Lecture 1: First-Order Logic

2-AIN-108 Computational Logic

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Definition (Alphabet)

An alphabet contains

- Set of variables
  $V = \{x, y, z, \dots\}$
- Set of function symbols
  $F = \{f, g, h, \dots\}$
- Set of predicate symbols
  $P = \{p, q, r, \dots\}$
- Logical connectives
  $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$
- Quantifiers
  $\forall \quad \exists$
- Auxiliary symbols
  ( ) ,

FOL: Syntax (cont.)

Definition (Arity)

Given an alphabet with function symbols F and predicate symbols P, arity is any function $arity : F \cup P \mapsto \mathbb{N}_0$.

Note:

- Arity specifies how many "arguments" each function and predicate requires.
- Functions (predicates) of arity 0, 1, 2, 3, and so on are called: nullary, unary, binary, ternary, etc.

FOL: Syntax (cont.)

Definition (Term)

Given an alphabet and an arity function, a term is any of the following:

- a variable;
- an expression $f(t_1, \ldots, t_n)$ if f is a function symbol with arity n and $t_1, \ldots, t_n$ are terms.

Definition (Atom)

Given an alphabet and an arity function, an atomic formula (atom) is an expression $p(t_1, \ldots, t_n)$ where p is a predicate symbol with arity n and $t_1, \ldots, t_n$ are terms.

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Note: Given nullary f, p, the term $f()$ is called a constant and the atom $p()$ is called a propositional variable.
Note: In such a case we often omit the brackets and write just f, p instead of $f()$, $p()$.

FOL: Syntax (cont.)

Definition (Formula)

Given an alphabet and an arity function, a formula is any expression of the following forms:

- an atom;
- $\neg\Phi$;
- $(\Phi \land \Psi)$;
- $(\Phi \lor \Psi)$;
- $(\Phi \rightarrow \Psi)$;
- $(\Phi \leftrightarrow \Psi)$;
- $(\forall x)\Phi$;
- $(\exists x)\Phi$;

where Φ, Ψ are formulae, and x is a variable.

Note: Any occurrence of a variable x in quantified formulae $(\forall x)\Phi$, $(\exists x)\Phi$ is an occurrence within the scope of the respective quantifier.

FOL: Syntax (cont.)

Definition (Language of FOL)

The language of First Order Logic over some alphabet and the respective arity function is the set $\mathcal{L}$ of all formulae.

FOL: Syntax (cont.)

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Note: from now on we will always assume some fixed FOL language $\mathcal{L}$ over some alphabet with the respective arity function.

FOL: Syntax (cont.)

Definition (Free vs. bounded variable occurrence)

An occurrence of some variable x in a formula Φ is free if it is not within the scope of any quantifier. The occurrence is bounded otherwise.

Definition (Ground term)

A term t is ground if it does not contain any variable.

Definition (Ground formula)

A formula Φ is ground if it does not contain any free occurrence of any variable.

Note: Ground formulae are also called closed formulae or sentences.
Note: from now on we will assume that all formulae are ground.

Definition (Theory)

A first order theory (or just theory) T is a finite set of (ground) formulae.

Note: we will look at theories as knowledge bases: a theory T is a set of formulae that describes some situation or some problem.

Example

Let us assume the following situation: *Jack killed John. If someone killed somebody else, he is a murderer. Murderers go to jail.* We may encode this in FOL theory T:

$$\mathrm{Killed}(\mathrm{Jack}, \mathrm{John})$$
$$(\forall x)((\exists y)\mathrm{Killed}(x, y) \rightarrow \mathrm{Murderer}(x))$$
$$(\forall x)(\mathrm{Murderer}(x) \rightarrow \mathrm{Jail}(x))$$

FOL: Semantics

Definition (First order structure)

A structure is a pair $\mathcal{D} = (D, I)$ where

- D, called domain, is a nonempty set;
- I, called interpretation, is a function s.t.:
  - $I(f)$ is a function $f^I : D^{arity(f)} \to D$;
  - $I(t)$ is $t^I = f^I(t_1^I, \ldots, t_n^I)$ for any ground term of the form $t = f(t_1, \ldots, t_n)$;
  - $I(p)$ is a relation $p^I \subseteq D^{arity(p)}$.

Note: $D^0 = \{\langle\rangle\}$, hence there are two possible interpretations of each propositional variable p: either $p^I = \{\langle\rangle\}$ (i.e., p is *true*) or $p^I = \emptyset$ (i.e., p is false).

Note: similarly for a constant c: $c^I : D^0 \to D$, i.e., each constant term is interpreted by a constant function which returns one of the elements of D.

FOL: Semantics (cont.)

Definition (Structure extension)

An extension of a structure $\mathcal{D} = (D, I)$ w.r.t. a variable x is a structure $\mathcal{D}' = (D, I')$ where I' is identical to I except for in addition $I'(x) = x^{I'} = d$ for some element $d \in D$.

FOL: Semantics (cont.)

Definition (Satisfaction $\models$)

A formula Π is satisfied w.r.t. a structure $\mathcal{D} = (D, I)$ (denoted by $\mathcal{D} \models \Pi$) based type of Π:

$p(t_1, \ldots, t_n)$: $\mathcal{D} \models p(t_1, \ldots, t_n)$ iff $(t_1^I, \ldots, t_n^I) \in p^I$;

$\neg\Phi$: $\mathcal{D} \models \neg\Phi$ iff $\mathcal{D} \not\models \Phi$;

$\Phi \wedge \Psi$: $\mathcal{D} \models (\Phi \wedge \Psi)$ iff $\mathcal{D} \models \Phi$ and $\mathcal{D} \models \Psi$;

if $\Phi \vee \Psi$: $\mathcal{D} \models (\Phi \vee \Psi)$ iff $\mathcal{D} \models \Phi$ or $\mathcal{D} \models \Psi$;

$\Phi \rightarrow \Psi$: $\mathcal{D} \models (\Phi \rightarrow \Psi)$ iff $\mathcal{D} \not\models \Phi$ or $\mathcal{D} \models \Psi$;

$\Phi \leftrightarrow \Psi$: $\mathcal{D} \models (\Phi \leftrightarrow \Psi)$ iff $\mathcal{D} \models (\Phi \rightarrow \Psi)$ and $\mathcal{D} \models (\Psi \rightarrow \Phi)$;

$(\exists x)\Phi$: $\mathcal{D} \models (\exists x)\Phi$ iff $\mathcal{D}' \models \Phi$ for some ext. $\mathcal{D}'$ of $\mathcal{D}$ w.r.t. x;

$(\forall x)\Phi$: $\mathcal{D} \models (\forall x)\Phi$ iff $\mathcal{D}' \models \Phi$ for all ext. $\mathcal{D}'$ of $\mathcal{D}$ w.r.t. x;

where Φ, Ψ are any formulae and $p(t_1, \ldots, t_n)$ is any ground atom.

Semantics (cont.)

Definition (Model)

A structure $\mathcal{D}$ is a model of Φ if $\mathcal{D} \models \Phi$; $\mathcal{D}$ is a model of a theory T (denoted $\mathcal{D} \models T$) if $\mathcal{D} \models \Phi$ for all $\Phi \in T$.

Definition (Satisfiability)

A formula (or theory) is satisfiable, if it has a model.

Definition (Entailment)

A theory T entails a formula Φ (denoted $T \models \Phi$) if for each model $\mathcal{D}$ of T we have $\mathcal{D} \models \Phi$.

Example (cont.)

Is there a model of our theory T? T was:

$$\mathrm{Killed}(\mathrm{Jack}, \mathrm{John})$$
$$(\forall x)((\exists y)\mathrm{Killed}(x, y) \rightarrow \mathrm{Murderer}(x))$$
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Example (cont.)

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Let us construct $\mathcal{D} = (\{s\}, I)$ with:

$$\begin{aligned}
\mathrm{Jack}^I &= s \\
\mathrm{John}^I &= s \\
\mathrm{Killed}^I &= \{\langle s, s\rangle\} \\
\mathrm{Murderer}^I &= \{\langle s\rangle\} \\
\mathrm{Jail}^I &= \{\langle s\rangle\}
\end{aligned}$$

Example (cont.)

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Is $\mathcal{D}$ a model of T?

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Is it our indented model of T?

Example (cont.)

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Does it hold $T \models \text{Jail}(\text{Jack})$?