Computational Logic

Reasoning with $\mathcal{ALC}$

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Basic Reasoning Tasks

C is satisfiable if there is an interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} \neq \emptyset$

C is subsumed by D if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ in every interpretation $\mathcal{I}$

Reasoning Tasks for $\mathcal{T}$

C is satisfiable w.r.t. $\mathcal{T}$ if there is a model $\mathcal{I}$ of $\mathcal{T}$ such that $C^{\mathcal{I}} \neq \emptyset$

$\mathcal{T}$ entails $C \sqsubseteq D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ in every model $\mathcal{I}$ of $\mathcal{T}$

Reasoning Tasks for $\mathcal{T}$ and $\mathcal{A}$

C is satisfiable w.r.t. $\langle \mathcal{T}, \mathcal{A} \rangle$ if there is a model $\mathcal{I}$ of $\langle \mathcal{T}, \mathcal{A} \rangle$ such that $C^{\mathcal{I}} \neq \emptyset$

$\langle \mathcal{T}, \mathcal{A} \rangle$ entails $C \sqsubseteq D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ in every model $\mathcal{I}$ of $\langle \mathcal{T}, \mathcal{A} \rangle$

Reasoning Tasks for $\mathcal{T}$ and $\mathcal{A}$

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$\mathcal{A}$ is consistent w.r.t. $\mathcal{T}$ if if there is a model $\mathcal{I}$ of $\langle\mathcal{T},\mathcal{A}\rangle$

Theorem:

1. C is satisfiable iff $C \not\sqsubseteq \bot$
2. $C \sqsubseteq D$ iff $C \sqcap \neg D$ is unsatisfiable
3. C is satisfiable w.r.t. $\langle \mathcal{T}, \mathcal{A} \rangle$ iff $\mathcal{A} \cup \{C(a)\}$ is consistent w.r.t. $\mathcal{T}$ for some new constant a

Finite Tree Model Property

Theorem. If C is satisfiable then it is satisfiable by some interpretation $\mathcal{I}$ that is a finite tree

Negation Normal Form

C is in NNF if $\neg$ only occurs in front of atomic concept symbols inside C

Lemma. For every concept C there exists C' in NNF such that $C \equiv C'$

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Proof: (Sketch)

- $\neg(E \sqcap F) \equiv \neg E \sqcup \neg F$
- $\neg(E \sqcup F) \equiv \neg E \sqcap \neg F$
- $\neg\exists R.E \equiv \forall R.\neg E$
- $\neg\forall R.E \equiv \exists R.\neg E$

Completion Tree

Completion tree (CTree) is a triple $T = \langle V, E, \mathcal{L} \rangle$ such that $\langle V, E \rangle$ is a tree and $\mathcal{L}$ is a labeling function such that

- $\mathcal{L}(x)$ is a set of concepts for all $x \in V$
- $\mathcal{L}(\langle x, y \rangle)$ is a set of roles for all $\langle x, y \rangle \in E$

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$y \in V$ is an R-successor of $x \in V$ if $\langle x, y \rangle \in E$ and $R \in \mathcal{L}(\langle x, y \rangle)$

Clash

There is a clash in a CTree $T = \langle V, E, \mathcal{L} \rangle$ if for some $x \in V$ and for some concept C both $C \in \mathcal{L}(x)$ and $\neg C \in \mathcal{L}(x)$.

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Otherwise T is clash-free

Is C satisfiable?

Input: concept C in NNF
Output: answers if C is satisfiable or not
Algorithm:

1. Initialize a new CTree $T = \langle \{s_0\}, \emptyset, \{s_0 \mapsto \{C\}\} \rangle$;
2. Apply completion rules (next slide) while at least one rule is applicable;
3. If no rule is applicable, answer "Yes" if T is clash-free. Otherwise answer "No".

Tableaux Expansion Rules

$\sqcap$-rule: if $C_1 \sqcap C_2 \in \mathcal{L}(x)$, $x \in V$ and $\{C_1, C_2\} \not\subseteq \mathcal{L}(x)$
then $\mathcal{L}(x) := \mathcal{L}(x) \cup \{C_1, C_2\}$

$\sqcup$-rule: if $C_1 \sqcup C_2 \in \mathcal{L}(x)$, $x \in V$ and $\{C_1, C_2\} \cap \mathcal{L}(X) = \emptyset$
then either $\mathcal{L}(x) := \mathcal{L}(x) \cup \{C_1\}$ or $\mathcal{L}(x) := \mathcal{L}(x) \cup \{C_2\}$

$\forall$-rule: if $\forall R.C \in \mathcal{L}(x)$, $x, y \in V$, y R-successor of x, $C \notin \mathcal{L}(y)$
then $\mathcal{L}(y) := \mathcal{L}(y) \cup \{C\}$

$\exists$-rule: if $\exists R.C \in \mathcal{L}(x)$, $x \in V$ with no R-successor y s.t. $C \in \mathcal{L}(y)$
then $V := V \cup \{z\}$, $\mathcal{L}(z) := \{C\}$ and $\mathcal{L}(\langle x, z\rangle) := \{R\}$

Theorem. The tableaux algorithm for deciding satisfiability of concepts always terminates and it is sound and complete.

Correctness of the Tableaux Algorithm

Theorem. The tableaux algorithm for deciding satisfiability of concepts always terminates and it is sound and complete.

Proof: See:

- *Description Logics Handbook.* Baader, F., et al., Cambridge University Press, 2003
- *Semantic Investigations in Distributed Ontologies.* Homola, M., PhD. thesis, Comenius University, 2010

Reasoning with TBox

Lemma. $C \sqsubseteq D$ iff $\top \sqsubseteq \neg C \sqcup D$

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Idea:

- If $C \sqsubseteq D \in \mathcal{T}$ then $\neg C \sqcup D$ must be true for every $x \in \Delta$

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- If $C \sqsubseteq D \in \mathcal{T}$ then $\neg C \sqcup D$ must be true for every $x \in \Delta$
- Add $\neg C \sqcup D$ to $\mathcal{L}(x)$ for every $x \in V$

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- If $C \sqsubseteq D \in \mathcal{T}$ then $\neg C \sqcup D$ must be true for every $x \in \Delta$
- Add $\neg C \sqcup D$ to $\mathcal{L}(x)$ for every $x \in V$

$\mathcal{T}$-rule: if $C_1 \sqsubseteq C_2 \in \mathcal{T}$, $x \in V$ and $\mathrm{nnf}(\neg C_1 \sqcup C_2) \notin \mathcal{L}(x)$
then $\mathcal{L}(x) := \mathcal{L}(x) \cup \{\mathrm{nnf}(\neg C_1 \sqcup C_2)\}$

Problem with Termination

Let $\mathcal{T} = \{C \sqsubseteq \exists R.C\}$
Is C satisfiable?

Blocking

$x \in V$ is blocked if it has an ancestor y such that

- either $\mathcal{L}(x) \subseteq \mathcal{L}(y)$
- or y is blocked

Tableaux Expansion Rules with Blocking

$\sqcap$-rule: if $C_1 \sqcap C_2 \in \mathcal{L}(x)$, $x \in V$ and $\{C_1, C_2\} \not\subseteq \mathcal{L}(x)$
and x is not blocked
then $\mathcal{L}(x) := \mathcal{L}(x) \cup \{C_1, C_2\}$

$\sqcup$-rule: if $C_1 \sqcup C_2 \in \mathcal{L}(x)$, $x \in V$ and $\{C_1, C_2\} \cap \mathcal{L}(X) = \emptyset$
and x is not blocked
then either $\mathcal{L}(x) := \mathcal{L}(x) \cup \{C_1\}$ or $\mathcal{L}(x) := \mathcal{L}(x) \cup \{C_2\}$

$\forall$-rule: if $\forall R.C \in \mathcal{L}(x)$, $x, y \in V$, y R-successor of x, $C \notin \mathcal{L}(y)$
and x is not blocked
then $\mathcal{L}(y) := \mathcal{L}(y) \cup \{C\}$

$\exists$-rule: if $\exists R.C \in \mathcal{L}(x)$, $x \in V$ with no R-successor y s.t. $C \in \mathcal{L}(y)$
and x is not blocked
then $V := V \cup \{z\}$, $\mathcal{L}(z) := \{C\}$ and $\mathcal{L}(\langle x, z\rangle) := \{R\}$

$\mathcal{T}$-rule: if $C_1 \sqsubseteq C_2 \in \mathcal{T}$, $x \in V$ and $\mathrm{nnf}(\neg C_1 \sqcup C_2) \notin \mathcal{L}(x)$
and x is not blocked
then $\mathcal{L}(x) := \mathcal{L}(x) \cup \{\mathrm{nnf}(\neg C_1 \sqcup C_2)\}$

Correctness of the Tableaux Algorithm with TBoxes

Theorem. The tableaux algorithm for deciding satisfiability of concepts w.r.t. a TBox always terminates and it is sound and complete.

Reasoning with TBox and ABox

Is C satisfiable w.r.t. $\langle \mathcal{T}, \mathcal{A} \rangle$?

Change initialization:

- $V := \{a \mid \text{constant } a \text{ occurs in } \mathcal{A}\} \cup \{s_0\}$
- $E := \{\langle a, b \rangle \mid R(a, b) \in \mathcal{A} \text{ for some role } R\}$
- $\mathcal{L}(s_0) := \{C\}$
  $\mathcal{L}(a) := \{\text{nnf}(C) \mid C(a) \in \mathcal{A}\}$ for all $a \in V$
  $\mathcal{L}(\langle a, b \rangle) := \{R \mid R(a, b) \in \mathcal{A}\}$ for all $\langle a, b \rangle \in E$

Correctness of the Tableaux Algorithm with TBox and ABox

Theorem. The tableaux algorithm for deciding satisfiability of concepts w.r.t. TBox and ABox always terminates and it is sound and complete.