mathcal{I}'} \mid \exists y\ (x,y) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$ from (†,2)
$= \{x \in \Delta^{\mathcal{I}} \mid \exists y\ (x,y) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$ by $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ and $\Delta^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}'}$
$= \exists R.C^{\mathcal{I}}$ by definition of $\exists$

if $X = \exists R.C$ and $R \in \Sigma_e$, then $f^*(\exists R.C)^{\mathcal{I}'} =$
$= f(\top) \sqcap \exists f(R).f^*(C)^{\mathcal{I}'}$ by definition of f^* and $R \in \Sigma_e$
$= f(\top)^{\mathcal{I}'} \cap \exists f(R).f^*(C)^{\mathcal{I}'}$ by interpretation of $\sqcap$
$= \Delta^{\mathcal{I}} \cap \exists f(R).f^*(C)^{\mathcal{I}'}$ from (‡)
$= \Delta^{\mathcal{I}} \cap \{x \in \Delta^{\mathcal{I}'} \mid \exists y\ (x,y) \in f(R)^{\mathcal{I}'} \wedge y \in f^*(C)^{\mathcal{I}'}\}$ by interpretation of $\exists$
$= \Delta^{\mathcal{I}} \cap \{x \in \Delta^{\mathcal{I}'} \mid \exists y\ (x,y) \in f(R)^{\mathcal{I}'} \wedge y \in C^{\mathcal{I}}\}$ by induction
$= \{x \in \Delta^{\mathcal{I}} \mid \exists y\ (x,y) \in f(R)^{\mathcal{I}'} \cap \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$ since $C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$
$= \{x \in \Delta^{\mathcal{I}} \mid \exists y\ (x,y) \in f(R)^{\mathcal{I}'} \cap f(\top)^{\mathcal{I}'} \times f(\top)^{\mathcal{I}'} \wedge y \in C^{\mathcal{I}}\}$ from (‡)
$= \{x \in \Delta^{\mathcal{I}} \mid \exists y\ (x,y) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$ from (†,4)
$= \exists R.C^{\mathcal{I}}$ by interpretation of $\exists$

if $X = \exists R.\text{Self}$ and $R \in \Sigma_c$, then $f^*(\exists R.\text{Self})^{\mathcal{I}'} =$
$= \exists f(R).\text{Self}^{\mathcal{I}'}$ by definition of f^* and $R \in \Sigma_c$
$= \{x \in \Delta^{\mathcal{I}'} \mid (x,x) \in f(R)^{\mathcal{I}'}\}$ by interpretation of $\exists R.\text{Self}$
$= \{x \in \Delta^{\mathcal{I}'} \mid (x,x) \in R^{\mathcal{I}}\}$ by (†,2)
$= \{x \in \Delta^{\mathcal{I}} \mid (x,x) \in R^{\mathcal{I}}\}$ due to $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ and $\Delta^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}'}$
$= \exists R.\text{Self}^{\mathcal{I}}$ by definition of $\exists R.\text{Self}$

if $X = \exists R.\text{Self}$ and $R \in \Sigma_e$, then $f^*(\exists R.\text{Self})^{\mathcal{I}'} =$
$= f(\top) \sqcap \exists f(R).\text{Self}^{\mathcal{I}'}$ by the definition of f^* with $R \in \Sigma_e$
$= f(\top)^{\mathcal{I}'} \cap \{x \in \Delta^{\mathcal{I}'} \mid (x,x) \in f(R)^{\mathcal{I}'}\}$ by interpretation of $\sqcap$ and $\exists R.\text{Self}$
$= \Delta^{\mathcal{I}} \cap \{x \in \Delta^{\mathcal{I}'} \mid (x,x) \in f(R)^{\mathcal{I}'}\}$ from (‡)
$= \{x \in \Delta^{\mathcal{I}} \mid (x,x) \in f(R)^{\mathcal{I}'}\}$ as $\Delta^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}'}$ by (‡)
$= \{x \in \Delta^{\mathcal{I}} \mid (x,x) \in f(R)^{\mathcal{I}'} \cap \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}\}$
$= \{x \in \Delta^{\mathcal{I}} \mid (x,x) \in f(R)^{\mathcal{I}'} \cap f(\top)^{\mathcal{I}'} \times f(\top)^{\mathcal{I}'}\}$ from (‡)
$= \{x \in \Delta^{\mathcal{I}} \mid (x,x) \in R^{\mathcal{I}}\}$ from (†,4)
$= \exists R.\text{Self}^{\mathcal{I}}$ by definition of $\exists R.\text{Self}$

if $X = \geqslant nR.C$ and $R \in \Sigma_c$, then $f^*(\geqslant nR.C)^{\mathcal{I}'} =$
$= \geqslant nf(R).f^*(C)^{\mathcal{I}'}$ by definition of f^* and $R \in \Sigma_c$
$= \{x \in \Delta^{\mathcal{I}'} \mid \exists_{1 \leq i \leq n}\, y_i\ (x,y_i) \in f(R)^{\mathcal{I}'} \wedge y_i \in f^*(C)^{\mathcal{I}'}\}$ interpretation of $\geqslant n$
$= \{x \in \Delta^{\mathcal{I}'} \mid \exists_{1 \leq i \leq n}\, y_i\ (x,y_i) \in f(R)^{\mathcal{I}'} \wedge y_i \in C^{\mathcal{I}}\}$ by induction
$= \{x \in \Delta^{\mathcal{I}'} \mid \exists_{1 \leq i \leq n}\, y_i\ (x,y_i) \in R^{\mathcal{I}} \wedge y_i \in C^{\mathcal{I}}\}$ by (†,2)
$= \{x \in \Delta^{\mathcal{I}} \mid \exists_{1 \leq i \leq n}\, y_i\ (x,y_i) \in R^{\mathcal{I}} \wedge y_i \in C^{\mathcal{I}}\}$ as $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, $\Delta^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}'}$
$= \geqslant nR.C^{\mathcal{I}}$ by interpretation of $\geqslant n$

if $X = \geqslant nR.C$ and $R \in \Sigma_e$, then $f^*(\geqslant nR.C)^{\mathcal{I}'} =$
$= f(\top) \sqcap \geqslant nf(R).f^*(C)^{\mathcal{I}'}$ by definition of f^* and $R \in \Sigma_e$
$= f(\top)^{\mathcal{I}'} \cap \{x \in \Delta^{\mathcal{I}'} \mid \exists_{1 \leq i \leq n}\, y_i\ (x,y_i) \in f(R)^{\mathcal{I}'} \wedge y_i \in f^*(C)^{\mathcal{I}'}\}$ by $\sqcap$, $\geqslant n$
$= \Delta^{\mathcal{I}} \cap \{x \in \Delta^{\mathcal{I}'} \mid \exists_{1 \leq i \leq n}\, y_i\ (x,y_i) \in f(R)^{\mathcal{I}'} \wedge y_i \in f^*(C)^{\mathcal{I}'}\}$ by (‡)
$= \{x \in \Delta^{\mathcal{I}} \mid \exists_{1 \leq i \leq n}\, y_i\ (x,y_i) \in f(R)^{\mathcal{I}'} \wedge y_i \in f^*(C)^{\mathcal{I}'}\}$ as $\Delta^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}'}$ by (‡)
$= \{x \in \Delta^{\mathcal{I}} \mid \exists_{1 \leq i \leq n}\, y_i\ (x,y_i) \in f(R)^{\mathcal{I}'} \wedge y_i \in C^{\mathcal{I}}\}$ by induction
$= \{x \in \Delta^{\mathcal{I}} \mid \exists_{1 \leq i \leq n}\, y_i\ (x,y_i) \in f(R)^{\mathcal{I}'} \cap \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \wedge y_i \in C^{\mathcal{I}}\}$ as $C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$
$= \{x \in \Delta^{\mathcal{I}} \mid \exists_{1 \leq i \leq n}\, y_i\ (x,y_i) \in f(R)^{\mathcal{I}'} \cap f(\top)^{\mathcal{I}'} \times f(\top)^{\mathcal{I}'} \wedge y_i \in C^{\mathcal{I}}\}$ by (‡)
$= \{x \in \Delta^{\mathcal{I}} \mid \exists_{1 \leq i \leq n}\, y_i\ (x,y_i) \in R^{\mathcal{I}} \wedge y_i \in C^{\mathcal{I}}\}$ by (†,4)
$= \geqslant nR.C^{\mathcal{I}}$ by definition of $\geqslant n$

if $X = \{a\}$, then $f^*(\{a\})^{\mathcal{I}'} =$
$= \{f(a)\}^{\mathcal{I}'}$ by definition of f^*
$= \{f(a)^{\mathcal{I}'}\}$ by interpretation of nominals
$= \{a^{\mathcal{I}}\}$ from (†,1)
$= \{a\}^{\mathcal{I}}$ by interpretation of nominals

if $X = R \in \Sigma_c$, then $f^*(R)^{\mathcal{I}'} =$
$= f(R)^{\mathcal{I}'}$ by definition of f^*
$= R^{\mathcal{I}}$ by (†,2)

if $X = R \in \Sigma_e$, then $f^*(R)^{\mathcal{I}'} =$
$= f(I) \circ f(R) \circ f(I)^{\mathcal{I}'}$ by definition of f^*
$= \{(u,v) \mid \exists (x,y) \in f(R)^{\mathcal{I}'} \wedge (u,x),(v,y) \in f(I)^{\mathcal{I}'}\}$ by interpretation of $\circ$
$= \{(x,y) \mid (x,y) \in f(R)^{\mathcal{I}'} \wedge (x,x),(y,y) \in f(I)^{\mathcal{I}'}\}$ as $I^{\mathcal{I}'}$ is identity on $f(\top)^{\mathcal{I}'}$
$= \{(x,y) \mid (x,y) \in f(R)^{\mathcal{I}'} \cap f(\top)^{\mathcal{I}'} \times f(\top)^{\mathcal{I}'}\}$ as $I^{\mathcal{I}'}$ is identity on $f(\top)^{\mathcal{I}'}$
$= \{(x,y) \mid (x,y) \in f(R)^{\mathcal{I}}\}$ from (†,4)
$= f(R)^{\mathcal{I}}$

if $X = R \circ S$, then $f^*(R \circ S)^{\mathcal{I}'} =$
$= f^*(R) \circ f^*(S)^{\mathcal{I}'}$ by definition of f^*
$= f^*(R)^{\mathcal{I}'} \circ f^*(S)^{\mathcal{I}'}$ by interpretation of $\circ$
$= R^{\mathcal{I}} \circ S^{\mathcal{I}}$ by induction
$= R \circ S^{\mathcal{I}}$ by interpretation of $\circ$

Let us now continue with the second proposition of the lemma, i.e., that for every axiom ϕ, $\mathcal{I} \models \phi$ if and only if $\mathcal{I}' \models f^*(\phi)$. We must consider all the cases corresponding to the different forms of ϕ:

1. $\phi = C(a)$, $f^*(\phi) = f^*(C)(f(a))$: from (†,1) we have $a^{\mathcal{I}} = f(a)^{\mathcal{I}'}$ and we have proved above that $C^{\mathcal{I}} = f^*(C)^{\mathcal{I}'}$. Therefore $\mathcal{I} \models C(a)$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$ iff $f(a)^{\mathcal{I}'} \in f^*(C)^{\mathcal{I}'}$ iff $\mathcal{I}' \models f^*(C)(f(a))$;
2. $\phi = R(a,b)$, $f^*(\phi) = f(R)(f(a),f(b))$: from (†,1) we have $a^{\mathcal{I}} = f(a)^{\mathcal{I}'}$, $b^{\mathcal{I}} = f(b)^{\mathcal{I}'}$ and from (†,4) we have $R^{\mathcal{I}} = f(R)^{\mathcal{I}'}$. The rest of the proof is analogous to the previous case;
3. $\phi = \neg R(a,b)$, $f^*(\phi) = \neg f(R)(f(a),f(b))$: since $a^{\mathcal{I}} = f(a)^{\mathcal{I}'}$, $b^{\mathcal{I}} = f(b)^{\mathcal{I}'}$ and $R^{\mathcal{I}} = f(R)^{\mathcal{I}'}$, we have $\mathcal{I} \models \neg R(a,b)$ iff $(a^{\mathcal{I}}, b^{\mathcal{I}}) \notin R^{\mathcal{I}}$ iff $(f(a)^{\mathcal{I}'}, f(b)^{\mathcal{I}'}) \notin f(R)^{\mathcal{I}'}$ iff $\mathcal{I}' \models \neg f(R)(f(a),f(b))$;
4. $\phi = C \sqsubseteq D$, $f^*(\phi) = f^*(C) \sqsubseteq f^*(D)$: as we have already proved $C^{\mathcal{I}} = f^*(C)^{\mathcal{I}'}$ and $D^{\mathcal{I}} = f^*(D)^{\mathcal{I}'}$. Therefore $\mathcal{I} \models C \sqsubseteq D$ iff $C^{\mathcal{I}} \subseteq B^{\mathcal{I}}$ iff $f^*(C)^{\mathcal{I}'} \subseteq f^*(D)^{\mathcal{I}'}$ iff $\mathcal{I}' \models f^*(C) \sqsubseteq f^*(D)$;

5. $\phi = R \sqsubseteq S$, $f^*(\phi) = f^*(R) \sqsubseteq f(S)$: we have proved that $R^{\mathcal{I}} = f^*(R)^{\mathcal{I}'}$. If $S \in \Sigma_c$, we have from (†,2) that $S^{\mathcal{I}} = f(S)^{\mathcal{I}'}$. The proof of this case is analogous to the previous case.
   If $S \in \Sigma_e$ then from (†,4) and from (‡) we have $S^{\mathcal{I}} = f(S)^{\mathcal{I}'} \cap \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. For this case, let us first prove the if part: suppose $\mathcal{I}' \models f^*(R) \sqsubseteq f(S)$ and therefore $f^*(R)^{\mathcal{I}'} \subseteq f(S)^{\mathcal{I}'}$. Then $R^{\mathcal{I}} = R^{\mathcal{I}} \cap \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} = f^*(R)^{\mathcal{I}'} \cap \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \subseteq f(S)^{\mathcal{I}'} \cap \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} = S^{\mathcal{I}}$. Which amounts to $\mathcal{I} \models R \sqsubseteq S$. The only-if part: Suppose $\mathcal{I} \models R \sqsubseteq S$, that is, $R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$. It follows that $f^*(R)^{\mathcal{I}'} = R^{\mathcal{I}} \subseteq S^{\mathcal{I}} = f(S)^{\mathcal{I}'} \cap \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \subseteq f(S)^{\mathcal{I}'}$.
6. ϕ is $a = b$, i.e., $f^*(\phi)$ is $f(a) = f(b)$: we know from (†,1) that $a^{\mathcal{I}} = f(a)^{\mathcal{I}'}$ and $b^{\mathcal{I}} = f(b)^{\mathcal{I}'}$. Hence $\mathcal{I} \models a = b$ iff $a^{\mathcal{I}} = b^{\mathcal{I}}$ iff $f(a)^{\mathcal{I}'} = f(b)^{\mathcal{I}'}$ iff $\mathcal{I}' \models f(a) = f(b)$;
7. ϕ is $a \neq b$, i.e., $f^*(\phi)$ is $f(a) \neq f(b)$: as a consequence of the previous case we have $\mathcal{I} \models a \neq b$ iff $\mathcal{I} \not\models a = b$ iff $\mathcal{I}' \not\models f(a) = f(b)$ iff $\mathcal{I}' \models f(a) \neq f(b)$.

Appendix A.2. Proof of Theorem 1

Theorem 1 (Soundness and Completeness). *For every CKR $\mathfrak{K}$ over $\langle\Gamma, \Sigma\rangle$, for every $\mathbf{d} \in \mathfrak{D}_\Gamma$, and for every formula ϕ over Σ, $\mathfrak{K} \vdash \mathbf{d} : \phi$ if and only if $\mathfrak{K} \models \mathbf{d} : \phi$.*

Appendix A.2.1. Soundness.

We ought to prove that if $\mathfrak{K} \vdash \mathbf{d} : \phi$ then also $\mathfrak{K} \models \mathbf{d} : \phi$. We will prove this by showing that all calculus rules are sound. For each rule ρ, which in general is of the form:

$$\frac{\alpha_1 \quad \cdots \quad \alpha_n \quad \begin{matrix}[B_{n+1}]\\ \alpha_{n+1}\end{matrix} \quad \cdots \quad \begin{matrix}[B_{n+m}]\\ \alpha_{n+m}\end{matrix}}{\alpha}\rho$$

we have to prove that if $\mathfrak{K}, B_i \vdash \alpha_i$ holds for each i, $n < i \leq n+m$, then it is also true that $\mathfrak{K}, \{\alpha_1, \ldots, \alpha_{n+m}\} \models \alpha$. More formally, we have to prove the implication:

$$\mathfrak{K}, B_{n+1} \vdash \alpha_{n+1} \quad \wedge \quad \cdots \quad \wedge \quad \mathfrak{K}, B_{n+m} \vdash \alpha_{n+m} \implies \mathfrak{K}, \{\alpha_1, \ldots, \alpha_{n+m}\} \models \alpha$$

This has to be proved separately for each of the calculus rules, as listed in Table 3. Due to the rules with discharges, the proof is done by structural induction. The base of the induction is formed by the cases of the rules without discharges. The inductive step comprises the remaining cases, in which the induction hypothesis allows us to derive $\mathfrak{K}, B_i \models \alpha_i$ from $\mathfrak{K}, B_i \vdash \alpha_i$, for each premise α_i with discharges B_i of the rule ρ in question, because the proof respective to this subproof is a sub-tree of the overall proof of the conclusion of ρ. The proof for each type of rule follows:

LReas: let $\mathfrak{I}$ be a model of $\mathfrak{K} \cup \{\mathbf{d} : \phi_1, \ldots, \mathbf{d} : \phi_n\}$. This implies that it also satisfies $\mathcal{I}_\mathbf{d} \models_{\mathrm{DL}} \phi_i$, for every i, $1 \leq i \leq n$. From the assumptions of the LReas rule, $\{\phi_1, \ldots, \phi_n\} \models_{\mathrm{DL}} \phi$ (i.e., the conclusion is only derived by this rule when this entailment universally holds). From monotonicity of DL this implies $\mathcal{I}_\mathbf{d} \models_{\mathrm{DL}} \phi$ and hence also $\mathfrak{I} \models \mathbf{d} : \phi$;

Top: this rule has three independent forms. The first form assumes $\mathbf{d} \leq \mathbf{e}$ on the meta knowledge and concludes $\mathbf{f} : A_\mathbf{d} \sqsubseteq \top_\mathbf{e}$. Let $C_\mathbf{d}$ and $C_\mathbf{e}$ be two contexts of $\mathfrak{K}$ such that $\mathbf{d} \leq \mathbf{e}$ and let $\mathfrak{I}$ be any model of $\mathfrak{K}$. By Condition 2 of Definition 9 we have $A_\mathbf{d}^{\mathcal{I}_\mathbf{f}} \subseteq \top_\mathbf{d}^{\mathcal{I}_\mathbf{f}}$. By Condition 1 of the same definition $\top_\mathbf{d}^{\mathcal{I}_\mathbf{f}} \subseteq \top_\mathbf{e}^{\mathcal{I}_\mathbf{f}}$. Together this implies $\mathcal{I}_\mathbf{f} \models_{\mathrm{DL}} A_\mathbf{d} \sqsubseteq \top_\mathbf{e}$ and hence $\mathfrak{I} \models \mathbf{f} : A_\mathbf{d} \sqsubseteq \top_\mathbf{e}$.

The second form of the Top rule concludes $\mathbf{f} : \exists R_\mathbf{d}.\top \sqsubseteq \top_\mathbf{d}$ without any premises. We therefore have to show that $\mathfrak{I} \models \mathbf{f} : \exists R_\mathbf{d}.\top \sqsubseteq \top_\mathbf{d}$, i.e., that the domain of the binary relation $R_\mathbf{d}^{\mathcal{I}_\mathbf{f}}$ is under $\top_\mathbf{d}^{\mathcal{I}_\mathbf{f}}$, in every model $\mathfrak{I}$ of every CKR $\mathfrak{K}$. This is a direct consequence of Condition 3 of Definition 9. The case of the third form of the Top rule is exactly analogous except it involves the range of $R_\mathbf{d}^{\mathcal{I}_\mathbf{f}}$;

Bot: there is no model of a CKR $\mathfrak{K}$ in which $\mathbf{d} : \bot(a)$ the premise of the Bot rule is true. This is due to the fact that $\bot(a)$ is not satisfied in any DL-interpretation, not even in the special one with empty domain. Therefore, trivially, it is true that the conclusion $\mathbf{e} : \top \sqsubseteq \bot$ is satisfied by all models of $\mathfrak{K}$;

Push: let $\mathfrak{I}$ be any model of $\mathfrak{K} \cup \{\mathbf{e} : \phi@\mathbf{d}, \mathbf{e} : \top_\mathbf{d}(a_1), \ldots, \mathbf{e} : \top_\mathbf{d}(a_n)\}$ where $a_1, \ldots, a_n$ are all constants occurring in ϕ. This implies that $\mathcal{I}_\mathbf{e} \models_{\mathrm{DL}} \top_\mathbf{d}(a_i)$, for every i, $1 \leq i \leq n$, and therefore $a_i \in \top_\mathbf{d}^{\mathcal{I}_\mathbf{e}}$. This implies $a_i \in \top_\mathbf{d}^{\mathcal{I}_\mathbf{d}}$ and hence $a_i \in \Delta_\mathbf{d}$ (if $\mathbf{d} < \mathbf{e}$ it follows from Definition 9, if $\mathbf{d} = \mathbf{e}$ it is trivial). This assures that ϕ is defined in $\mathcal{I}_\mathbf{d}$, as all constants occurring in it are defined. Since $\mathbf{d} \leq \mathbf{e}$, from Lemma 2 we know that $\mathcal{I}_\mathbf{d}$ and $\mathcal{I}_\mathbf{e}$ comply with the embedding respective to $@\mathbf{d}$, and since $\mathcal{I}_\mathbf{e} \models_{\mathrm{DL}} \phi@\mathbf{d}$, it follows from Lemma 1 that $\mathcal{I}_\mathbf{d} \models_{\mathrm{DL}} \phi$. Hence $\mathfrak{I} \models: \mathbf{d} : \phi$;

Pop: let $\mathfrak{I}$ be a model of $\mathfrak{K} \cup \{\mathbf{d} : \phi\}$. This implies $\mathfrak{I} \models \mathbf{d} : \phi$, i.e., $\mathcal{I}_\mathbf{d} \models_{\mathrm{DL}} \phi$. From $\mathbf{d} \leq \mathbf{e}$ and from Lemmata 1, 2 this implies $\mathcal{I}_\mathbf{e} \models_{\mathrm{DL}} \phi@\mathbf{d}$ and hence $\mathfrak{I} \models \mathbf{e} : \phi@\mathbf{d}$;

aE: the rule assumes that $\mathfrak{K}, \mathbf{d} : \top(a) \vdash \mathbf{d} : \top \sqsubseteq \bot$, for any a that does not appear in $\mathfrak{K}$, and under this assumption we have to prove that $\mathfrak{K} \models \mathbf{d} : \top \sqsubseteq \bot$. Let us assume the contrary, i.e., $\mathfrak{K} \not\models \mathbf{d} : \top \sqsubseteq \bot$. In this case, there must be a model $\mathfrak{I}$ of $\mathfrak{K}$ in which $\top^{\mathcal{I}_\mathbf{d}} \not\subseteq \bot^{\mathcal{I}_\mathbf{d}}$. Then there is $x \in \top^{\mathcal{I}_\mathbf{d}}$ (such that $x \notin \bot^{\mathcal{I}_\mathbf{d}}$). Since a does not appear in $\mathfrak{K}$, if we construct $\mathfrak{I}'$ by extending $\mathfrak{I}$ with the assignment $a^{\mathcal{I}_\mathbf{f}} = x$ for all $\mathbf{f} \in \mathfrak{D}_\Gamma$ such that $x \in \Delta_\mathbf{f}$ (i.e., certainly for $\mathbf{f} = \mathbf{d}$), then $\mathfrak{I}' \models \mathfrak{K}$ as it satisfies all axioms thereof. In addition $\mathfrak{I}' \models \mathbf{d} : \top(a)$ as $a^{\mathcal{I}_\mathbf{d}} = x \in \top^{\mathcal{I}_\mathbf{d}} = \top^{\mathcal{I}'_\mathbf{d}}$. From the assumption of the rule, and from the induction hypothesis $\mathfrak{K}, \mathbf{d} : \top(a) \models \mathbf{d} : \top \sqsubseteq \bot$ and since $\mathfrak{I}' \models \mathfrak{K} \cup \{\mathbf{d} : \top(a)\}$, it must be the case that $\top^{\mathcal{I}_\mathbf{d}} = \top^{\mathcal{I}'_\mathbf{d}} \subseteq \bot^{\mathcal{I}'_\mathbf{d}} = \bot^{\mathcal{I}_\mathbf{d}}$ which is a contradiction;

$\sqcup$E: the rule assumes that $\mathfrak{K}, \mathbf{d} : C(x) \vdash \mathbf{e} : \phi$ and $\mathfrak{K}, \mathbf{d} : D(x) \vdash \mathbf{e} : \phi$. By structural induction we have $\mathfrak{K}, \mathbf{d} : C(x) \models \mathbf{e} : \phi$ and $\mathfrak{K}, \mathbf{d} : D(x) \models \mathbf{e} : \phi$ (from the induction hypothesis). Under these assumptions we need to prove $\mathfrak{K}, \mathbf{d} : C \sqcup D(x) \models \mathbf{e} : \phi$. Let $\mathfrak{I}$ be any model of $\mathfrak{K} \cup \{\mathbf{d} : C \sqcup D(x)\}$. In this model, $\mathcal{I}_\mathbf{d} \models_{\mathrm{DL}} C \sqcup D(x)$. By basic properties of DL-interpretations it must either hold $\mathcal{I}_{\mathbf{d}} \models_{\mathrm{DL}} C(x)$ or $\mathcal{I}_{\mathbf{d}} \models_{\mathrm{DL}} D(x)$. Let us assume the first case. Since in this case $\mathfrak{I}$ is a model of $\mathfrak{K} \cup \{\mathbf{d} : C(x)\}$, from the fact that $\mathfrak{K}, \mathbf{d} : C(x) \models \mathbf{e} : \phi$ we get $\mathfrak{I} \models \mathbf{e} : \phi$. The in the other case analogously $\mathfrak{I} \models \mathbf{e} : \phi$ due to $\mathfrak{K}, \mathbf{d} : D(x) \models \mathbf{e} : \phi$;

$\exists$E: the rule assumes $\mathfrak{K}, \{\mathbf{d} : R(x, y), \mathbf{d} : C(y)\} \vdash \mathbf{e} : \phi$. By induction hypothesis we have $\mathfrak{K}, \{\mathbf{d} : R(x, y), \mathbf{d} : C(y)\} \models \mathbf{e} : \phi$. Under this assumption we ought to prove $\mathfrak{K}, \mathbf{d} : \exists R.C(x) \models \mathbf{e} : \phi$. Let $\mathfrak{I}$ by any model of $\mathfrak{K} \cup \{\mathbf{d} : \exists R.C(x)\}$. This implies that there must be an element $x' \in \Delta_{\mathbf{d}}$ such that $x^{\mathcal{I}_{\mathbf{d}}} = x'$ and $x' \in \exists R.C^{\mathcal{I}_{\mathbf{d}}}$. From the properties of DL-interpretations, there must also by $y' \in \Delta_{\mathbf{d}}$ such that $\langle x', y' \rangle \in R^{\mathcal{I}_{\mathbf{d}}}$ and $y' \in C^{\mathcal{I}_{\mathbf{d}}}$. Since the individual y does not appear anywhere in $\mathfrak{K}$, $y^{\mathcal{I}_{\mathbf{d}}}$ is undefined. Let us construct $\mathfrak{I}'$ by extending $\mathfrak{I}$ with the assignment $y^{\mathcal{I}_{\mathbf{f}}} = y'$ for all $\mathbf{f} \in \mathfrak{D}_\Gamma$ such that $y' \in \Delta_{\mathbf{f}}$. Since all axioms of $\mathfrak{K}$ and also all conditions of Definition 9 are satisfied by $\mathfrak{I}'$, and in addition also $\mathfrak{K} \models \mathbf{d} : R(x, y)$, and $\mathfrak{K} \models \mathbf{d} : C(y)$, then in fact $\mathfrak{I}' \models \mathfrak{K} \cup \{\mathbf{d} : R(x, y), \mathbf{d} : C(y)\}$. From the assumptions we now know that $\mathfrak{I}' \models \mathbf{e} : \phi$. Since the individual y does not occur in ϕ, this implies that $\mathfrak{I} \models \mathbf{e} : \phi$ as well, because the only difference between $\mathfrak{I}$ and $\mathfrak{I}'$ is the interpretation of y;

$(\geqslant n)$E: the case of this rule is analogous to the previous case. From the fact that $\mathfrak{I} \models \mathfrak{K} \cup \{\mathbf{d} : \geqslant nR.C(x)\}$ we are able to find n distinct elements $y'_1, \ldots, y'_n \in \Delta_{\mathbf{d}}$ with the required properties. We then extend the model $\mathfrak{I}$ by assigning these elements to the individuals $y_1, \ldots, y_n$ respectively, and prove that the extended model entails $\mathbf{e} : \phi$ using the induction hypothesis. As all the constants in question are new, this also implies $\mathfrak{I} \models \mathbf{e} : \phi$.

Appendix A.2.2. Completeness.

We now prove the completeness of the axiomatization, i.e., that $\mathfrak{K} \models \mathbf{d} : \phi$ implies $\mathfrak{K} \vdash \mathbf{d} : \phi$. Relying on the fact, that for any DL-formula ϕ over Σ the problem of entailment is reducible to (un)satisfiability, it suffices to prove the statement:

If $\mathfrak{K}$ is unsatisfiable then $\mathfrak{K} \vdash \mathbf{d} : \top \sqsubseteq \bot$.

This is justified as follows. If $\mathfrak{K} \models \mathbf{d} : \phi$ then there exists a CKR $\mathfrak{K}'$ (constructed by the respective reduction). The statement above shows that in this case we are able to prove by the calculus that $\mathfrak{K}'$ is $\mathbf{d}$-inconsistent. Since this holds for any formula ϕ, the calculus is complete. We give an indirect proof by proving the contrapositive of the statement:

If $\mathfrak{K} \nvdash \mathbf{d} : \top \sqsubseteq \bot$ then $\mathfrak{K}$ is $\mathbf{d}$-satisfiable.

The proof is a variation of the Henkin construction of a model based on constants (see e.g. [24]). We will show that a model of $\mathfrak{K}$ can be constructed by gradually enriching $\mathfrak{K}$ with additional assertions that are compatible with it. Once this is exhaustingly done, the model can be constructed from the enriched version of $\mathfrak{K}$. Since a CKR need not to have a finite model (because $\mathcal{SROIQ}$ knowledge bases need not to have finite models) we will also enrich the object alphabet Σ with an infinite set of constants Ξ so that all elements in the interpretation domain of the model that is being constructed are covered by constants. Thus the model will be encoded inside the enriched version of $\mathfrak{K}$.

Since in each CKR model, a local model $\mathcal{I}_{\mathbf{f}}$ is totally encoded inside each $\mathcal{I}_{\mathbf{e}}$ such that $\mathbf{f} \prec \mathbf{e}$, to construct any model, we need to pay special attention the so called roof contexts of $\mathfrak{K}$, i.e., those which have no super-context. We will denote this set of contexts by E. Finally, in the following definition we construct the CKR $\mathfrak{K}^E$ by enriching $\mathfrak{K}$ as discussed above. Later on we will show how model of $\mathfrak{K}$ is encoded within it.

Definition 22 ($\mathfrak{K}^E$)**.** *Given a CKR over $\langle \Gamma, \Sigma \rangle$, let $E = \{\mathbf{e} \in \mathfrak{D}_\Gamma \mid (\forall \mathbf{f} \in \mathfrak{D}_\Gamma)\ \mathbf{e} \not\preceq \mathbf{f}\}$. Let $\Xi = \bigcup_{\mathbf{e} \in E} \Xi_{\mathbf{e}}$, where for each $\mathbf{e} \in E$, $\Xi_{\mathbf{e}} = \{x_i^{\mathbf{e}} | i \geq 0\}$ is a countably infinite set of new constants not appearing in Σ. Let us inductively construct $\mathfrak{K}^E$ as follows:*

$\mathfrak{K}^\emptyset = \mathfrak{K} \cup \{\mathbf{d} : \top(x_0^{\mathbf{d}})\}$;

given $F \subset E$ and $\mathbf{e} \in E$ such that $\mathbf{e} \notin F$, if $\mathfrak{K}^F \vdash \mathbf{e} : \top \sqsubseteq \bot$ then $\mathfrak{K}^{F \cup \{\mathbf{e}\}} = \mathfrak{K}^F$, otherwise $\mathfrak{K}^{F \cup \{\mathbf{e}\}}$ in constructed in two steps:

step 1: we add witnesses for all existential statements. Let $\phi_1, \phi_2, \ldots$ be an exhaustive enumeration of all assertions of the form $\exists R.C(a)$ or $\geqslant nR.C(a)$ in the vocabulary Σ extended with $\Xi_{\mathbf{e}}$. Note that R is possibly an inverse role and C is any complex concept. We inductively construct $\mathfrak{K}^{F,m}$ for $m \geq 0$ as follows:

$$\mathfrak{K}^{F,0} = \mathfrak{K}^F$$

$$\mathfrak{K}^{F,m+1} = \begin{cases} \mathfrak{K}^{F,m} \cup \{\mathbf{e} : \neg\exists R.C \sqcup \exists R.(\{x_k^{\mathbf{e}}\} \sqcap C)(a)\} \\ \qquad \text{if } \phi_{m+1} \text{ is of the form } \exists R.C(a) \\ \mathfrak{K}^{F,m} \cup \{\mathbf{e} : \neg{\geqslant} nR.C \sqcup {\geqslant} nR.(\{x_k^{\mathbf{e}}, \ldots, x_{k+n-1}^{\mathbf{e}}\} \sqcap C)(a)\} \\ \qquad \text{if } \phi_{m+1} \text{ is of the form } {\geqslant} nR.C(a) \end{cases}$$

where $x_k^{\mathbf{e}}, \ldots, x_{k+n-1}^{\mathbf{e}}$ are constants of $\Xi^{\mathbf{e}}$ not appearing in $\mathfrak{K}^{F,m}$. The result of this step is $\mathfrak{K}^{F,*} = \bigcup_{m \geq 0} \mathfrak{K}^{F,m}$;

step 2: we saturate $\mathfrak{K}^{F \cup \{e\}}$ with respect to all atomic assertions on the constants of Σ extended with with $\Xi^{\mathbf{e}}$. Let $\phi_1, \phi_2, \ldots$ be a complete enumeration of assertions of the form $A(a)$, $R(a, b)$, or $a = b$, where A and R are are atomic concepts and roles of Σ, and a and b are constants of Σ or $\Xi_{\mathbf{e}}$ such that $\mathfrak{K}^{F,*} \vdash \mathbf{e} : \top(a)$ and $\mathfrak{K}^{F,*} \vdash \mathbf{e} : \top(b)$. We inductively construct $\mathfrak{K}_m^{F,*}$ for $m \geq 0$, as follows:

$$\mathfrak{K}_0^{F,*} = \mathfrak{K}^{F,*}$$

$$\mathfrak{K}_{m+1}^{F,*} = \begin{cases} \mathfrak{K}_m^{F,*} \cup \{\mathbf{e} : \phi\} & \text{if } \mathfrak{K}_m^{F,*} \cup \{\mathbf{e} : \phi_{m+1}\} \text{ is } \mathbf{e}\text{-consistent} \\ \mathfrak{K}_m^{F,*} \cup \{\mathbf{e} : \neg\phi_{m+1}\} & \text{otherwise} \end{cases}$$

where by $\neg\, a = b$ we mean the non-equality assertion $a \neq b$ for any a and b. The result of this step is $\mathfrak{K}^{F \cup \{\mathbf{e}\}} = \bigcup_{m \geq 0} \mathfrak{K}_m^{F,*}$.

Thus at the end of the construction given in the previous definition we reach the knowledge base $\mathfrak{K}^E$. The following two lemmata show that $\mathfrak{K}^E$ is $\mathbf{d}$-consistent. This is an important step in order to guarantee that the model of $\mathfrak{K}$ encoded in $\mathfrak{K}^E$ is valid. The first lemma shows that adding existential witnesses in the construction has no influence at the $\vdash$ consequence at all. The second lemma then proves that starting from **d**-consistent knowledge base $\mathfrak{K}$, we end up with a **d**-consistent knowledge base $\mathfrak{K}^E$.

Lemma 5. *For every assertion ϕ in Σ that does not contain any occurrence of $x_i^{\mathbf{e}}$, for every $F \subseteq E$, and for every $\mathbf{f} \in \mathfrak{D}_\Gamma$: $\mathfrak{K}^F \vdash \mathbf{f} : \phi$ iff $\mathfrak{K}^{F,*} \vdash \mathbf{f} : \phi$.*

Proof. The "only if" direction trivially follows due to monotonicity of the language. As $\mathfrak{K}^F \subseteq \mathfrak{K}^{F,*}$, everything that is proved from $\mathfrak{K}^F$ is also proved from its superset $\mathfrak{K}^{F,*}$.

The "if" direction. Suppose $\mathfrak{K}^{F,*} \vdash \mathbf{f} : \phi$. We first show, that there is a finite subset of S of $\mathfrak{K}^{F,*}$ such that $S \vdash \mathbf{f} : \phi$. This is due to the length of the proof of $\mathbf{f} : \phi$ from $\mathfrak{K}^{F,*}$ is finite and in each step we use exactly one inference rule which derives its conclusion from a finite number of premises. Let us denote the set of all premises used by all the inference rules in the proof by P. Obviously the set P is finite and $P \vdash \mathbf{f} : \phi$. The formulae in P are either from $\mathfrak{K}^{F,*}$ or were derived as the proof goes on. Let $S = \{\phi \in P | \phi \in \mathfrak{K}^{F,*}\}$. Since all the formulae which we discarded are consecutively derived as the proof goes on, then also $S \vdash \mathbf{f} : \phi$, and by construction $S \subseteq \mathfrak{K}^{F,*}$.

Since for every finite subset of $\mathfrak{K}^{F,*}$ there is a $\mathfrak{K}^{F,m}$ that contains such a subset, we have that $\mathfrak{K}^{F,m} \vdash \mathbf{f} : \phi$, for some m; let us denote this deduction by Π. If $m = 0$ then $\mathfrak{K}^{F,m} = \mathfrak{K}^F$ and we are done. If $m > 0$ we will show that in this case also $\mathfrak{K}^{F,m-1} \vdash \mathbf{f} : \phi$. We distinguish two cases:

<u>case 1:</u> $\mathfrak{K}^{F,m} = \mathfrak{K}^{F,m-1} \cup \{\mathbf{e} : \neg\exists R.C \sqcup \exists R.(\{x_k^{\mathbf{e}}\} \sqcap C)(a)\}$. Then starting from Π we construct a deduction of $\mathbf{f} : \phi$ from $\mathfrak{K}^{F,m-1}$:

(1)	$\mathbf{e} : \neg\exists R.C \sqcup \exists R.C(a)$	$\mathfrak{K}^{F,m-1}$	LReas
(2)	$\mathbf{e} : \neg\exists R.C(a)$	(2)	assumption
(3)	$\mathbf{e} : \neg\exists R.C \sqcup \exists R.(\{x_k^{\mathbf{e}}\} \sqcap C)(a)$	(2)	from (2) by LReas
(4)	$\mathbf{f} : \phi$	(2), $\mathfrak{K}^{F,m-1}$	from (3) by Π
(5)	$\mathbf{e} : \exists R.C(a)$	(5)	assumption
(6)	$\mathbf{e} : R(a, x_k^{\mathbf{e}})$	(6)	assumption
(7)	$\mathbf{e} : C(x_k^{\mathbf{e}})$	(7)	assumption
(8)	$\mathbf{e} : \neg\exists R.C \sqcup \exists R.(\{x_k^{\mathbf{e}}\} \sqcap C)(a)$	(6), (7)	from (6,7) by LReas
(9)	$\mathbf{f} : \phi$	(6), (7), $\mathfrak{K}^{F,m-1}$	from (8) by Π
(10)	$\mathbf{f} : \phi$	(5), $\mathfrak{K}^{F,m-1}$	from (5,9) by $\exists$E disc. (6) and (7)
(11)	$\mathbf{f} : \phi$	$\mathfrak{K}^{F,m-1}$	from (1,4,10) by $\sqcup$E disc. (2) and (5)

<u>case 2:</u> $\mathfrak{K}^{F,m} = \mathfrak{K}^{F,m-1} \cup \{\mathbf{e} : \neg\geqslant nR.C \sqcup \geqslant nR.(\{x_k^{\mathbf{e}}, \ldots, x_{k+n-1}^{\mathbf{e}}\} \sqcap C)(a)\}$. As before, starting from Π, we construct a deduction of $\mathbf{f} : \phi$ from $\mathfrak{K}^{F,m-1}$:

(1)	$\mathbf{e} : \neg\geqslant nR.C \sqcup \geqslant nR.C(a)$	$\mathfrak{K}^{F,m-1}$	LReas
(2)	$\mathbf{e} : \neg\geqslant nR.C(a)$	(2)	assumption
(3)	$\mathbf{e} : \neg\geqslant nR.C \sqcup \geqslant nR.(\{x_k^{\mathbf{e}}, \ldots, x_{k+n-1}^{\mathbf{e}}\} \sqcap C)(a)$	(2)	from (2) by LReas
(4)	$\mathbf{f} : \phi$	(2), $\mathfrak{K}^{F,m-1}$	from (3) by Π
(5)	$\mathbf{e} : \geqslant nR.C(a)$	(5)	assumption
(6_i)	$\mathbf{e} : R(a, x_{k+i}^{\mathbf{e}})$	(6_i)	assumption $(0 \leq i < n)$
(7_i)	$\mathbf{e} : C(x_{k+i}^{\mathbf{e}})$	(7_i)	assumption $(0 \leq i < n)$
(8_{ij})	$\mathbf{e} : x_i^{\mathbf{e}} \neq x_j^{\mathbf{e}}$	(8_{ij})	assumption $(0 \leq i < n, i \neq j)$
(9)	$\mathbf{e} : \neg\geqslant nR.C \sqcup \geqslant nR.(\{x_k^{\mathbf{e}}, \ldots, x_{k+n-1}^{\mathbf{e}}\} \sqcap C)(a)$	$(6_i), (7_i), (8_i)_{0 \leq i \neq j < n}$	from (6,7) by LReas
(10)	$\mathbf{f} : \phi$	$(6_i), (7_i), (8_i)_{0 \leq i \neq j < n}, \mathfrak{K}^{F,m-1}$	from (9) by Π
(11)	$\mathbf{f} : \phi$	(5), $\mathfrak{K}^{F,m-1}$	from (5,10) by $(\geqslant n)$E disc. $(6_i, 7_i, 8_{ij})$
(12)	$\mathbf{f} : \phi$	$\mathfrak{K}^{F,m-1}$	from (1,4,11) by $\sqcup$E disc. (2) and (5)

In both cases $\mathfrak{K}^{F,m-1} \vdash \mathbf{f} : \phi$. Since this holds for any $m > 0$ it follows by induction that $\mathfrak{K}^F \vdash \mathbf{f} : \phi$. □

Lemma 6. *For every $F \subseteq E$, $\mathfrak{K}^F$ is* **d***-consistent.*

Proof. The proof is by induction on F. In the base case, $F = \emptyset$. From the construction $\mathfrak{K}^{\emptyset} = \mathfrak{K} \cup \{\mathbf{d} : \top(x_0^{\mathbf{d}})\}$, where $x_0^{\mathbf{d}}$ does not appear in $\mathfrak{K}$. Suppose the contrary, i.e., $\mathfrak{K}, \mathbf{d} : \top(x_0^{\mathbf{d}}) \vdash \top \sqsubseteq \bot$. As $x_0^{\mathbf{d}}$ does not appear in $\mathfrak{K}$, by application of the aE rule we have $\mathfrak{K} \vdash \top \sqsubseteq \bot$ which is a contradicts the assumption that $\mathfrak{K}$ is **d**-consistent. Therefore it must be case that $\mathfrak{K}^{\emptyset}$ is **d**-consistent.

In the induction step, $F \subseteq E$ is nonempty. Hence there is some $\mathbf{e} \in F$. Let us denote by $G = F \setminus \{\mathbf{e}\}$. From the induction hypothesis $\mathfrak{K}^G$ is **d**-consistent. We first prove by induction on m that $\mathfrak{K}_m^{G,*}$ is **d**-consistent. For $m = 0$, $\mathfrak{K}_0^{G,*} = \mathfrak{K}^{G,*}$ which **d**-consistent, as follows from Lemma 5: $\mathfrak{K}^G \nvdash \top \sqsubseteq \bot$, and $\mathfrak{K}^G \vdash \phi$ iff $\mathfrak{K}^{G,*} \vdash \phi$, therefore $\mathfrak{K}^{G,*} \nvdash \top \sqsubseteq \bot$.

Now suppose that $\mathfrak{K}_m^{G,*}$ is **d**-consistent, and let us show that $\mathfrak{K}_{m+1}^{G,*}$ is **d**-consistent. Suppose by contradiction that $\mathfrak{K}_{m+1}^{G,*}$ is not **d**-consistent. By definition this means that both $\mathfrak{K}_m^{G,*} \cup \{\mathbf{e} : \phi\}$ and $\mathfrak{K}_m^{G,*} \cup \{\mathbf{e} : \neg\phi\}$ are not **d**-consistent. Let Π_1 and Π_2 be two deductions of $\mathbf{d} : \top \sqsubseteq \bot$ from $\mathfrak{K}_m^{G,*} \cup \{\mathbf{e} : \phi\}$ and $\mathfrak{K}_m^{G,*} \cup \{\mathbf{e} : \neg\phi\}$ respectively. We will show that there exists also deduction of $\mathbf{d} : \top \sqsubseteq \bot$ from $\mathfrak{K}_m^{G,*}$. We distinguish three cases:

<u>case 1:</u> if ϕ is $A(a)$

(1)	$\mathbf{e} : \top(a)$	$\mathfrak{K}_m^{G,*}$	by construction, for evey constant x occurring in ϕ we have $\mathfrak{K}_m^{G,*} \vdash \mathbf{e} : \top(x)$
(2)	$\mathbf{e} : A \sqcup \neg A(a)$	$\mathfrak{K}_m^{G,*}$	from (1) by LReas
(3)	$\mathbf{e} : A(a)$	(3)	assumption
(4)	$\mathbf{d} : \top \sqsubseteq \bot$	(3), $\mathfrak{K}_m^{G,*}$	From (3) by Π_1
(5)	$\mathbf{e} : \neg A(a)$	(5)	assumption
(6)	$\mathbf{d} : \top \sqsubseteq \bot$	(5), $\mathfrak{K}_m^{G,*}$	From (5) by Π_2
(7)	$\mathbf{d} : \top \sqsubseteq \bot$	$\mathfrak{K}_m^{G,*}$	From (2), (4), and (6) by $\sqcup$E, disc. (3) and (5)

case 2: if ϕ is $R(a,b)$

(1)	$\mathbf{e} : \top(a)$	$\mathfrak{K}_m^{G,*}$	by construction, for evey constant x occurring in ϕ, $\mathfrak{K}_m^{G,*} \vdash \mathbf{e} : \top(x)$
(2)	$\mathbf{e} : \top(b)$	$\mathfrak{K}_m^{G,*}$	as for (1)
(3)	$\mathbf{e} : (\exists R.\{b\} \sqcup \neg\exists R.\{b\})(a)$	$\mathfrak{K}_m^{G,*}$	from (1) and (2) by LReas
(4)	$\mathbf{e} : \exists R.\{b\}(a)$	(4)	assumption
(5)	$\mathbf{e} : R(a,b)$	(4)	from (4) by LReas
(6)	$\mathbf{d} : \top \sqsubseteq \bot$	(4), $\mathfrak{K}_m^{G,*}$	From (5) by Π_1
(7)	$\mathbf{e} : \neg\exists R.\{b\}(a)$	(7)	assumption
(8)	$\mathbf{e} : \neg R(a,b)$	(7)	from (7) by LReas
(9)	$\mathbf{d} : \top \sqsubseteq \bot$	(7), $\mathfrak{K}_m^{G,*}$	From (7) by Π_2
(10)	$\mathbf{d} : \top \sqsubseteq \bot$	$\mathfrak{K}_m^{G,*}$	From (3), (6), and (9) by $\sqcup$E, disc. (4) and (7)

case 3: if ϕ is $a = b$

(1)	$\mathbf{e} : \top(a)$	$\mathfrak{K}_m^{G,*}$	by construction, for all constants x occurring in ϕ we have $\mathfrak{K}_m^{G,*} \vdash \mathbf{e} : \top(x)$
(2)	$\mathbf{e} : \top(b)$	$\mathfrak{K}_m^{G,*}$	as for (1)
(3)	$\mathbf{e} : \{b\} \sqcup \neg\{b\}(a)$	$\mathfrak{K}_m^{G,*}$	from (1) and (2) by LReas
(4)	$\mathbf{e} : \{b\}(a)$	(4)	assumption
(5)	$\mathbf{e} : a = b$	(4)	from (4) by LReas
(6)	$\mathbf{d} : \top \sqsubseteq \bot$	(4), $\mathfrak{K}_m^{G,*}$	From (5) by Π_1
(7)	$\mathbf{e} : \neg\{b\}(a)$	(7)	assumption
(8)	$\mathbf{e} : \neg a = b$	(7)	from (7) by LReas
(9)	$\mathbf{d} : \top \sqsubseteq \bot$	(7), $\mathfrak{K}_m^{G,*}$	From (7) by Π_2
(10)	$\mathbf{d} : \top \sqsubseteq \bot$	$\mathfrak{K}_m^{G,*}$	From (3), (6), (9) by $\sqcup$E, disc. (4) and (7)

We therefore conclude that if $\mathfrak{K}_{m+1}^{G,*} \vdash \mathbf{d} : \top \sqsubseteq \bot$, then $\mathfrak{K}_m^{G,*} \vdash \mathbf{d} : \top \sqsubseteq \bot$, which is a contradiction because by the induction hypothesis $\mathfrak{K}_m^{G,*}$ is $\mathbf{d}$-consistent. We have now showed that $\mathfrak{K}_m^{G,*}$ is $\mathbf{d}$-consistent for all $m \geq 0$. From the construction $\mathfrak{K}_m^{G,*} \subseteq \mathfrak{K}_{m+1}^{G,*}$, and therefore $\mathfrak{K}^F = \bigcup_{m \geq 0} \mathfrak{K}_m^{G,*}$ is $\mathbf{d}$-consistent too. □

We have showed that $\mathfrak{K}^E$ is $\mathbf{d}$-consistent. As the next step, in order to prove that a model of $\mathfrak{K}$ is encoded within it, we will show that for any ABox assertion ϕ of the form $C(a)$ or $R(a,b)$ we are able to verify by the CKR calculus if ϕ holds or if $\neg\phi$ holds. Recall that we have assured this by construction of $\mathfrak{K}^E$ for all atomic assertions. For non-atomic assertions it is shown in the following lemma. Combined with the fact that $\mathfrak{K}^E$ contains a constant for every domain element of the model, we will subsequently be able to retrieve and construct the model.

Lemma 7. *Given a CKR $\mathfrak{K}$ over $\langle\Gamma, \Sigma\rangle$, and $\mathfrak{K}^E$ constructed as in Definition 22. Let C and R be a possibly complex concept and a possibly inverse role over Σ, and let a, b any individuals of Σ such that $\mathfrak{K}^E \vdash \mathbf{e} : \top(a)$ and $\mathfrak{K}^E \vdash \mathbf{e} : \top(b)$. Then for any $\mathbf{e} \in \mathfrak{D}_\Gamma$:*

1. *if $\mathfrak{K}^E \vdash \mathbf{e} : \top(x)$ for all individuals x in C, then either $\mathfrak{K}^E \vdash \mathbf{e} : C(a)$ or $\mathfrak{K}^E \vdash \mathbf{e} : \neg C(a)$;*
2. *either $\mathfrak{K}^E \vdash \mathbf{e} : R(a,b)$ or $\mathfrak{K}^E \vdash \mathbf{e} : \neg R(a,b)$.*

Proof. The proof is by structural induction C and R:

1. $C = A$ is atomic: then the lemma follows directly from the construction of $\mathfrak{K}^E$;
2. $C = \neg D$: by induction either $\mathfrak{K}^E \vdash \mathbf{e} : D(a)$ or $\mathfrak{K}^E \vdash \mathbf{e} : \neg D(a)$, which implies either $\mathfrak{K}^E \vdash \mathbf{e} : \neg\neg D(a)$ or $\mathfrak{K}^E \vdash \mathbf{e} : \neg D(a)$ by LReas, that is, either $\mathfrak{K}^E \vdash \mathbf{e} : \neg C(a)$ or $\mathfrak{K}^E \vdash \mathbf{e} : C(a)$;
3. $C = F \sqcap G$: suppose $\mathfrak{K}^E \nvdash \mathbf{e} : F \sqcap G(a)$. Then $\mathfrak{K}^E \nvdash \mathbf{e} : F(a)$ or $\mathfrak{K}^E \nvdash \mathbf{e} : G(a)$ due to LReas. By induction hypothesis $\mathfrak{K}^E \vdash \mathbf{e} : \neg F(a)$ or $\mathfrak{K}^E \vdash \mathbf{e} : \neg G(a)$. Finally $\mathfrak{K}^E \vdash \mathbf{e} : \neg(F \sqcap G)(a)$ by LReas;
4. $C = F \sqcup G$: suppose $\mathfrak{K}^E \nvdash \mathbf{e} : F \sqcup G(a)$. This implies that $\mathfrak{K}^E \nvdash \mathbf{e} : F(a)$ and $\mathfrak{K}^E \nvdash \mathbf{e} : G(a)$ by LReas. By induction hypothesis we have $\mathfrak{K}^E \vdash \mathbf{e} : \neg F(a)$ and $\mathfrak{K}^E \vdash \mathbf{e} : \neg G(a)$ which implies $\mathfrak{K}^E \vdash \mathbf{e} : \neg(F \sqcup G)(a)$ by LReas;
5. $C = \exists R.D$: from the construction of $\mathfrak{K}^E$ we have $\mathfrak{K}^E \vdash \mathbf{e} : \neg\exists R.D \sqcup \exists R.(\{x^{\mathbf{e}}\} \sqcap D)(a)$ for some x^e.
   From the construction of $\mathfrak{K}^E$ we know that either $\mathfrak{K}^E \vdash \mathbf{e} : \phi$ or $\mathfrak{K}^E \vdash \mathbf{e} : \neg\phi$ for any assertion ϕ. Since $R(a, x^{\mathbf{e}})$ and $D(x^{\mathbf{e}})$ are assertions, one of the three cases must occur:
   - $\mathfrak{K}^E \vdash \mathbf{e} : R(a, x^{\mathbf{e}})$ and $\mathfrak{K}^E \vdash \mathbf{e} : D(x^{\mathbf{e}})$: in this case $\mathfrak{K}^E \vdash \mathbf{e} : \exists R.D(a)$ directly by LReas;
   - $\mathfrak{K}^E \vdash \mathbf{e} : \neg R(a, x^{\mathbf{e}})$: in this case $\mathfrak{K}^E \vdash \neg\exists R.(\{x^{\mathbf{e}}\} \sqcap D)(a)$ and since we have $\mathfrak{K}^E \vdash \mathbf{e} : \neg\exists R.D \sqcup \exists R.(\{x^{\mathbf{e}}\} \sqcap D)(a)$ then $\mathfrak{K}^E \vdash \mathbf{e} : \neg\exists R.D$, both steps by LReas;
   - $\mathfrak{K}^E \vdash \mathbf{e} : \neg D(x^{\mathbf{e}})$: in this case again $\mathfrak{K}^E \vdash \neg\exists R.(\{x^{\mathbf{e}}\} \sqcap D)(a)$ and hence $\mathfrak{K}^E \vdash \mathbf{e} : \neg\exists R.D$ by LReas;
6. $C = \geqslant nR.D$: analogously to the previous case;
7. $C = \forall R.D$: can be rewritten as $\neg\exists R.\neg D$;
8. $C = \leqslant nR.D$: can be rewritten as $\neg \geqslant n{+}1 R.D$;
9. $C = \exists R.\mathrm{Self}$: by construction we have that $\mathfrak{K}^E \vdash \mathbf{e} : R(a,a)$ or that $\mathfrak{K}^E \vdash \mathbf{e} : \neg R(a,a)$. By LReas this implies that $\mathfrak{K}^E \vdash \mathbf{e} : \exists R.\mathrm{Self}(a)$ or $\mathfrak{K}^E \vdash \mathbf{e} : \neg\exists R.\mathrm{Self}(a)$;
10. $C = \{x\}$ is a nominal: by construction we have $\mathfrak{K}^E \vdash \mathbf{e} : a = x$ or $\mathfrak{K}^E \vdash \mathbf{e} : \neg a = x$. By LReas this gives us $\mathfrak{K}^E \vdash \mathbf{e} : \{x\}(a)$ or $\mathfrak{K}^E \vdash \mathbf{e} : \neg\{x\}(a)$;
11. $R = S^-$ is an inverse role: by induction we have either $\mathfrak{K}^E \vdash \mathbf{e} : S(b,a)$ or $\mathfrak{K}^E \vdash \mathbf{e} : \neg S(b,a)$, by LReas we get either $\mathfrak{K}^E \vdash \mathbf{e} : R(a,b)$ or $\mathfrak{K}^E \vdash \mathbf{e} : \neg R(a,b)$.

□

The next step of the proof is the construction of a CKR interpretation from $\mathfrak{K}^E$ in that will then be shown to be a model of $\mathfrak{K}$. The basic idea is to take the constants that appear in $\mathfrak{K}^E$ as the interpretation domain. Relying on the previous lemma, we would then construct the interpretation of of each concept C in $C_{\mathbf{e}}$ by querying the calculus whether $\mathbf{e} : C(a)$ holds or not for each constant a, and analogously for roles. A minor problem with this approach is that in a CKR interpretation, if two constants are equal, they are interpreted by the same element of the interpretation domain.

Therefore we have to modify the nave construction as outlined above, and add just one domain element for all constants that are equal. We will achieve this by introducing an equivalence relation on constants and use the equivalence classes as domain elements.

Definition 23 ($\sim_{\mathbf{e}}$). *Let $\mathfrak{K}$ be a CKR over $\langle\Gamma, \Sigma\rangle$ and let E, Ξ, and $\mathfrak{K}^E$ be as constructed in Definition 22. For each $\mathbf{e} \in \mathfrak{D}_\Gamma$, the binary relation $\sim_{\mathbf{e}}$ on the set of constants of Σ extended with Ξ is defined as follows:*

$$a \sim_{\mathbf{e}} b \text{ iff } \mathfrak{K}^E \vdash \mathbf{e} : a = b$$

The equivalence class of $\sim_{\mathbf{e}}$ respective to a constant x will be denoted by $[x]_{\mathbf{e}}$, i.e., $[x]_{\mathbf{e}} = \{y | \mathfrak{K}^E \vdash \mathbf{e} : x = y\}$.

We now show that $\sim_{\mathbf{e}}$ is defined exactly on the set of constants that are relevant with respect to $C_{\mathbf{e}}$. And that it is really an equivalence relation on this set. This will justify the construction of the model that will then follow.

By constants relevant to $C_{\mathbf{e}}$ we mean all constants a such that the calculus proves $\mathfrak{K}^E \vdash \mathbf{e} : \top(a)$. The following lemma shows that for any such constant there is the respective equivalence class of $\sim_{\mathbf{e}}$ is nonempty. This ensures that we will find a domain element to interpret each constant when we construct the model below.

Lemma 8. *Given a CKR $\mathfrak{K}$ over $\langle\Gamma,\Sigma\rangle$, its extension $\mathfrak{K}^E$, and the relation $\sim_{\mathbf{e}}$ as defined in Definitions 22 and 23, then for any constant a of Σ and for any $\mathbf{e} \in \mathfrak{D}_\Gamma$ the following holds:*

$$[a]_{\mathbf{e}} \neq \emptyset \iff \mathfrak{K}^E \vdash \mathbf{e} : \top(a)$$

Proof. If $[a]_{\mathbf{d}} \neq \emptyset$ then there is a b such that $\mathfrak{K}^E \vdash \mathbf{e} : a = b$, which implies that $\mathfrak{K}^E \vdash \mathbf{e} : \top(a)$. Vice versa, if $\mathfrak{K}^E \vdash \mathbf{e} : \top(a)$ then by LReas we have $\mathfrak{K}^E \vdash \mathbf{e} : a = a$. Therefore by definition $a \in [a]_{\mathbf{e}}$, which implies $[a]_{\mathbf{e}} \neq \emptyset$. □

To justify the existence of equivalence classes we must also show that $\sim_{\mathbf{e}}$ is an equivalence relation.

Lemma 9. *The relation $\sim_{\mathbf{e}}$ is an equivalence relation on the set $\{a \mid \mathfrak{K}^E \vdash \mathbf{e} : \top(a)\}$.*

Proof. Equivalence is defined as a reflexive, symmetric, and transitive binary relation. Let us show that $\sim_{\mathbf{e}}$ has all these properties:

- reflexivity: $\mathfrak{K}^E \vdash \mathbf{e} : \top(a)$ implies $\mathfrak{K}^E \vdash \mathbf{e} : a = a$, which implies that $a \sim_{\mathbf{e}} a$;
- symmetry: $a \sim_{\mathbf{e}} b$ implies $\mathfrak{K}^E \vdash \mathbf{e} : a = b$, by LReas this implies that $\mathfrak{K}^E \vdash \mathbf{e} : b = a$, and therefore $b \sim_{\mathbf{e}} a$;
- transitivity: $a \sim_{\mathbf{e}} b$ and $b \sim_{\mathbf{e}} c$ imply $\mathfrak{K}^E \vdash \mathbf{e} : a = b$ and $\mathfrak{K}^E \vdash \mathbf{e} : b = c$. By LReas we have that $\mathfrak{K}^E \vdash \mathbf{e} : a = c$, and therefore also $a \sim_{\mathbf{e}} c$.

□

Finally we construct a model $\mathfrak{I}$ of $\mathfrak{K}$, using the equivalence classes of $\sim_{\mathbf{e}}$ as domain elements for each local interpretation and retrieving the interpretation of concepts and roles from $\mathfrak{K}^E$ by the CKR calculus.

Definition 24 (Model construction)**.** *Given a CKR $\mathfrak{K}$ over $\langle\Gamma,\Sigma\rangle$, given $\mathfrak{K}^E$, and the equivalence relations $\sim_{\mathbf{e}}$ for each $\mathbf{e} \in \mathfrak{D}_\Gamma$ as defined in Definitions 22 and 23, let us construct an interpretation $\mathfrak{I} = \{\mathcal{I}_{\mathbf{e}}\}_{\mathbf{e}\in\mathfrak{D}_\Gamma}$ where for each $\mathbf{e} \in \mathfrak{D}_\Gamma$, the local interpretation $\mathcal{I}_{\mathbf{e}} = \left\langle \Delta_{\mathbf{e}}, \cdot^{\mathcal{I}_{\mathbf{e}}} \right\rangle$ is defined as follows:*

$$\Delta_{\mathbf{e}} = \{[x]_{\mathbf{e}} \mid \mathfrak{K}^E \vdash \mathbf{e} : \top(x)\}$$

$$a^{\mathcal{I}_{\mathbf{e}}} = \begin{cases} [a]_{\mathbf{e}} & \textit{if } \mathfrak{K}^E \vdash \mathbf{e} : \top(a) \\ \textit{undefined} & \textit{otherwise} \end{cases}$$

$$A^{\mathcal{I}_{\mathbf{e}}} = \{[x]_{\mathbf{e}} \mid \mathfrak{K}^E \vdash \mathbf{e} : A(x)\}$$

$$R^{\mathcal{I}_{\mathbf{e}}} = \{\langle [x]_{\mathbf{e}}, [y]_{\mathbf{e}} \rangle \mid \mathfrak{K}^E \vdash \mathbf{e} : R(x,y)\}$$

for any constant, atomic concept, and atomic role a, A, R. For complex concepts and roles the interpretation is inductively defined as given in Table 1.

We first prove that complex concepts and roles are well defined with respect to the CKR calculus, that is, any ABox assertion is satisfied by the constructed model if and only if it is proved by the calculus.

Lemma 10. *Given a CKR $\mathfrak{K}$ over $\langle\Gamma,\Sigma\rangle$, and given $\mathfrak{K}^E$ and $\mathfrak{I}$ as constructed in Definitions 22 and 24, then for all $\mathbf{e} \in \mathfrak{D}_\Gamma$, all constants a, b, all complex concepts C, and all possibly inverse roles R of Σ the following holds:*

- $[a]_{\mathbf{e}} \in C^{\mathcal{I}_{\mathbf{e}}}$ *iff* $\mathfrak{K}^E \vdash \mathbf{e} : C(a)$*;*
- $\langle [a]_{\mathbf{e}}, [b]_{\mathbf{e}} \rangle \in R^{\mathcal{I}_{\mathbf{e}}}$ *iff* $\mathfrak{K}^E \vdash \mathbf{e} : R(a,b)$*.*

Proof. By structural induction:

1. $C = A$ is atomic: this case follows directly from the construction;
2. $C = \neg D$: $[a]_{\mathbf{e}} \in (\neg D)^{\mathcal{I}_{\mathbf{e}}}$ iff $[a]_{\mathbf{e}} \in \Delta_{\mathbf{e}} \setminus D^{\mathcal{I}_{\mathbf{e}}}$ iff $\mathfrak{K}^E \vdash \mathbf{e} : \top(a)$ and $\mathfrak{K}^E \nvdash \mathbf{e} : D(a)$ (first from the construction, second from the induction hypothesis) iff $\mathfrak{K}^E \vdash \mathbf{e} : \top(a)$ and $\mathfrak{K}^E \vdash \mathbf{e} : \neg D(a)$ (from Lemma 7) iff $\mathfrak{K}^E \vdash \mathbf{e} : \neg D(a)$ (by LReas);
3. $C = F \sqcap G$: $[a]_{\mathbf{e}} \in (F \sqcap G)^{\mathcal{I}_{\mathbf{e}}}$ iff $[a]_{\mathbf{e}} \in F^{\mathcal{I}_{\mathbf{e}}}$ and $[a]_{\mathbf{e}} \in G^{\mathcal{I}_{\mathbf{e}}}$ iff $\mathfrak{K}^E \vdash \mathbf{e} : F(a)$ and $\mathfrak{K}^E \vdash \mathbf{e} : G(a)$ (by induction hypothesis) iff $\mathfrak{K}^E \vdash \mathbf{e} : F \sqcap G(a)$ (by LReas);
4. $C = F \sqcup G$: $[a]_{\mathbf{e}} \in (F \sqcup G)^{\mathcal{I}_{\mathbf{e}}}$ iff $[a]_{\mathbf{e}} \in F^{\mathcal{I}_{\mathbf{e}}}$ or $[a]_{\mathbf{e}} \in G^{\mathcal{I}_{\mathbf{e}}}$ iff $\mathfrak{K}^E \vdash \mathbf{e} : F(a)$ or $\mathfrak{K}^E \vdash \mathbf{e} : G(a)$ (by induction hypothesis) iff $\mathfrak{K}^E \vdash \mathbf{e} : F \sqcup G(a)$ (by LReas);
5. $C = \exists R.D$: $[a]_{\mathbf{e}} \in (\exists R.D)^{\mathcal{I}_{\mathbf{e}}}$ iff for some $[b]_{\mathbf{e}}$ $\langle [a]_{\mathbf{e}}, [b]_{\mathbf{e}} \rangle \in R^{\mathcal{I}_{\mathbf{e}}}$ and $[b]_{\mathbf{e}} \in D^{\mathcal{I}_{\mathbf{e}}}$ iff for some b we have $\mathfrak{K}^E \vdash \mathbf{e} : R(a,b)$ and $\mathfrak{K}^E \vdash \mathbf{e} : D(b)$ (by the construction and induction hypothesis) iff $\mathfrak{K}^E \vdash \mathbf{e} : \exists R.D(a)$ (by LReas);
6. $C = \geqslant nR.D$: analogously to the previous case;
7. $C = \forall R.D$: can be rewritten as $\neg\exists R.\neg D$;
8. $C = \leqslant nR.D$: can be rewritten as $\neg\geqslant n{+}1R.D$;
9. $C = \exists R.\mathrm{Self}$: $[a]_{\mathbf{e}} \in (\exists R.\mathrm{Self})^{\mathcal{I}_{\mathbf{e}}}$ iff $\langle [a]_{\mathbf{e}}, [a]_{\mathbf{e}} \rangle \in R^{\mathcal{I}_{\mathbf{e}}}$ iff $\mathfrak{K}^E \vdash \mathbf{e} : R(a,a)$ (from the construction) iff $\mathfrak{K}^E \vdash \mathbf{e} : \exists R.\mathrm{Self}(a)$ (by LReas);
10. $C = \{b\}$: $[a]_{\mathbf{e}} \in \{b\}^{\mathcal{I}_{\mathbf{e}}}$ iff $[a]_{\mathbf{e}} = [b]_{\mathbf{e}}$ iff $a \sim_{\mathbf{e}} b$ iff $\mathfrak{K}^E \vdash \mathbf{e} : a = b$ (by definition of $\sim_{\mathbf{e}}$) iff $\mathfrak{K}^E \vdash \mathbf{e} : \{b\}(a)$;
11. R is atomic: this case follows directly from the construction;
12. $R = S^-$ is an inverse role: $\langle [a]_{\mathbf{e}}, [b]_{\mathbf{e}} \rangle \in R^{\mathcal{I}_{\mathbf{e}}}$ iff $\langle [b]_{\mathbf{e}}, [e]_{\mathbf{e}} \rangle \in S^{\mathcal{I}_{\mathbf{e}}}$ iff $\mathfrak{K}^E \vdash \mathbf{e} : S(b,a)$ (by induction hypothesis) iff $\mathfrak{K}^E \vdash \mathbf{e} : R(a,b)$ (by LReas).

□

The last lemma that we need before we prove that $\mathfrak{I}$ is a model of $\mathfrak{K}$ is the following one which shows that the interpretation of constants match between $I_{\mathbf{e}}$ and $\mathcal{I}_{\mathbf{f}}$ for constants defined in $\Delta_{\mathbf{d}}$.

Lemma 11. *If $[a]_{\mathbf{e}} \neq \emptyset$ and $\mathbf{e} \prec \mathbf{f}$, then $[a]_{\mathbf{e}} = [a]_{\mathbf{f}}$.*

Proof. Let us prove that $[a]_{\mathbf{e}} \subseteq [a]_{\mathbf{f}}$: if $b \in [a]_{\mathbf{e}}$ then $\mathfrak{K}^E \vdash \mathbf{e} : a = b$ (by the construction), then also $\mathfrak{K}^E \vdash \mathbf{f} : a = b$ (by Pop), which finally gives us $b \in [a]_{\mathbf{f}}$ (again by the construction).

Vice versa, let us prove that $[a]_{\mathbf{f}} \subseteq [a]_{\mathbf{e}}$: if $b \in [a]_{\mathbf{f}}$ then $\mathfrak{K}^E \vdash \mathbf{f} : a = b$ from the construction of $\sim_{\mathbf{f}}$. We have assumed $[a]_{\mathbf{e}} \neq \emptyset$, hence by Lemma 8 we get $\mathfrak{K} \vdash \mathbf{e} : \top(a)$, which gives us $\mathfrak{K} \vdash \mathbf{f} : \top_{\mathbf{e}}(a)$ by Pop. Since $\mathfrak{K}^E \vdash \mathbf{f} : a = b$, we get $\mathfrak{K} \vdash \mathbf{f} : \top_{\mathbf{e}}(b)$ by LReas, which finally allows us to apply Push at $\mathfrak{K}^E \vdash \mathbf{f} : a = b$ and thus derive $\mathfrak{K}^E \vdash \mathbf{e} : a = b$ and hence from the construction of $\sim_{\mathbf{e}}$ we have $b \in [a]_{\mathbf{e}}$. □

We will now prove that $\mathfrak{I}$ is a CKR model for $\mathfrak{K}^E$, by showing that all conditions of Definition 9 are satisfied. Note that for sake of clarity in the following enumeration we will keep the same notation as in Definition 9 (i.e., the previously bound $\mathbf{d}$ is now any $\mathbf{d} \in \mathfrak{D}_\Gamma$):

1. $(\top_{\mathbf{d}})^{\mathcal{I}_{\mathbf{f}}} \subseteq (\top_{\mathbf{e}})^{\mathcal{I}_{\mathbf{f}}}$ if $\mathbf{d} \prec \mathbf{e}$: $[x]_{\mathbf{f}} \in (\top_{\mathbf{d}})^{\mathcal{I}_{\mathbf{f}}}$ iff $\mathfrak{K}^E \vdash \mathbf{f} : \top_{\mathbf{d}}(x)$. Since $\mathfrak{K}^E \vdash \mathbf{f} : \top_{\mathbf{d}} \sqsubseteq \top_{\mathbf{e}}$ by LReas, we have that $\mathfrak{K}^E \vdash \mathbf{f} : \top_{\mathbf{e}}(x)$, which implies that $[x]_{\mathbf{f}} \in (\top_{\mathbf{e}})^{\mathcal{I}_{\mathbf{f}}}$
2. $(C_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} \subseteq (\top_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}}$: $[x]_{\mathbf{d}} \in (C_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}}$ iff $\mathfrak{K}^E \vdash \mathbf{d} : C_{\mathbf{f}}(x)$. Since $\mathfrak{K}^E \vdash \mathbf{d} : C_{\mathbf{f}} \sqsubseteq \top_{\mathbf{f}}$ by LReas, we have that $\mathfrak{K}^E \vdash \mathbf{d} : \top_{\mathbf{f}}(x)$, which implies that $[x]_{\mathbf{d}} \in (\top_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}}$.
3. $(R_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} \subseteq (\top_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} \times (\top_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}}$: If $\langle [x]_{\mathbf{d}}, [y]_{\mathbf{d}} \rangle \in (R_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}}$, then $\mathfrak{K}^E \vdash \mathbf{d} : R_{\mathbf{f}}(x, y)$. Furthermore we have that $\mathfrak{K}^E \vdash \mathbf{d} : \top \sqsubseteq \forall R_{\mathbf{f}}.\top_{\mathbf{f}}$, and $\mathfrak{K}^E \vdash \mathbf{d} : \exists R_{\mathbf{f}}.\top \sqsubseteq \top_{\mathbf{f}}$, which implies that $\mathfrak{K}^E \vdash \mathbf{d} : \top_{\mathbf{f}}(x)$ and $\mathfrak{K}^E \vdash \mathbf{d} : \top_{\mathbf{f}}(y)$. This implies that $\langle [x]_{\mathbf{d}}, [y]_{\mathbf{d}} \rangle \in (\top_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}}$.
4. if $\mathbf{d} \prec \mathbf{e}$, and $a^{\mathcal{I}_{\mathbf{e}}} \in \Delta_{\mathbf{d}}$ then $a^{\mathcal{I}_{\mathbf{e}}} = a^{\mathcal{I}_{\mathbf{d}}}$. $[a]_{\mathbf{e}} \in \Delta_{\mathbf{d}}$ iff there is a b, such that $[b]_{\mathbf{d}} = [a]_{\mathbf{e}}$, and $[b]_{\mathbf{d}} \in \Delta_{\mathbf{d}}$. By Lemma 11 we have that $[b]_{\mathbf{d}} = [b]_{\mathbf{e}}$ which implies that $[b]_{\mathbf{e}} = [a]_{\mathbf{e}}$. This implies that $\mathfrak{K}^E \vdash \mathbf{e} : a = b$, and therefore that $\mathfrak{K}^E \vdash \mathbf{d} : a = b$, which implies $[b]_{\mathbf{d}} = [a]_{\mathbf{d}}$. Summing up, $a^{\mathcal{I}_{\mathbf{e}}} = [a]_{\mathbf{e}} = [b]_{\mathbf{d}} = [a]_{\mathbf{d}} = a^{\mathcal{I}_{\mathbf{d}}}$.
5. $(X_{\mathbf{d_B}})^{\mathcal{I}_{\mathbf{e}}} = (X_{\mathbf{d_B+e}})^{\mathcal{I}_{\mathbf{e}}}$: Let X be a concept C. $[x]_{\mathbf{e}} \in C_{\mathbf{d_B}}$, iff $\mathfrak{K}^E \vdash \mathbf{e} : C_{\mathbf{d_B}}(x)$. Since $\mathfrak{K}^E \vdash \mathbf{e} : C_{\mathbf{d_B}} \equiv C_{\mathbf{d_B+e}}$, we have that $\mathfrak{K}^E \vdash \mathbf{e} : C_{\mathbf{d_B}}(x)$ iff $\mathfrak{K}^E \vdash \mathbf{e} : C_{\mathbf{d_B+e}}(x)$, which holds iff $[x]_{\mathbf{e}} \in C_{\mathbf{d_B+e}}$. An analogous argument can be done if X is a role symbol.
6. $(X_{\mathbf{d}})^{\mathcal{I}_{\mathbf{e}}} = (X_{\mathbf{d}})^{\mathcal{I}_{\mathbf{d}}}$ if $\mathbf{d} \prec \mathbf{e}$. Let X be a concept symbol C. $[x]_{\mathbf{e}} \in (C_{\mathbf{d}})^{\mathcal{I}_{\mathbf{e}}}$ iff $\mathfrak{K}^E \vdash \mathbf{e} : C_{\mathbf{d}}(x)$ iff $\mathfrak{K}^E \vdash \mathbf{d} : C_{\mathbf{d}}(x)$ iff $[x]_{\mathbf{e}} \in (C_{\mathbf{d}})^{\mathcal{I}_{\mathbf{d}}}$. An analogous argument can be done if X is a role symbol.
7. $(C_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} = (C_{\mathbf{f}})^{\mathcal{I}_{\mathbf{e}}} \cap \Delta_{\mathbf{d}}$, if $\mathbf{d} \prec \mathbf{e}$: $[x]_{\mathbf{d}} \in (C_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}}$ iff $\mathfrak{K}^E \vdash \mathbf{d} : C_{\mathbf{f}}(x)$ iff $\mathfrak{K}^E \vdash \mathbf{e} : \top_{\mathbf{d}} \sqcap C_{\mathbf{f}}(x)$ iff $\mathfrak{K}^E \vdash \mathbf{e} : \top_{\mathbf{d}}$ and $\mathfrak{K}^E \vdash \mathbf{e} : C_{\mathbf{f}}(x)$ iff $[x]_{\mathbf{e}} \in (C_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}}$ and $[x]_{\mathbf{e}} \in (\top_{\mathbf{d}})^{\mathcal{I}_{\mathbf{d}}}$ iff $[x]_{\mathbf{e}} \in (C_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} \cap \Delta_{\mathbf{d}}$ iff $[x]_{\mathbf{d}} \in (C_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} \cap \Delta_{\mathbf{d}}$.
8. $(R_{\mathbf{f}})^{\mathcal{I}_{\mathbf{d}}} = (R_{\mathbf{f}})^{\mathcal{I}_{\mathbf{e}}} \cap (\Delta_{\mathbf{d}} \times \Delta_{\mathbf{d}})$, if $\mathbf{d} \prec \mathbf{e}$ The same argument as the previous point.
9. $\mathcal{I}_{\mathbf{d}} \models \phi$ for all $\mathbf{d} : \phi \in \mathfrak{K}^E$. Consider the four different axioms:
   (a) ϕ is $C(a)$: in this case $\mathfrak{K}^E \vdash \mathbf{d} : C(a)$, therefore $[a]_{\mathbf{d}} \in C^{\mathcal{I}_{\mathbf{d}}}$ follows from Lemma 10;
   (b) ϕ is $R(a, b)$ and ϕ is $\neg R(a, b)$: by construction $\langle [a]_{\mathbf{d}}, [b]_{\mathbf{d}} \rangle \in R^{\mathcal{I}_{\mathbf{d}}}$ or $\langle [a]_{\mathbf{d}}, [b]_{\mathbf{d}} \rangle \notin R^{\mathcal{I}_{\mathbf{d}}}$;
   (c) ϕ is $C \sqsubseteq D$: if $[x]_{\mathbf{d}} \in C^{\mathcal{I}_{\mathbf{d}}}$, then $\mathfrak{K}^E \vdash \mathbf{d} : C(x)$ by Lemma 10. Since in this case $\mathfrak{K}^E \vdash \mathbf{d} : C \sqsubseteq D$ by LReas we have that $\mathfrak{K}^E \vdash \mathbf{d} : D(x)$ and therefore that $[x]_{\mathbf{d}} \in D^{\mathcal{I}_{\mathbf{d}}}$ again by Lemma 10;
   (d) ϕ is $R_1 \circ \cdots \circ R_n \sqsubseteq R$: if $\langle [x]_{\mathbf{d}}, [y]_{\mathbf{d}} \rangle \in (R_1 \circ \cdots \circ R_n)^{\mathcal{I}_{\mathbf{d}}}$ then there must be $[z_1]_{\mathbf{d}}, \ldots, [z_{n-1}]_{\mathbf{d}}$ such that $\langle [x]_{\mathbf{d}}, [z_1]_{\mathbf{d}} \rangle \in R_1^{\mathcal{I}_{\mathbf{d}}}$, $\langle [z_1]_{\mathbf{d}}, [z_2]_{\mathbf{d}} \rangle \in R_2^{\mathcal{I}_{\mathbf{d}}}$, $\ldots$, $\langle [z_{n-1}]_{\mathbf{d}}, [y]_{\mathbf{d}} \rangle \in R_n^{\mathcal{I}_{\mathbf{d}}}$. From the construction we have $\mathfrak{K}^E \vdash \mathbf{d} : R_1(x, z_1)$, $\mathfrak{K}^E \vdash \mathbf{d} : R_2(z_1, z_2)$, $\ldots$, $\mathfrak{K}^E \vdash \mathbf{d} : R_n(z_{n-1}, y)$. Since in this case also $\mathfrak{K}^E \vdash \mathbf{d} : R_1 \circ \cdots \circ R_n \sqsubseteq R$ than by LReas we have $\mathfrak{K}^E \vdash R(x, y)$ and therefore $\langle [x]_{\mathbf{d}}, [y]_{\mathbf{d}} \rangle \in R$. Please note that this holds also in case that any of $R, R_1, \ldots, R_n$ is an inverse role.

We have just showed that $\mathfrak{I}$ is a model of $\mathfrak{K}$. During the construction of $\mathfrak{K}^E$ (Definition 22) we have added the formula $\mathbf{d} : \top(x_0^{\mathbf{d}})$ into $\mathfrak{K}^E$, therefore by LReas we have $\mathfrak{K}^E \vdash \mathbf{d} : \top(x_0^{\mathbf{d}})$. By the construction of the model $\mathfrak{I}$ (Definition 24), this implies $[x_0^{\mathbf{d}}] \in \Delta_{\mathbf{d}}$. Therefore $\mathfrak{K}$ is $\mathbf{d}$-consistent.

Appendix A.3. Proof of Lemma 3

Lemma 3. *If $\mathfrak{K}$ is $\mathbf{d}$-satisfiable then $\#(\mathfrak{K})$ is satisfiable.*

As $\mathfrak{K}$ is $\mathbf{d}$-satisfiable, there exists a model $\mathfrak{I}$ be a model of $\mathfrak{K}$ with $\Delta_{\mathbf{d}} \neq \emptyset$. Let us construct a DL interpretation $\mathcal{I} = \left\langle \Delta, \cdot^{\mathcal{I}} \right\rangle$ over $\#(\Gamma, \Sigma)$ as follows:

1. $\Delta = \bigcup_{\mathbf{d} \in \mathfrak{D}_\Gamma} \Delta_{\mathbf{d}} \cup \{x_{\mathsf{undef}}\}$ where x_{undef} is a new element not occurring in $\Delta_{\mathbf{d}}$ for all $\mathbf{d} \in \mathfrak{D}_\Gamma$;
2. $(a^{\mathbf{d}})^{\mathcal{I}} = a^{\mathcal{I}_{\mathbf{d}}}$ if $a^{\mathcal{I}_{\mathbf{d}}}$ is defined otherwise $(a^{\mathbf{d}})^{\mathcal{I}} = x_{\mathsf{undef}}$ for every individual a and for every $\mathbf{d} \in \mathfrak{D}_\Gamma$;
   $\mathsf{undef}^{\mathcal{I}} = x_{\mathsf{undef}}$;
3. $(A_{\mathbf{e}}^{\mathbf{d}})^{\mathcal{I}} = C_{\mathbf{e}}^{\mathcal{I}_{\mathbf{d}}}$ for every atomic concept C of Σ and for every $\mathbf{d}, \mathbf{e} \in \mathfrak{D}_\Gamma$;
4. $(R_{\mathbf{e}}^{\mathbf{d}})^{\mathcal{I}} = R_{\mathbf{e}}^{\mathcal{I}_{\mathbf{d}}}$ for every atomic role R of Σ and for every $\mathbf{d}, \mathbf{e} \in \mathfrak{D}_\Gamma$;
   $(S_R^{\mathbf{d},\mathbf{e},\mathbf{f}})^{\mathcal{I}} = (I_{\mathbf{d}}^{\mathbf{d}})^{\mathcal{I}} \circ (R_{\mathbf{f}}^{\mathbf{e}})^{\mathcal{I}}$ for every role R and for all $\mathbf{d}, \mathbf{e}, \mathbf{f} \in \mathfrak{D}_\Gamma$;

First of all it is apparent from the construction that $\Delta \neq \emptyset$. It remains to prove that $\mathcal{I}$ satisfies all axioms of $\#(\mathfrak{K})$ as given in Definition 21. The satisfaction of the axioms introduced in items 1–3 and 5–7 follows directly from the construction and from the corresponding condition of $\mathcal{I}$ being a CKR-model (Definition 9). Let us now show the satisfaction of the remaining axioms (items 4, 8, and 9).

The first type of axioms added in item 4 is $\top_{\mathbf{d}}^{\mathbf{d}} \sqcap \{a^{\mathbf{e}}\} \sqsubseteq \{a^{\mathbf{d}}\}$ for any individual a and for any two $\mathbf{d}, \mathbf{e} \in \mathfrak{D}_\Gamma$. Let $x \in (\top_{\mathbf{d}}^{\mathbf{d}} \sqcap \{a^{\mathbf{e}}\})^{\mathcal{I}}$, that is, $x \in (\top_{\mathbf{d}}^{\mathbf{d}})^{\mathcal{I}}$ and $x = (a^{\mathbf{e}})^{\mathcal{I}}$. Notice that $x \neq x_{\mathsf{undef}}$ because $x \in (\top_{\mathbf{d}}^{\mathbf{d}})^{\mathcal{I}} = \top_{\mathbf{d}}^{\mathcal{I}_{\mathbf{d}}} = \Delta_{\mathbf{d}}$ and the construction implies $x_{\mathsf{undef}} \notin \Delta_{\mathbf{d}}$. Altogether this implies that $a^{\mathcal{I}_{\mathbf{e}}} = (a^{\mathbf{e}})^{\mathcal{I}}$ is defined in $\mathfrak{I}$ and also $a^{\mathcal{I}_{\mathbf{e}}} \in \Delta_{\mathbf{d}} = (\top_{\mathbf{d}}^{\mathbf{d}})^{\mathcal{I}}$. From Condition 4 of the CKR model this implies that $x = a^{\mathcal{I}_{\mathbf{e}}} = a^{\mathcal{I}_{\mathbf{d}}}$ and therefore $x \in (\{a^{\mathbf{d}}\})^{\mathcal{I}}$ and hence the axiom is satisfied.

The second type of axioms added in item 4 is $\{a^{\mathbf{d}}\} \sqsubseteq \{a^{\mathbf{e}}, \mathsf{undef}\}$ for any individual a and for any two $\mathbf{d}, \mathbf{e} \in \mathfrak{D}_\Gamma$. That is, we have to show that either $(a^{\mathbf{d}})^{\mathcal{I}} = (a^{\mathbf{e}})^{\mathcal{I}}$ of $(a^{\mathbf{d}})^{\mathcal{I}} = \mathsf{undef}^{\mathcal{I}}$. If $a^{\mathcal{I}_\mathbf{d}}$ is defined, then from Condition 4 of CKR models we have $a^{\mathcal{I}_\mathbf{d}} = a^{\mathcal{I}_\mathbf{e}}$ and hence from the construction $(a^{\mathbf{d}})^{\mathcal{I}} = (a^{\mathbf{e}})^{\mathcal{I}}$. If $a^{\mathcal{I}_\mathbf{d}}$ is undefined, then directly from the construction $(a^{\mathbf{d}})^{\mathcal{I}} = \mathsf{undef}^{\mathcal{I}}$.

The last type of axioms added in item 4 is $\neg\top^{\mathbf{d}}_{\mathbf{d}}(\mathsf{undef})$ for any $\mathbf{d} \in \mathfrak{D}_\Gamma$. This follows directly from the construction of $\mathcal{I}$ as $\mathsf{undef}^{\mathcal{I}} = x_{\mathsf{undef}} \notin (\top^{\mathbf{d}}_{\mathbf{d}})^{\mathcal{I}} = \Delta_\mathbf{d}$.

The first type of axioms introduced in item 8 is $I^{\mathbf{d}}_{\mathbf{d}} \circ R^{\mathbf{e}}_{\mathbf{f}} \circ I^{\mathbf{d}}_{\mathbf{d}} \sqsubseteq R^{\mathbf{d}}_{\mathbf{f}}$ for any role R and for any $\mathbf{d}, \mathbf{e}, \mathbf{f} \in \mathfrak{D}_\Gamma$ with $\mathbf{d} \prec \mathbf{e}$. Let us first realize that $(I^{\mathbf{d}}_{\mathbf{d}})^{\mathcal{I}} = I^{\mathcal{I}_\mathbf{d}}_{\mathbf{d}}$, that is, it is the identity relation on $\Delta_\mathbf{d}$. Hence $(I^{\mathbf{d}}_{\mathbf{d}} \circ R^{\mathbf{e}}_{\mathbf{f}} \circ I^{\mathbf{d}}_{\mathbf{d}})^{\mathcal{I}} = (R^{\mathbf{e}}_{\mathbf{f}})^{\mathcal{I}} \cap (\Delta_\mathbf{d} \times \Delta_\mathbf{d})$. The fact that this set is a subset of $(R^{\mathbf{d}}_{\mathbf{f}})^{\mathcal{I}}$ follows as a consequence of the construction of $\mathcal{I}$ and Condition 8 of CKR models.

The second type of axioms introduced in item 8 is $R^{\mathbf{d}}_{\mathbf{f}} \sqsubseteq R^{\mathbf{e}}_{\mathbf{f}}$ for any role R and for any $\mathbf{d}, \mathbf{e}, \mathbf{f} \in \mathfrak{D}_\Gamma$ with $\mathbf{d} \prec \mathbf{e}$. Similarly to the previous case, Condition 8 of CKR models gives us $R^{\mathcal{I}_\mathbf{d}}_{\mathbf{f}} = R^{\mathcal{I}_\mathbf{e}}_{\mathbf{f}} \cap (\Delta_\mathbf{d} \times \Delta_\mathbf{d})$ and from the construction of $\mathcal{I}$ it follows that $(R^{\mathbf{d}}_{\mathbf{f}})^{\mathcal{I}} = R^{\mathcal{I}_\mathbf{d}}_{\mathbf{f}} = R^{\mathcal{I}_\mathbf{e}}_{\mathbf{f}} \cap (\Delta_\mathbf{d} \times \Delta_\mathbf{d}) \subseteq R^{\mathcal{I}_\mathbf{e}}_{\mathbf{f}} = (R^{\mathbf{e}}_{\mathbf{f}})^{\mathcal{I}}$.

In addition, step 8 (b) introduces two new axioms of thew form $I^{\mathbf{d}}_{\mathbf{d}} \circ R^{\mathbf{e}}_{\mathbf{f}} \sqsubseteq S^{\mathbf{d},\mathbf{e},\mathbf{f}}_{R}$ and $S^{\mathbf{d},\mathbf{e},\mathbf{f}}_{R} \circ I^{\mathbf{d}}_{\mathbf{d}} \sqsubseteq R^{\mathbf{d}}_{\mathbf{f}}$ for each role R and for any $\mathbf{d}, \mathbf{e}, \mathbf{f} \in \mathfrak{D}_\Gamma$ with $\mathbf{d} \prec \mathbf{e}$. The first axiom is trivially satisfied due to the construction of $\mathcal{I}$ which implies that $(S^{\mathbf{d},\mathbf{e},\mathbf{f}}_{R})^{\mathcal{I}} = (I^{\mathbf{d}}_{\mathbf{d}})^{\mathcal{I}} \circ (R^{\mathbf{e}}_{\mathbf{f}})^{\mathcal{I}}$. Similarly for the second axiom, if $x \in (S^{\mathbf{d},\mathbf{e},\mathbf{f}}_{R} \circ I^{\mathbf{d}}_{\mathbf{d}})^{\mathcal{I}}$ then $x \in (I^{\mathbf{d}}_{\mathbf{d}} \circ R^{\mathbf{e}}_{\mathbf{f}} \circ I^{\mathbf{d}}_{\mathbf{d}})^{\mathcal{I}}$ and since we have already proven that $\mathcal{I} \models I^{\mathbf{d}}_{\mathbf{d}} \circ R^{\mathbf{e}}_{\mathbf{f}} \circ I^{\mathbf{d}}_{\mathbf{d}} \sqsubseteq R^{\mathbf{d}}_{\mathbf{f}}$ then $x \in (R^{\mathbf{d}}_{\mathbf{f}})^{\mathcal{I}}$.

Finally item 9, that is, the fact that $\mathcal{I}$ satisfies also the axioms $\phi\#\mathbf{d}$, $\phi \in \mathrm{K}(C_\mathbf{d})$, is a consequence of the fact that $(\cdot)\#\mathbf{d}$ is defined on the basis of an embedding of Σ into $\#(\Gamma, \Sigma)$, and that each pair of interpretations $\mathcal{I}_\mathbf{d}$ of Σ and $\mathcal{I}$ of $\#(\Gamma, \Sigma)$ complies with this embedding. Then this item is a direct consequence of Lemma 1.

Appendix A.4. Proof of Lemma 4

Lemma 4. *If there is a* $\mathbf{d}$ *such that* $\#(\mathfrak{K}) \not\models \top^{\mathbf{d}}_{\mathbf{d}} \sqsubseteq \bot$, *then* $\mathfrak{K}$ *is* $\mathbf{d}$*-satisfiable.*

Given a CKR $\mathfrak{K}$ let $\mathcal{I}$ be a model of $\#(\mathfrak{K})$ such that $\top^{\mathbf{d}}_{\mathbf{d}}$ is not empty. This model exists since by hypothesis $\#(\mathfrak{K}) \not\models \top^{\mathbf{d}}_{\mathbf{d}} \sqsubseteq \bot$. Let us construct the CKR model $\mathfrak{I} = \{\mathcal{I}_\mathbf{d}\}_{\mathbf{d} \in \mathfrak{D}_\Gamma}$, where for every $\mathbf{d} \in \mathfrak{D}_\Gamma$, $\mathcal{I}_\mathbf{d} = \langle \Delta_\mathbf{d}, \mathcal{I}_\mathbf{d} \rangle$ is defined as follows:

1. $\Delta_\mathbf{d} = (\top^{\mathbf{d}}_{\mathbf{d}})^{\mathcal{I}}$;
2. $a^{\mathcal{I}_\mathbf{d}} = (a^{\mathbf{d}})^{\mathcal{I}}$ if $(a^{\mathbf{d}})^{\mathcal{I}} \neq \mathsf{undef}^{\mathcal{I}}$ otherwise $a^{\mathcal{I}_\mathbf{d}}$ is undefined for every individual a;
3. $(X_{\mathbf{d}'_\mathbf{B}})^{\mathcal{I}_\mathbf{d}} = (X^{\mathbf{d}}_{\mathbf{d}'_\mathbf{B}+\mathbf{d}})^{\mathcal{I}}$ for every atomic concept/role $X_{\mathbf{d}'_\mathbf{B}}$.

We show that $\mathfrak{I}$ is a model of $\mathfrak{K}$. By construction we have that there is a $\mathbf{d}$ such that $\Delta_\mathbf{d}$ is not empty. Let us show that all the conditions of Definition 9 are satisfied by $\mathfrak{I}$.

1. If $\mathbf{d} \prec \mathbf{e}$, then $\mathcal{I} \models \top^{\mathbf{f}}_{\mathbf{d}} \sqsubseteq \top^{\mathbf{f}}_{\mathbf{e}}$. This implies that $(\top^{\mathbf{f}}_{\mathbf{d}})^{\mathcal{I}} \subseteq (\top^{\mathbf{f}}_{\mathbf{e}})^{\mathcal{I}}$, which implies $(\top_\mathbf{d})^{\mathcal{I}_\mathbf{f}} \subseteq (\top_\mathbf{e})^{\mathcal{I}_\mathbf{f}}$.
2. $\mathcal{I} \models C^{\mathbf{d}}_{\mathbf{e}} \sqsubseteq \top^{\mathbf{d}}_{\mathbf{e}}$ implies that $(C^{\mathbf{d}}_{\mathbf{e}})^{\mathcal{I}} \subseteq (\top^{\mathbf{d}}_{\mathbf{e}})^{\mathcal{I}}$, which implies $(C_\mathbf{e})^{\mathcal{I}_\mathbf{d}} \subseteq (\top_\mathbf{e})^{\mathcal{I}_\mathbf{d}}$.
3. $\mathcal{I} \models \exists R^{\mathbf{d}}_{\mathbf{e}}.\top \sqsubseteq \top^{\mathbf{d}}_{\mathbf{e}}$ and $\mathcal{I} \models \top \sqsubseteq \forall R^{\mathbf{d}}_{\mathbf{e}}.\top^{\mathbf{d}}_{\mathbf{e}}$ implies that $(R^{\mathbf{d}}_{\mathbf{e}})^{\mathcal{I}} \subseteq (\top^{\mathbf{d}}_{\mathbf{e}})^{\mathcal{I}} \times (\top^{\mathbf{d}}_{\mathbf{e}})^{\mathcal{I}}$, which implies $(R_\mathbf{e})^{\mathcal{I}_\mathbf{d}} \subseteq (\top_\mathbf{e})^{\mathcal{I}_\mathbf{d}} \times (\top_\mathbf{e})^{\mathcal{I}_\mathbf{d}}$.
4. Suppose $\mathbf{d} \prec \mathbf{e}$ and $a^{\mathcal{I}_\mathbf{d}}$ is defined. From the construction of $\#(\mathfrak{K})$ we have $\mathcal{I} \models \{a^{\mathbf{d}}\} \sqsubseteq \{a^{\mathbf{e}}, \mathsf{undef}\}$, that is, either $(a^{\mathbf{d}})^{\mathcal{I}} = (a^{\mathbf{e}})^{\mathcal{I}}$ or $(a^{\mathbf{d}})^{\mathcal{I}} = \mathsf{undef}^{\mathcal{I}}$. However, since $a^{\mathcal{I}_\mathbf{d}}$ is defined, due to the construction of $\mathfrak{I}$ it must be the case that $(a^{\mathbf{d}})^{\mathcal{I}} \neq \mathsf{undef}^{\mathcal{I}}$ and hence $(a^{\mathbf{d}})^{\mathcal{I}} = (a^{\mathbf{e}})^{\mathcal{I}}$. From the construction of $\mathfrak{I}$ we have $a^{\mathcal{I}_\mathbf{d}} = (a^{\mathbf{d}})^{\mathcal{I}} = (a^{\mathbf{e}})^{\mathcal{I}} = a^{\mathcal{I}_\mathbf{e}}$.
   Suppose the other case, i.e., $\mathbf{d} \prec \mathbf{e}$ and $a^{\mathcal{I}_\mathbf{e}}$ is defined and $a^{\mathcal{I}_\mathbf{e}} \in \Delta_\mathbf{d}$. From the construction of $\#(\mathfrak{K})$ we have $\mathcal{I} \models \top^{\mathbf{d}}_{\mathbf{d}} \sqcap \{a^{\mathbf{e}}\} \sqsubseteq \{a^{\mathbf{d}}\}$, that is, $(\top^{\mathbf{d}}_{\mathbf{d}})^{\mathcal{I}} \cap \{(a^{\mathbf{e}})^{\mathcal{I}}\} \subseteq \{(a^{\mathbf{d}})^{\mathcal{I}}\}$. We have assumed $a^{\mathcal{I}_\mathbf{e}} \in \Delta_\mathbf{d}$ and hence the construction of $\mathfrak{I}$ this implies $(a^{\mathbf{e}})^{\mathcal{I}} \in (\top^{\mathbf{d}}_{\mathbf{d}})^{\mathcal{I}}$ and hence the above inclusion reduces into $\{(a^{\mathbf{e}})^{\mathcal{I}}\} \subseteq \{(a^{\mathbf{d}})^{\mathcal{I}}\}$ which implies $(a^{\mathbf{e}})^{\mathcal{I}} = (a^{\mathbf{d}})^{\mathcal{I}}$ and from construction of $\mathfrak{I}$ also $a^{\mathcal{I}_\mathbf{d}} = a^{\mathcal{I}_\mathbf{e}}$.
5. By construction of $\mathfrak{I}$, we have that $(X_{\mathbf{d}_\mathbf{B}})^{\mathcal{I}_\mathbf{e}} = (X^{\mathbf{e}}_{\mathbf{d}_\mathbf{B}+\mathbf{e}})^{\mathcal{I}} = (X_{\mathbf{d}_\mathbf{B}+\mathbf{e}})^{\mathcal{I}_\mathbf{e}}$
6. We have that $\mathcal{I} \models X^{\mathbf{d}}_{\mathbf{d}} \equiv X^{\mathbf{e}}_{\mathbf{d}}$. This implies that $(X_\mathbf{d})^{\mathcal{I}_\mathbf{e}} = (X^{\mathbf{e}}_{\mathbf{d}})^{\mathcal{I}} = (X^{\mathbf{d}}_{\mathbf{d}})^{\mathcal{I}} = (X_\mathbf{d})^{\mathcal{I}_\mathbf{d}}$;
7. If $\mathbf{d} \prec \mathbf{e}$, we have that $\mathcal{I} \models C^{\mathbf{d}}_{\mathbf{f}} \equiv C^{\mathbf{e}}_{\mathbf{f}} \sqcap \top^{\mathbf{d}}_{\mathbf{d}}$. This implies that $(C^{\mathbf{d}}_{\mathbf{f}})^{\mathcal{I}} = (C^{\mathbf{e}}_{\mathbf{f}})^{\mathcal{I}} \cap (\top^{\mathbf{d}}_{\mathbf{d}})^{\mathcal{I}}$, which implies $(C_\mathbf{f})^{\mathcal{I}_\mathbf{d}} = (C_\mathbf{f})^{\mathcal{I}_\mathbf{e}} \cap \Delta_\mathbf{d}$;
8. $\mathcal{I} \models I^{\mathbf{d}}_{\mathbf{d}} \circ R^{\mathbf{e}}_{\mathbf{f}} \circ I^{\mathbf{d}}_{\mathbf{d}} \sqsubseteq R^{\mathbf{d}}_{\mathbf{f}}$ implies that $(R_\mathbf{f})^{\mathcal{I}_\mathbf{d}} \supseteq (R_\mathbf{f})^{\mathcal{I}_\mathbf{e}} \cap {\Delta_\mathbf{d}}^2$. The fact that $\mathcal{I} \models R^{\mathbf{d}}_{\mathbf{f}} \sqsubseteq R^{\mathbf{e}}_{\mathbf{f}}$, implies that $(R_\mathbf{f})^{\mathcal{I}_\mathbf{d}} \subseteq (R_\mathbf{f})^{\mathcal{I}_\mathbf{e}} \cap {\Delta_\mathbf{d}}^2$.
9. Let $\mathbf{d} = \dim(C)$, if $\phi \in \mathrm{K}(C)$, then we have that $\mathcal{I} \models \phi\#\mathbf{d}$. Similarly to the previous lemma, this again follows from the fact that $\mathcal{I}_\mathbf{d}$ and $\mathcal{I}$ comply with the embedding respective to the operator $(\cdot)\#\mathbf{d}$, and hence we obtain this condition by Lemma 1.

Appendix A.5. Proof of Theorem 2

Theorem 2. *If* $\mathfrak{K}$ *is* $\precsim$*-stratified, then checking if* $\mathfrak{K} \models \mathbf{d} : \phi$ *is decidable with the complexity upper-bound of* 2NExpTime.

The decidability follows from Lemmata 3 and 4, as the problem of checking $\mathfrak{K} \models \mathbf{e} : \phi$ can be rewritten into the problem of checking if $\#(\mathfrak{K}) \models \phi\sharp\mathbf{d}$ and $\#(\mathfrak{K})$ is $\precsim$-stratified. We will show that the transformation $\#(\cdot)$ is polynomial. Since $\#(\mathfrak{K})$ is $\mathcal{SROIQ}$ knowledge base and deciding entailment id 2NExpTime-hard for $\mathcal{SROIQ}$ [35] it follows that checking if $\mathfrak{K} \models \mathbf{d} : \phi$ is possible within the upper bound of 2NExpTime worst case complexity.

Without loss of generality, we will consider the size of the input to be the total number of occurrences of all symbols from Σ and Γ in both $\mathfrak{K}$ and ϕ summed together with the total number of all DL constructors in $\mathfrak{K}$ and ϕ and with the number of formulae in $\mathfrak{K}$ and ϕ. We will denote this number by m. The real size of input to be processed depends on the encoding of symbols. As the number of symbols used in any particular knowledge base is always finite, suitable encoding can always be found such that the real size of input is $c \times m$ for some constant c [36].

As explained before, the number of contextual dimensions is assumed to be a fixed constant k. While in theory the number of possible dimensional values may be large, in practise the number of contexts n is always smaller than m. This is because whenever a new context C is initialized, also k new formulae are added in the meta knowledge, by which the dimensional values are associated with C.

Let us now determine the size of $\#(\mathfrak{K})$. We will go through the construction in Definition 21 point by point:

1. one axiom $\top^{\mathbf{f}}_{\mathbf{d}} \sqsubseteq \top^{\mathbf{f}}_{\mathbf{e}}$ for any three initialized contexts $C_{\mathbf{d}}$, $C_{\mathbf{e}}$ and $C_{\mathbf{f}}$, with $\mathbf{d} \prec \mathbf{e}$. These are maximum n^3 of axioms of size 3, i.e., with total size bounded with $3 \times n^3$;
2. one axiom $A^{\mathbf{d}}_{\mathbf{e}} \sqsubseteq \top^{\mathbf{d}}_{\mathbf{e}}$ for any two initialized contexts $C_{\mathbf{d}}$, $C_{\mathbf{e}}$ and for any $A_{\mathbf{e}}$ occurring in $\mathfrak{K}$. Note that $A_{\mathbf{e}}$ occurs in $\mathfrak{K}$ whenever $A_{\mathbf{e_B}}$ occurs in $C_{\mathbf{g}}$ with $\mathbf{e} = \mathbf{e_B}{+}\mathbf{g}$ (below this sense will be also used w.r.t. roles). This means that for each such occurrence of $A_{\mathbf{e}}$ in $\mathfrak{K}$ there is at least one actual occurrence of some $A_{\mathbf{e_B}}$ with possibly incomplete dimensional vector. Therefore at most m atomic symbols (concepts, roles and individuals) in total occur in $\mathfrak{K}$ in this sense. This implies that most $m \times n^2$ axioms of size 3 are added in this step, with total size bounded with $3 \times m \times n^2$;
3. a pair of axioms $\exists R^{\mathbf{d}}_{\mathbf{e}}.\top \sqsubseteq \top^{\mathbf{d}}_{\mathbf{e}}$ and $\top \sqsubseteq \forall R^{\mathbf{d}}_{\mathbf{e}}.\top^{\mathbf{d}}_{\mathbf{e}}$; for any two initialized contexts $C_{\mathbf{d}}$, $C_{\mathbf{e}}$ and for any $R_{\mathbf{e}}$ occurring in $\mathfrak{K}$. Similarly to the previous step this yields at most $2 \times m \times n^2$ axioms of size 5, i.e., with total size bounded with $10 \times m \times n^2$;
4. two axioms $\top^{\mathbf{d}}_{\mathbf{d}} \sqcap \{a^{\mathbf{e}}\} \sqsubseteq \{a^{\mathbf{d}}\}$ and $\{a^{\mathbf{d}}\} \sqsubseteq \{a^{\mathbf{e}}, \mathsf{undef}\}$ for any two initialized contexts $C_{\mathbf{d}}$, $C_{\mathbf{e}}$ with $\mathbf{d} \prec \mathbf{e}$ and for any constant a, and one axiom $\neg\top^{\mathbf{d}}_{\mathbf{d}}(\mathsf{undef})$ for any initialized context $C_{\mathbf{d}}$ This leads to maximum of $m{\times}n^2$ axioms of size 7, maximum of $m \times n^2$ axioms of size 8, and maximum of n axioms of size 3. The total sum of all these axioms is bounded under $15 \times m \times n^2 + 3 \times n$;
6. one axiom $X^{\mathbf{d}}_{\mathbf{d}} \equiv X^{\mathbf{e}}_{\mathbf{d}}$ for any two initialized contexts $C_{\mathbf{d}}$, $C_{\mathbf{e}}$ with $\mathbf{d} \preceq \mathbf{e}$ and for any atomic concept or role $X_{\mathbf{d}}$ occurring in $\mathfrak{K}$. This again leads to the maximum of $m \times n^2$ axioms of size 3 with total size bounded with $3 \times m \times n^2$;
7. $A^{\mathbf{d}}_{\mathbf{f}} \equiv A^{\mathbf{e}}_{\mathbf{f}} \sqcap \top^{\mathbf{d}}_{\mathbf{d}}$ for any two initialized contexts $C_{\mathbf{d}}$, $C_{\mathbf{e}}$ with $\mathbf{d} \preceq \mathbf{e}$ and for any atomic concept $A_{\mathbf{f}}$ occurring in $\mathfrak{K}$. This leads to the maximum of $m{\times}n^2$ axioms of size 5, i.e., with total size bounded with $5 \times m \times n^2$;
8. four axioms $I^{\mathbf{d}}_{\mathbf{d}} \circ R^{\mathbf{e}}_{\mathbf{f}} \circ I^{\mathbf{d}}_{\mathbf{d}} \sqsubseteq R^{\mathbf{d}}_{\mathbf{f}}$, $R^{\mathbf{d}}_{\mathbf{f}} \sqsubseteq R^{\mathbf{e}}_{\mathbf{f}}$, $I^{\mathbf{d}}_{\mathbf{d}} \circ R^{\mathbf{e}}_{\mathbf{f}} \sqsubseteq S^{\mathbf{d,e,f}}_{R}$, and $S^{\mathbf{d,e,f}}_{R} \circ I^{\mathbf{d}}_{\mathbf{d}} \sqsubseteq R^{\mathbf{d}}_{\mathbf{f}}$ for any two initialized contexts $C_{\mathbf{d}}$, $C_{\mathbf{e}}$ with $\mathbf{d} \prec \mathbf{e}$ and for any $R_{\mathbf{f}}$ occurring in $\mathfrak{K}$. This leads to the maximum of $m \times n^2$ axioms of size 7, $m \times n^2$ axioms of size 3, and twice $m \times n^2$ axioms size 5. Total size of both these sets together is therefore bounded with $20 \times m \times n^2$;
9. one axiom $\phi\#\mathbf{d}$ for every axiom ϕ occurring in any context $\mathrm{K}(C)$ of $\mathfrak{K}$. In this step less than m axioms are added. All of these axioms are transformed by the $\#(\cdot)$ operator which yields to a blow up in the axiom size because each symbol may be replaced by up to 5 new symbols (i.e., the transformation is linear). Therefore the total size of the axioms added in this step is bounded with $5 \times m$.

Summing up, the transformed knowledge base $\#(\mathfrak{K})$ is bounded in size with $59 \times m \times n^2 + 8 \times m$ which is under $O(m^3)$ since $n \leq m$.

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