Lecture 2: Reasoning with FOL

2-AIN-108 Computational Logic

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Calculus

Intuitions:

1. Formula $P \wedge (P \rightarrow Q) \rightarrow Q$ is a tautology.
2. Hence for any theory T: if $T \models P$ and $T \models P \rightarrow Q$ we can conclude $T \models Q$.
3. We express this with with the derivation rule Modus Ponens:

$$\frac{P, P \rightarrow Q}{Q}$$

Note: A tautology is a formula that is satisfied by any first order structure. A contradiction is a formula that is unsatisfiable.

Calculus (cont.)

1. Calculus is a system which allows us to derive formulae by derivation rules.
2. Derivation of a formula Φ from T is called a proof of Φ from T.
3. We denote by $T \vdash \phi$ if formula Φ is derived from T by the calculus.

Calculus (cont.)

Definition (Soundness)

A calculus is sound iff for all theories T and for all formulae Φ, $T \vdash \Phi$ implies $T \models \Phi$.

Definition (Completeness)

A calculus is complete iff for all theories T and for all formulae Φ, $T \models \Phi$ implies $T \vdash \Phi$.

Calculus (cont.)

Definition (Substitution)

The formula resulting from Φ by substitution of a variable x by some term t (denoted $\Phi\{x/t\}$) is a formula Ψ obtained from Φ by replacing every free occurrence of x by t.
A term t is substitutable for a variable x in a formula Φ iff no occurrence of a variable in t becomes bounded after the substitution.

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Intuition: x is not substitutable for y in the following example:

$$\begin{aligned} \Phi &= (\exists x)(y < x) \\ \Phi\{y/x\} &= (\exists x)(x < x) \end{aligned}$$

Hilbert Calculus

Axioms

1. $(P \rightarrow (Q \rightarrow P))$
2. $((P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R)))$
3. $((\neg P \rightarrow \neg Q) \rightarrow (Q \rightarrow P))$
4. $((\forall x)P \rightarrow P\{x/t\})$
   where term t is substitutable for x in P
5. $((\forall x)(P \rightarrow Q) \rightarrow (P \rightarrow (\forall x)Q))$
   where x does not occur free in P

Inference Rules

- Modus Ponens (MP):

$$\frac{P, (P \rightarrow Q)}{Q}$$

- Generalization (G):

$$\frac{P}{(\forall x)P}$$

Hilbert Calculus (cont.)

A proof of Φ from T in Hilbert Calculus is a sequence $\langle \Phi_1, \Phi_2, \dots, \Phi_n \rangle$ s.t. $\Phi_n = \Phi$ and for all $1 \leq i \leq n$ one of the following holds:

1. Φ_i instantiates an axiom;
2. $\Phi_i \in T$;
3. Φ_i is derived from the formulae $\Phi_1, \dots, \Phi_{i-1}$ by one of the derivation rules.

We write $T \vdash \Phi$ if there exists a proof from of Φ from T.

Example

Prove:

$$(P(t) \rightarrow (\exists x)P(x)) \quad \text{i.e.} \quad (P(t) \rightarrow \neg(\forall x)\neg P(x))$$

where t is substitutable for x in P.

Example

Prove:

$$(P(t) \to (\exists x)P(x)) \quad \text{i.e.} \quad (P(t) \to \neg(\forall x)\neg P(x))$$

where t is substitutable for x in P.

Proof:

1. $((\forall x)\neg P(x) \to \neg P(t))$ (Axiom 4)
2. $(((\forall x)\neg P(x) \to \neg P(t)) \to (P(t) \to \neg(\forall x)\neg P(x)))$ (Axiom 3)
3. $(P(t) \to \neg(\forall x)\neg P(x))$ (MP)

Hilbert Calculus (cont.)

Theorem (Soundness & completeness)

Hilbert calculus for FOL is sound and complete.

Resolution

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$$P(x) \lor \neg Q(x) \tag{1}$$

$$P(x) \lor Q(x) \land S(x, y) \tag{2}$$

$$(\exists x)P(x) \tag{3}$$

$$(\forall x)(\neg P(x) \lor Q(x)) \tag{4}$$

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Note: we will understand clauses as closed, universally quantified formulae, but we will omit the quantifiers.

Resolution (cont)

Definition (Complementary literals)

Given any atom A, we say that the two literals A and $\neg A$ are complementary.

Intuition: (Simplified) resolution rule:

$$\frac{P \vee Q, \neg P \vee R}{Q \vee R} \qquad \frac{Q \vee P, R \vee \neg P}{Q \vee R}$$

Note: we say that the two clauses $P \vee Q$ and $\neg P \vee R$ ($R \vee \neg P$) containing complementary literals P and $\neg P$ resolve into the single clause $Q \vee R$.

Negation Normal Form

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Transform any formula into NNF:

- Double negative law:
  $\neg\neg P / P$
- De Morgan's law:
  $\neg(P \wedge Q)/(\neg P \vee \neg Q)$
  $\neg(P \vee Q)/(\neg P \wedge \neg Q)$
- Quantifiers:
  $\neg(\forall x)P/(\exists x)\neg P$
  $\neg(\exists x)P/(\forall x)\neg P$

Prenex Normal Form

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A formula is in prenex normal form (PNF) iff it is of the form $(Q_1 x_1) \dots (Q_n x_n) F$, $n \geq 0$, where Q_i is a quantifier, x_i is a variable and F is quantifier-free formula.

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Transform a formula in NNF into PNF – push quantifiers outwards:

- Conjunction:
  $((\forall x)P \wedge Q)/(\forall x)(P \wedge Q) \quad (Q \wedge (\forall x)P)/(\forall x)(Q \wedge P)$
  $((\exists x)P \wedge Q)/(\exists x)(P \wedge Q) \quad (Q \wedge (\exists x)P)/(\exists x)(Q \wedge P)$
  if x does not appear as free variable in Q
- Disjunction:
  $((\forall x)P \vee Q)/(\forall x)(P \vee Q) \quad (Q \vee (\forall x)P)/(\forall x)(Q \vee P)$
  $((\exists x)P \vee Q)/(\exists x)(P \vee Q) \quad (Q \vee (\exists x)P)/(\exists x)(Q \vee P)$
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Skolemize a formula in PNF:

1. Given $\Phi = (\forall x_1) \ldots (\forall x_n)(\exists y)\Psi$, replace $(\exists y)\Psi$ with Ψ' in which every occurrence of y is replaced by $f(x_1, \ldots, x_n)$ where f is a new function symbol.
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Note: Φ and the resulting formula Φ' are equisatisfiable (i.e., one is satisfiable iff the other one is). They are not necessarily equivalent.

Note: the new function f is called Skolem function. If f is nullary, it is called Skolem constant.

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Transform Φ into CNF:

1. Reduce eqivalence and implication:
   $(P \leftrightarrow Q)/(P \rightarrow Q) \wedge (Q \rightarrow P)$
   $(P \rightarrow Q)/(\neg P \vee Q)$
2. Negation Normal Form
3. Prenex Normal Form
4. Skolem Normal Form
5. Apply distributive law:
   $((P \wedge Q) \vee R)/((P \vee R) \wedge (Q \vee R))$
   $(P \vee (Q \wedge R))/((P \vee Q) \wedge (P \vee R))$

Resolution (cont.)

Definition (Unification)

Given two literals P, Q and a substitution θ, we say that $\mathit{Unify}(P, Q, \theta)$ is true if $P\theta = Q\theta$.

Resolution (cont.)

Resolution rule:

$$\frac{P_1 \vee \cdots \vee P_i \vee \cdots \vee P_m,\ Q_1 \vee \cdots \vee Q_j \vee \cdots \vee Q_n,\ Unify(P_i, \neg Q_j, \theta)}{P_1 \vee \cdots \vee P_{i-1} \vee P_{i+1} \vee \cdots \vee P_m \vee Q_1 \vee \cdots \vee Q_{j-1} \vee Q_{j+1} \vee \cdots \vee Q_n\theta}$$

where for all k, l: P_k, Q_l are literals.

Resolution (cont.)

Theorem

Given a first order theory T and any formula ϕ we have: $T \models \phi$ iff $T \cup \{\neg\phi\}$ is unsatisfiable.

Resolution

Algorithm: Resolution
Input: FOL theory T, formula ϕ
Output: True if $T \models \phi$

1. Transform $T \cup \{\neg\phi\}$ into CNF, yielding a set of clauses.
2. Exhaustively apply the resolution rule to all possible clauses that contain complementary literals
   - all repeated literals are removed
   - all clauses with complementary literals are discarded
3. if *empty clause* is derived answer "True" $T \wedge \neg\phi$ is not satisfiable; answer "False" if it is not possible to resolve any more clauses.

Resolution (cont.)

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The resolution algorithm is sound and complete, i.e., given input T and Φ, if the algorithm answers "True" then $T \models \Phi$ (soundness), and, if $T \models \Phi$ the algorithm answers "True" (completeness).

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If $T \models \Phi$ then the resolution algorithm eventually terminates, given T and Φ on input.

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Note: the resolution algorithm may not terminate, if $T \not\models \Phi$ – the algorithm is semidecidable.