Computer Graphics Course

Three-Dimensional Modeling

Lecture 12

"Three-Dimensional Transformations"

Three-Dimensional Transformations

- Types of transformations
- Affine transformations
  (translation, rotation, scaling)
- Deformations (twisting, bending, tapering)
- Composite transformations
- Set-theoretic operations
- Offsetting and blending
- Metamorphosis
- Collision detection

Types of transformations

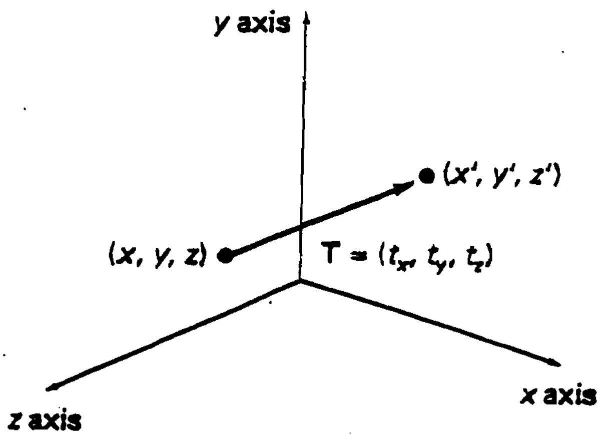
- **Change of parameters**
  Example: radius of a sphere, positions of control points of a parametric surface;

- **Mapping** (coordinate transformation)
  Sets one-to-one correspondance between space points (x, y, z) -> (x', y', z')
  Example: affine transformations, deformations;

- **Set-theoretic operations**
  Example: union;

- **Change of a function**
  Example: offsetting, blending, metamorphosis;

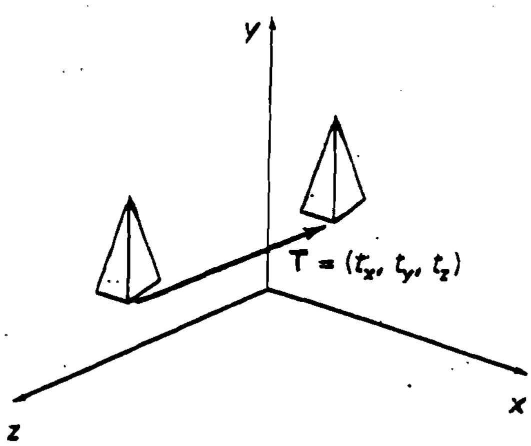
Affine transformations
Translation



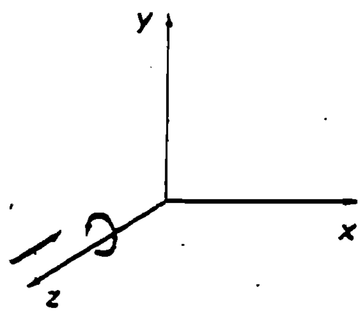
$$x' = x + t_x,$$
$$y' = y + t_y,$$
$$z' = z + t_z$$

In a three-dimensional homogeneous coordinate representation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



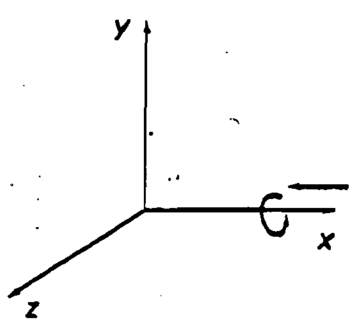
Affine transformations
Coordinate-axes rotations



z-axis rotation

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$
$$z' = z$$

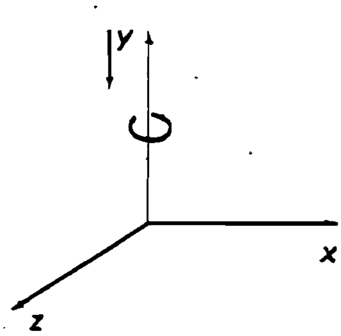
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



x-axis rotation

$$y' = y\cos\theta - z\sin\theta$$
$$z' = y\sin\theta + z\cos\theta$$
$$x' = x$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



y-axis rotation

$$z' = z\cos\theta - x\sin\theta$$
$$x' = z\sin\theta + x\cos\theta$$
$$y' = y$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Affine transformations
Scaling

$$x' = x \cdot s_x,$$

$$y' = y \cdot s_y,$$

$$z' = z \cdot s_z$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Scaling with respect to a selected fixed position (x_f, y_f, z_f) can be represented with the following transformation sequence:

1. Translate the fixed point to the origin
2. Scale the object relative to the coordinate origin
3. Translate the fixed point back to its original position

$$T(x_f, y_f, z_f) \cdot S(s_x, s_y, s_z) \cdot T(-x_f, -y_f, -z_f) = \begin{bmatrix} s_x & 0 & 0 & (1 - s_x)x_f \\ 0 & s_y & 0 & (1 - s_y)y_f \\ 0 & 0 & s_z & (1 - s_z)z_f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Deformations

Author: Alan Barr

(x,y,z) - original point

(X,Y,Z) - point of a deformed object

Forward mapping

For polygonal and parametric forms

$$\Phi: (x,y,z) \rightarrow (X,Y,Z) \text{ or}$$

$$(X,Y,Z) = (\phi_1(x,y,z), \phi_2(x,y,z), \phi_3(x,y,z))$$

Inverse mapping

For implicit form

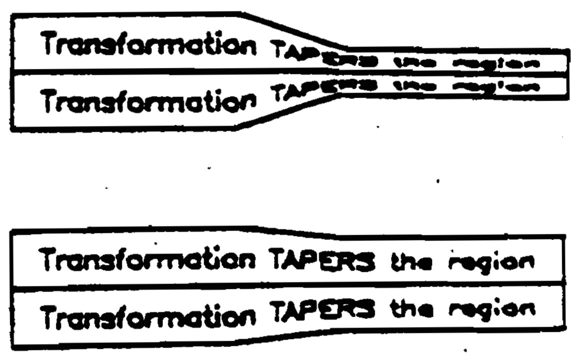
$$\Phi^{-1}: (X,Y,Z) \rightarrow (x,y,z) \text{ or}$$

$$(x,y,z) = (\phi^{-1}_1(X,Y,Z), \phi^{-1}_2(X,Y,Z), \phi^{-1}_3(X,Y,Z))$$

Deformations: tapering

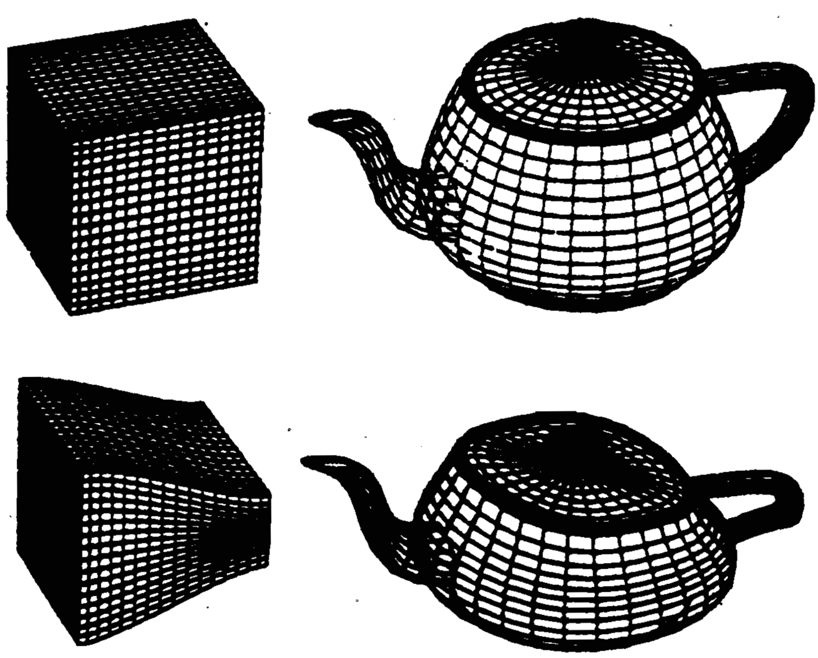
Forward mapping

$$r = f(z),$$
$$X = rx,$$
$$Y = ry,$$
$$Z = z$$



Inverse mapping

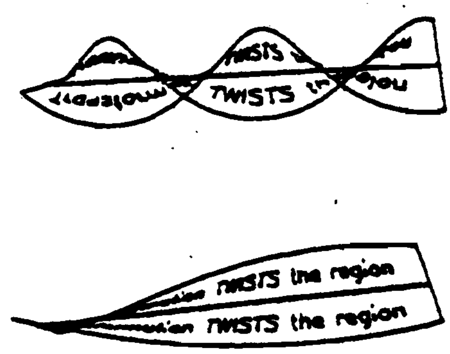
$$r(Z) = f(Z),$$
$$x = X/r,$$
$$y = Y/r,$$
$$z = Z$$



Deformations: twisting

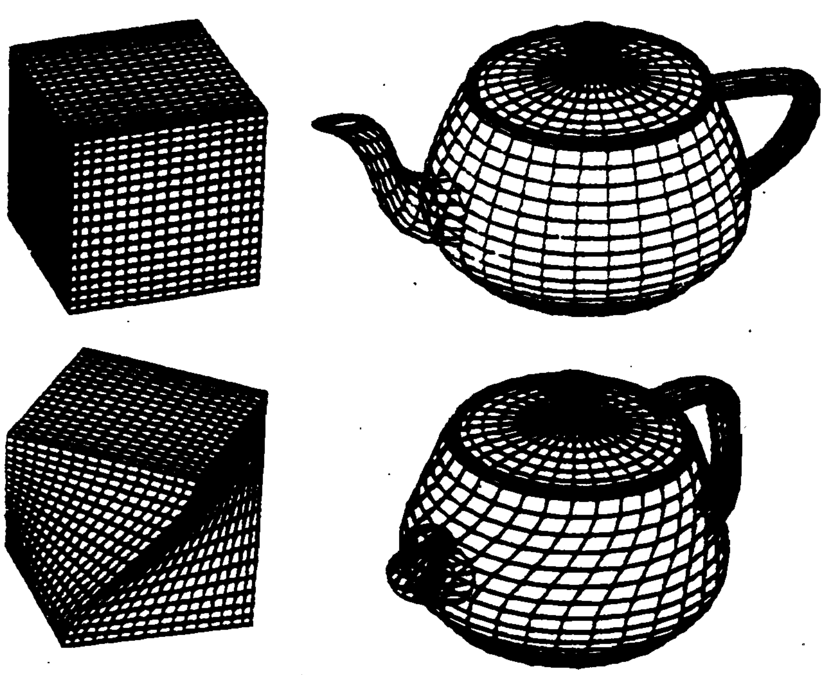
Forward mapping

$$\theta = f(z) \qquad X = xC_\theta - yS_\theta,$$
$$C_\theta = cos(\theta) \qquad Y = xS_\theta + yC_\theta,$$
$$S_\theta = sin(\theta) \qquad Z = z.$$



Inverse mapping

$$\theta = f(Z),$$
$$x = XC_\theta + YS_\theta,$$
$$y = -XS_\theta + YC_\theta,$$
$$z = Z$$



Deformations: bending

Forward mapping

The following equations represent an isotropic bend along a centerline parallel to the y-axis.

bending angle θ is given by:

$$\theta = k(\hat{y} - y_0),$$
$$C_\theta = cos(\theta),$$
$$S_\theta = sin(\theta),$$

$$\hat{y} = \begin{cases} y_{min}, & \text{if } y \leq y_{min} \\ y, & \text{if } y_{min} < y < y_{max} \\ y_{max}, & \text{if } y \geq y_{max} \end{cases}$$

$$X = x$$

$$Y = \begin{cases} -S_\theta(z - \frac{1}{k}) + y_0, & y_{min} \leq y \leq y_{max}, \\ -S_\theta(z - \frac{1}{k}) + y_0 + C_\theta(y - y_{min}), & y < y_{min} \\ -S_\theta(z - \frac{1}{k}) + y_0 + C_\theta(y - y_{max}), & y > y_{max} \end{cases}$$

$$Z = \begin{cases} C_\theta(z - \frac{1}{k}) + \frac{1}{k}, & y_{min} \leq y \leq y_{max}, \\ C_\theta(z - \frac{1}{k}) + \frac{1}{k} + S_\theta(y - y_{min}), & y < y_{min} \\ C_\theta(z - \frac{1}{k}) + \frac{1}{k} + S_\theta(y - y_{max}), & y > y_{max} \end{cases}$$

Inverse mapping

$$x = X$$

$$\theta_{min} = k(y_{min} - y_0)$$
$$\theta_{max} = k(y_{max} - y_0)$$
$$\hat{\theta} = -tan^{-1}\left(\frac{Y - y_0}{Z - \frac{1}{k}}\right)$$

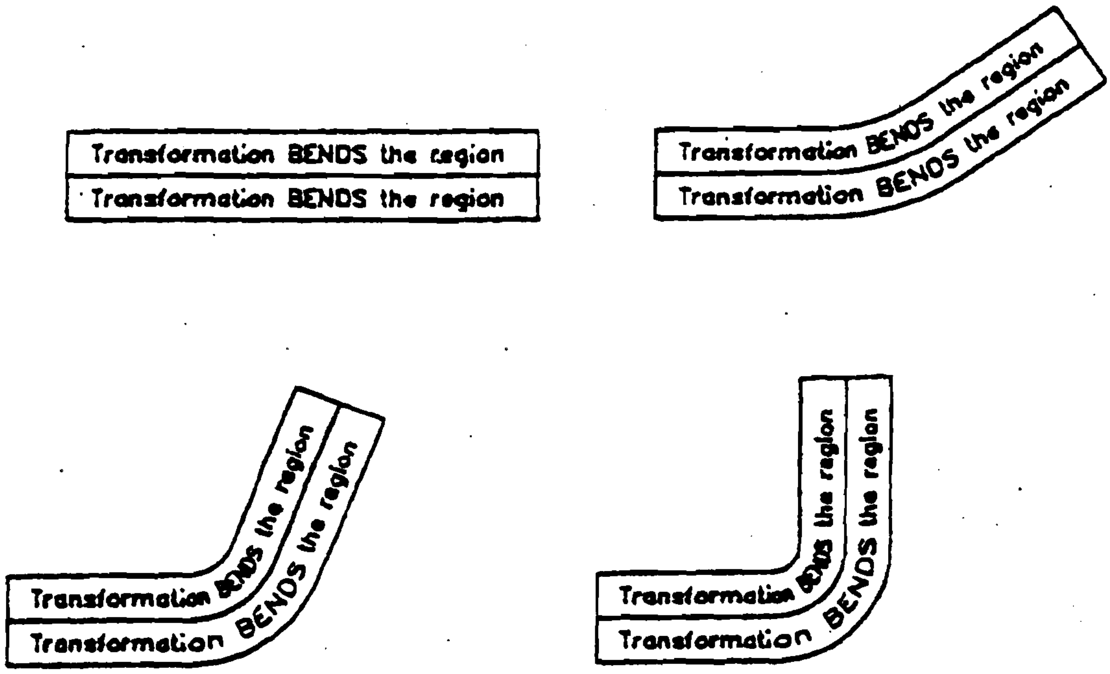
$$\theta = \begin{cases} \theta_{min}, & \text{if } \theta < \hat{\theta}_{min} \\ \hat{\theta}, & \text{if } \theta_{min} \leq \hat{\theta} \leq \theta_{max} \\ \theta_{max}, & \text{if } \hat{\theta} > \theta_{max} \end{cases}$$

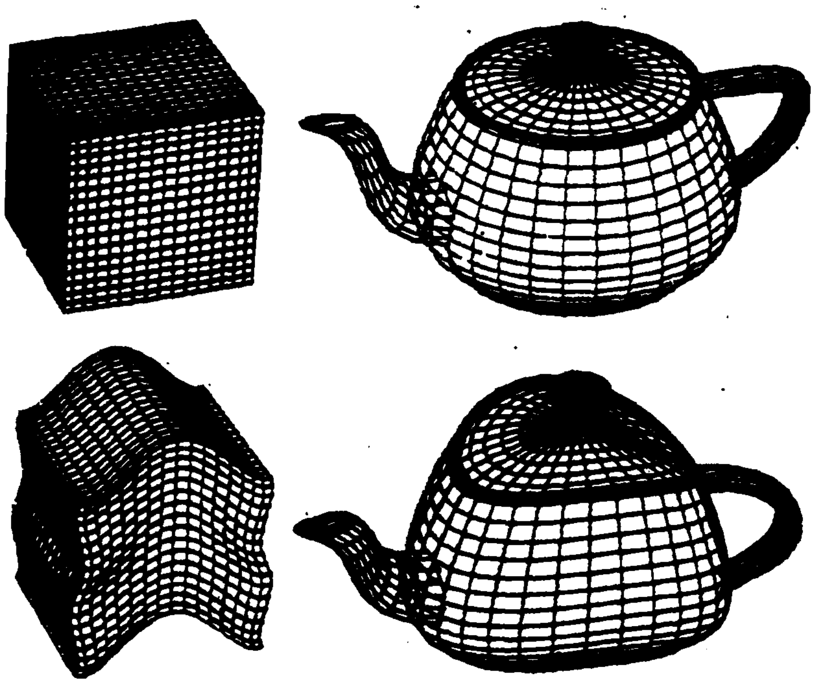
$$\hat{y} = \frac{\theta}{k} + y_0$$

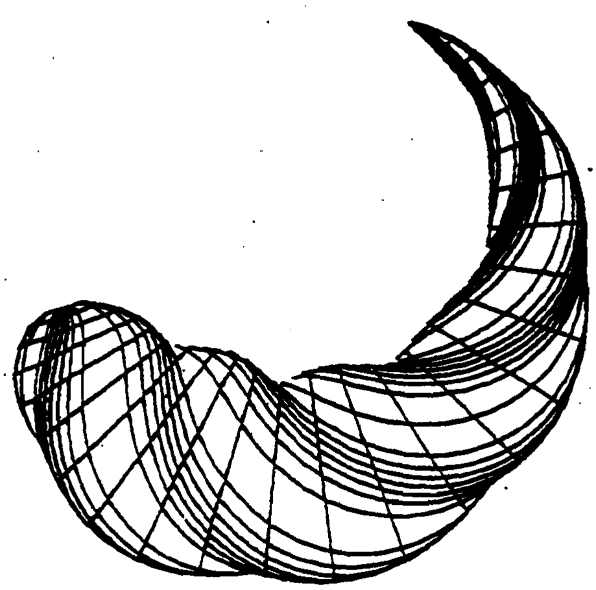
$$y = \begin{cases} \hat{y}, & y_{min} < \hat{y} < y_{max} \\ (Y - y_0)C_\theta + (z - \frac{1}{k})S_\theta + \hat{y}, & \hat{y} = y_{min} \text{ or } y_{max} \end{cases}$$

$$z = \begin{cases} \frac{1}{k} + ((Y - y_0)^2 + (Z - \frac{1}{k})^2)^{1/2}, & y_{min} < \hat{y} < y_{max} \\ -(Y - y_0)S_\theta + (z - \frac{1}{k})C_\theta + \hat{y}, & \hat{y} = y_{min} \text{ or } y_{max} \end{cases}$$

Deformations: bending
Examples







a Bent, Twisted, Tapered Primitive

Composite transformations

Example: tapering and translation of an ellipsoid

In "functional" terms:

Translation (Tapering (Ellipsoid))

For the implicit form inverse transformations are applied "from left to right". Let (X, Y, Z) be the given point.

1) **Translation:** the center is translated from (0, 0, 0) to (a, b, c)

$$X' = X - a$$
$$Y' = Y - b$$
$$Z' = Z - c$$

2) **Tapering:** scaling coefficient

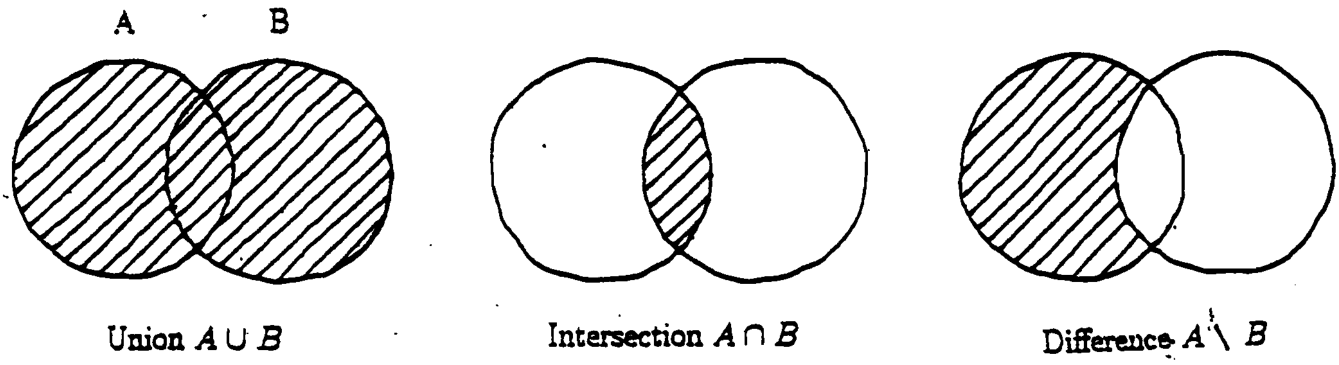
$r = 1$ for $z = z_{min}$ and $r = 0.5$ for $z = z_{max}$

$$s = (z_{max} - Z') / (z_{max} - z_{min})$$
$$r = 0.5\ (1+s)$$
$$x = X' / r$$
$$y = Y' / r$$
$$z = Z'$$

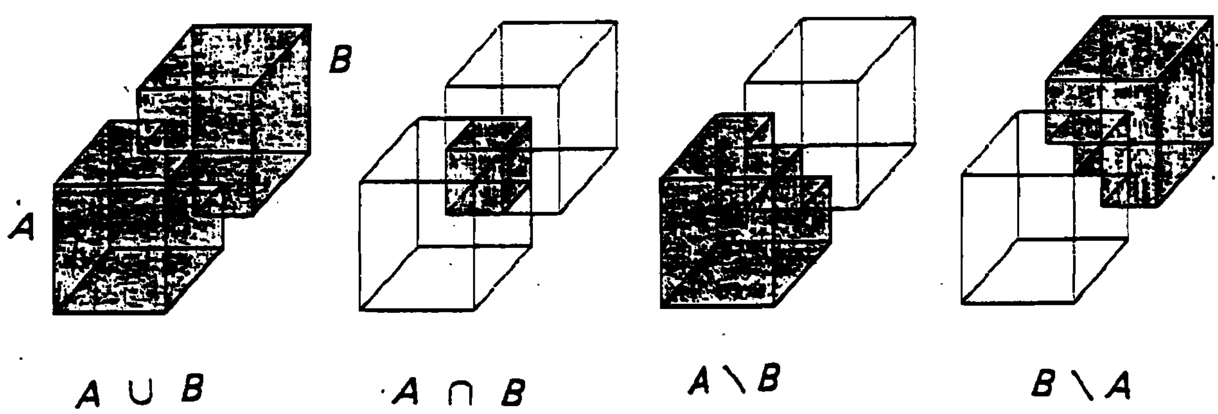
3) **Transformed ellipsoid**

$$f(x,y,z) = 1 - \left(\frac{x}{r_x}\right)^2 - \left(\frac{y}{r_y}\right)^2 - \left(\frac{z}{r_z}\right)^2$$

Set-theoretic (Boolean) operations



A Venn diagram showing the operators of set-theory



R-functions and set-theoretic operations

Geometric object in E^n :

$$f(x_1, x_2, ..., x_n) \geq 0$$

Binary operation on geometric objects:

$$F\,(\,f_1(X),\;f_2(X)\,) \geq 0$$

Resultant object:

$f_3 = f_1 \mid f_2$ for union;
$f_3 = f_1 \,\&\, f_2$ for intersection;
$f_3 = f_1 \setminus f_2$ for subtraction.

R-functions:

$$f_1 \mid f_2 = \frac{1}{1+a}\left(f_1 + f_2 + \sqrt{f_1^2 + f_2^2 - 2af_1f_2}\right)$$

$$f_1 \,\&\, f_2 = \frac{1}{1+a}\left(f_1 + f_2 - \sqrt{f_1^2 + f_2^2 - 2a\,f_1f_2}\right)$$

$$f_1 \setminus f_2 = f_1 \,\&\, (-f_2)$$

$$-1 < a(f_1, f_2) \leq 1, \quad a(f_1, f_2) = a(f_2, f_1) = a(-f_1, f_2) = a(f_1, -f_2).$$

Types of R-functions

For a = 1:

$$f_1 \mid f_2 = \max(f_1, f_2)$$

$$f_1 \,\&\, f_2 = \min(f_1, f_2)$$ C^1 discontinuity where $f_1 = f_2$.

For a=0:

$$f_1 \mid f_2 = f_1 + f_2 + \sqrt{f_1^2 + f_2^2}$$

$$f_1 \,\&\, f_2 = f_1 + f_2 - \sqrt{f_1^2 + f_2^2}$$ C^1 discontinuity where $f_1 = 0$ and $f_2 = 0$.

C^m continuity:

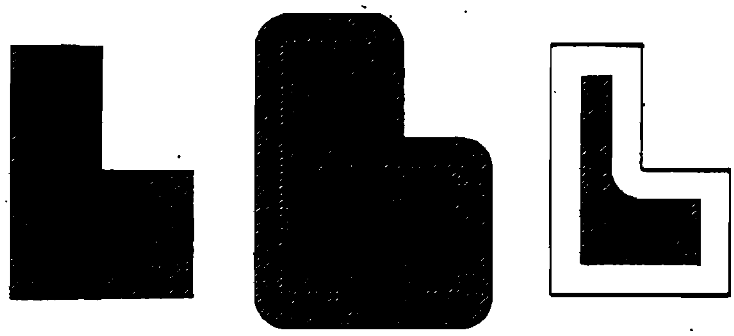
$$f_1 \mid f_2 = (f_1 + f_2 + \sqrt{f_1^2 + f_2^2})\ (f_1^2 + f_2^2)^{m/2}$$

$$f_1 \,\&\, f_2 = (f_1 + f_2 - \sqrt{f_1^2 + f_2^2})\ (f_1^2 + f_2^2)^{m/2}$$

Offsetting

Offset objects are expanded or contracted versions of an original object. To offset an object S by a distance d one adds to the object all the points that lie within a distance d of the boudary of S.

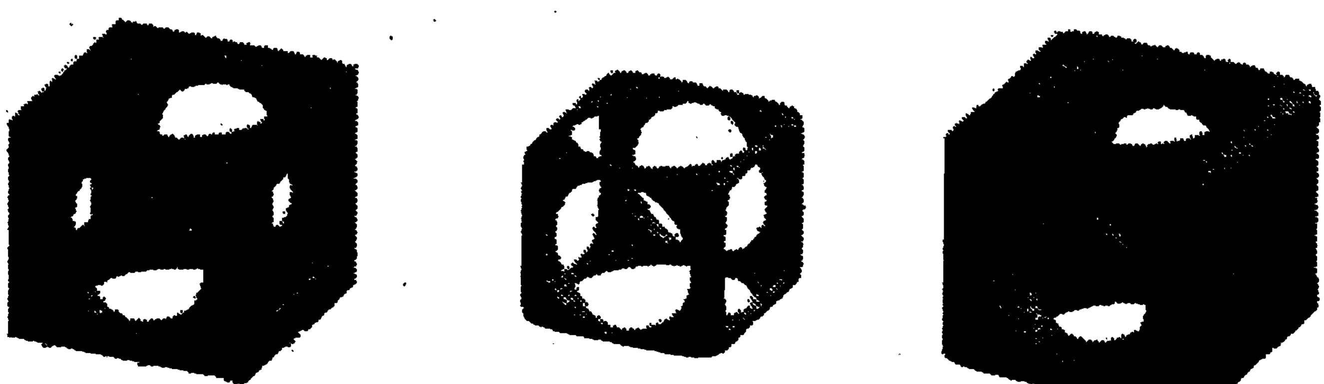
2D



(a) (b) (c)

A simple L-shaped object (a), a positive offset (b), and a negative offset (c).

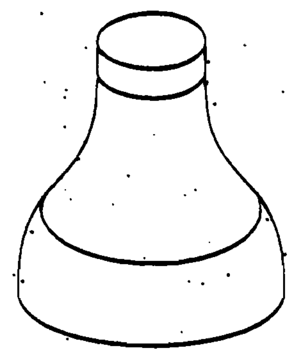
3D



(a) (b) (c)

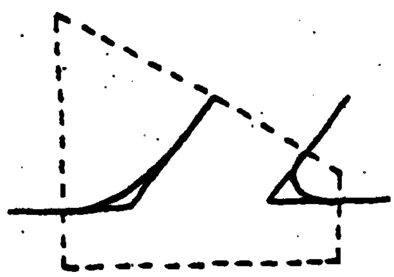
(a) Initial constructive solid (b) internal offset solid (c) external offset solid

Blending operations

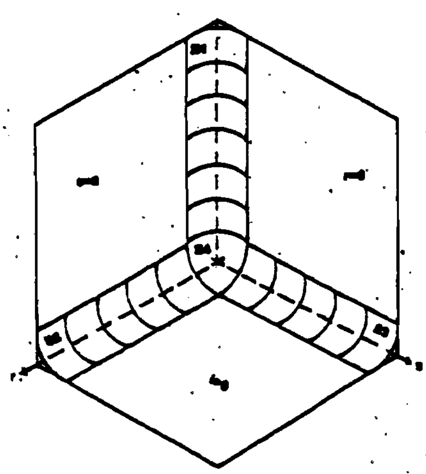


The operation joining several surfaces in a complex object with a smooth surface is called blending. The main difficulties and requirements to blending:

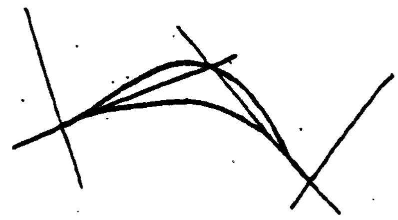
- Tangency of a blend surface with the base surfaces;



- Easy intuitive control of the blending surface shape;
- Necessity to perform for blended objects all the computations possible for unblended objects including set-theoretic operations;
- Blend interference or ability to blend on blends and as the particular case complex vertices (or corners) blending;



- At least C^1 continuous blending function in the entire domain of definition;
- Blending definition of basic set-theoretic operations: intersection, union and subtraction;
- Single edge blending or localizing the blend to a region about intersection curve of two faces;
- Added and subtracted material blends;



- The ability to produce constant-radius blending;
- No restriction of circular cross sections or the requirement of variable-radius blends;
- Exact representation for blends instead of any approximation;
- Automatic clipping of unwanted parts of the blending surface;
- Blending of two non-intersecting surfaces;
- Functional constraints;
- Aesthetic blends constrained by appearance.

Ricci [1973]:

A solid is defined as $f(P) \leq 1$.

Intersection:

$$I(f_1, f_2, \ldots, f_n) = (f_1^{\,p} + f_2^{\,p} + \ldots + f_n^{\,p})^{1/p}$$

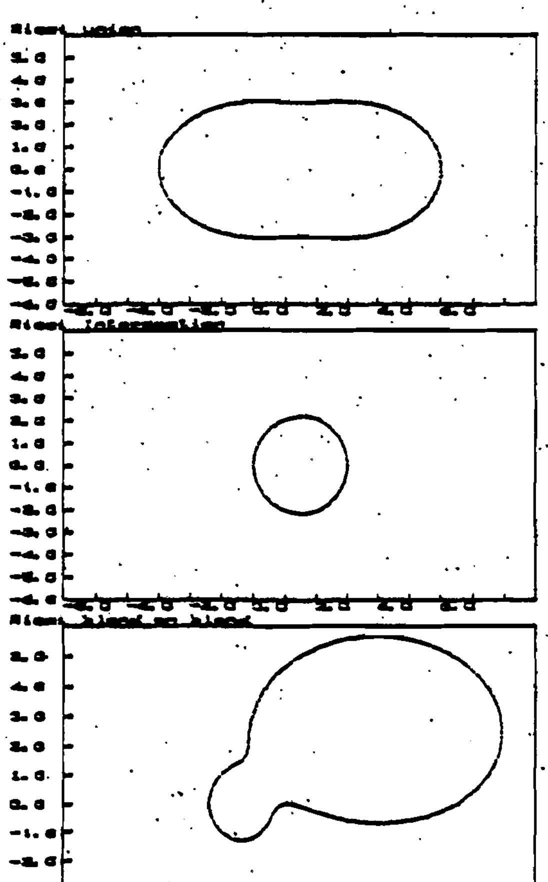
Union:

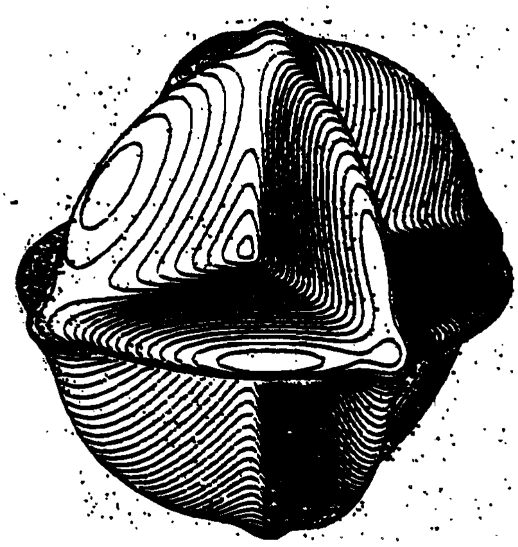
$$U(f_1, f_2, \ldots, f_n) = (f_1^{\,-p} + f_2^{\,-p} + \ldots + f_n^{\,-p})^{-1/p}$$

p is a positive real number.

$$\lim_{p \to \infty} I(f_1, f_2, \ldots, f_n) = \min(f_1, f_2, \ldots, f_n)$$

$$\lim_{p \to \infty} U(f_1, f_2, \ldots, f_n) = \max(f_1, f_2, \ldots, f_n)$$





for two disks;
Upper: Union,
Middle: Intersection,
Bottom: Blends on blend

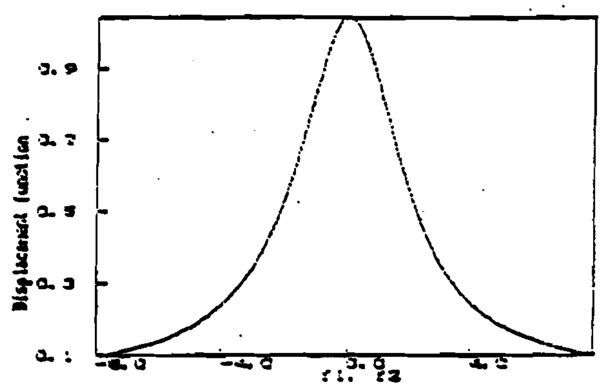
Blending set-theoretic operations

$$F(f_1,f_2) = R(f_1,f_2) + d(f_1,f_2)$$

R is a corresponding R-function,

d is a displacement function, $d(0,0) = \max d(f_1,f_2)$, $d \to 0$

$$d(f_1,f_2) = \frac{a_0}{1 + (f_1/a_1)^2 + (f_2/a_2)^2}$$



The shape of the sections $f_1=0$ and $f_2=0$ for the displacement function.

Blending intersection:

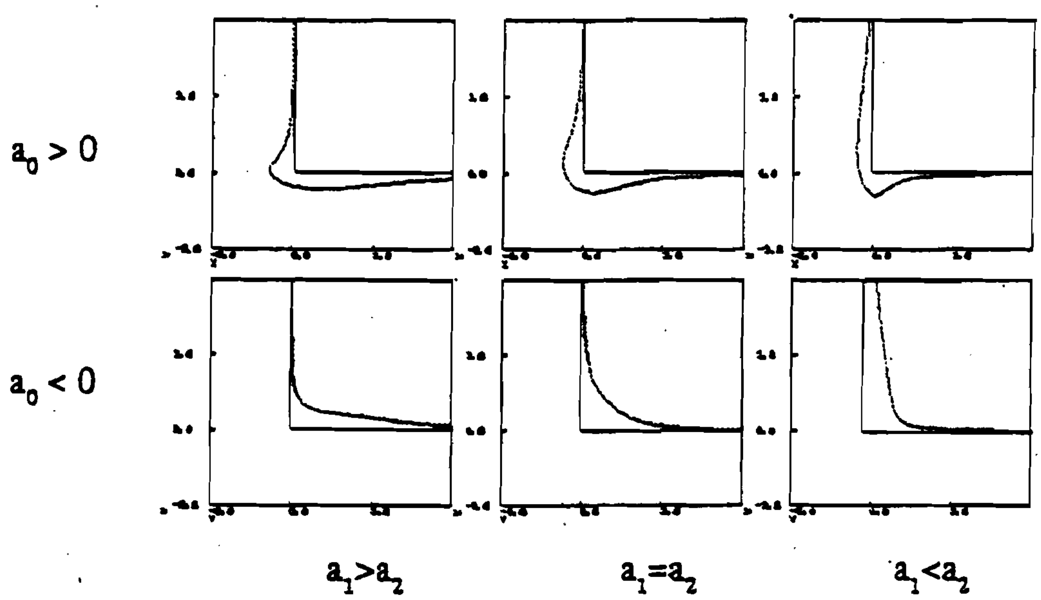
$$F(f_1, f_2) = f_1 + f_2 - \sqrt{f_1^2 + f_2^2} + \frac{a_0}{1 + (f_1/a_1)^2 + (f_2/a_2)^2}$$

C^1 discontinuity where $f_1=0$ and $f_2=0$.

Blending union

$$F(f_1, f_2) = f_1 + f_2 + \sqrt{f_1^2 + f_2^2} + \frac{a_0}{1 + (f_1/a_1)^2 + (f_2/a_2)^2}$$

Parameters of the displacement function for shape control



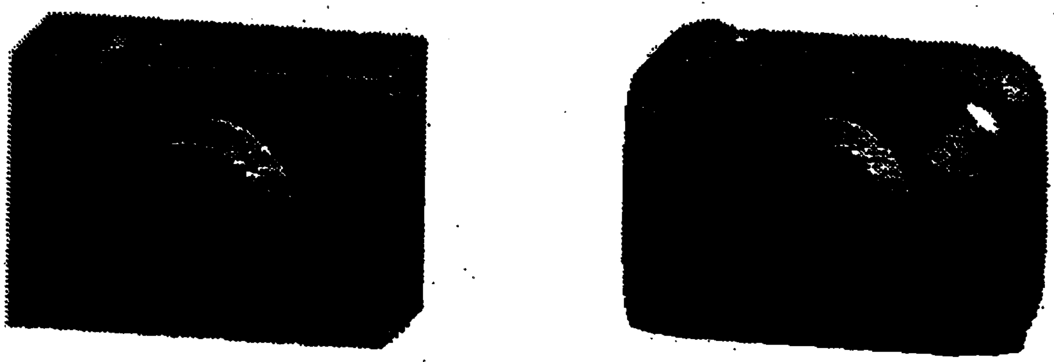
Influence of the displacement function parameters on the shape of blend. The basic set-theoretic operation is intersection of the 2D halfspaces $f_1(x,y)=x$ and $f_2(x,y)=y$.

- The absolute value of a_0 defines the total displacement of the blending surface from the two initial surfaces.

- $a_0=0$ means pure set-theoretic operation.

- A negative a_0 value gives subtracted material blend, and a positive a_0 value yields added material blend.

- The values of $a_1 > 0$ and $a_2 > 0$ are proportional to the distance between the blending surface and the original surfaces defined by f_2 and f_1 respectively.



(a) (b)

(a) initial CSG object

(b) CSG object with several blended edges and cylindrical hole

(a) (b)

(a) the body and the bottom of a wine glass to be connected with aesthetic blend defined by the stroke;
(b) the result of blending parameters estimation

Metamorphosis

Metamorphosis (morphing, warping, shape transformation) changes a geometric object from one given shape to another.
Applications: animation, design of objects that combine features of initial objects, 3D reconstruction from cross-sections.

Polygonal objects

Two steps: 1) search for correspondence between points;
2) interpolation between two surfaces.

Problems:
- different number of points in two objects;
- constant topology (for example, how to transform a sphere in three intersecting tori?);
- possible self-intersections.

Implicit form

Metamorphosis is defined as a transformation between two functions. The simplest form is

$$f_3(X) = f_1(X)\,(1-t) + f_2(X)\,t\,,$$

where $0 \le t \le 1$.

Metamorphosis

- Initial objects G_1 and G_2 are defined in E^{n-1}
- The resultant object G_3 is defined in E^n
- G_1 is a section of G_3 by the hyperplane $x_n = x_n^0$
- G_2 is a section of G_3 by the hyperplane $x_n = x_n^1$

$$f_3(x_1, x_2, \ldots, x_n) = f_1(x_1, x_2, \ldots, x_{n-1}) \cdot (1 - g(x_n)) + f_2(x_1, x_2, \ldots, x_{n-1}) \cdot g(x_n)$$

where $g(x_n)$ is a positive continuous function $g(x_n^0) = 0$ and $g(x_n^1) = 1$.