Roman Durikovic (323-B, roman@u-aizu.ac.jp, http://www.u-aizu.ac.jp/∼roman)

Surface Classification via Topological Surgery. Computer Representation of Polygonal Surfaces.

Topological Surgery

In this section we give topological classification of closed (without boundary), bounded, connected surfaces.

Constructing a planar model

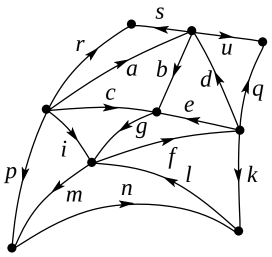
Consider a closed, bounded, connected surface S. By boundedness, S has a triangulation using only a finite number of triangles. Let us mark each edge of the triangulation with a letter, without repetition. We also draw an arrow along each edge.

Next we cut the surface along all the edges, decomposing S into a finite set of triangles, each side of which is marked with a letter and provided with an arrow.

Then we glue all these triangles together to obtain a plane polygon whose edges are marked with letters and arrows.

The edges of the polygon are divided into pairs of two types: those with arrows pointing in opposite directions around the polygon, and those with arrows pointing the same direction.

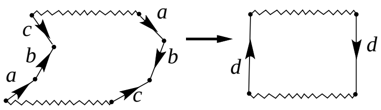
Throughout this lecture, diagrams are drawn with jagged edges to represent any unspecified edges that are not under current study.



Simplification of the planar model

We will simplify the planar model in several steps, always preserving the property that it represents surface S.

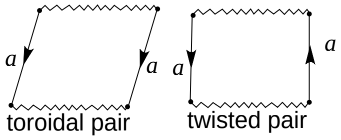
1. Shortcut. If a string of edges occurs twice in exactly the same order, taking into account the directions of the edges, we can consider the string as a single edge.



Algebraically the process depicted above can be written as

$$Xabc^{-1}Yabc^{-1}Z \longrightarrow XdYdZ$$

2. Eliminating adjacent toroidal pairs. Note the edges can occur in two forms: toroidal pairs and twisted pairs.



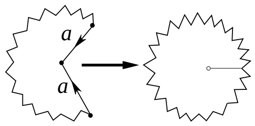
Algebraically an toroidal pair corresponds to

$$XaYa^{-1}Z$$

and a twisted pair corresponds to

$$XaYaZ$$

Adjacent toroidal edges can be eliminated by folding them in and gluing the edges together, as in the figure below.

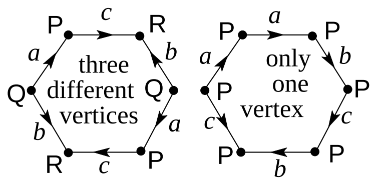


Algebraically this step can be written as

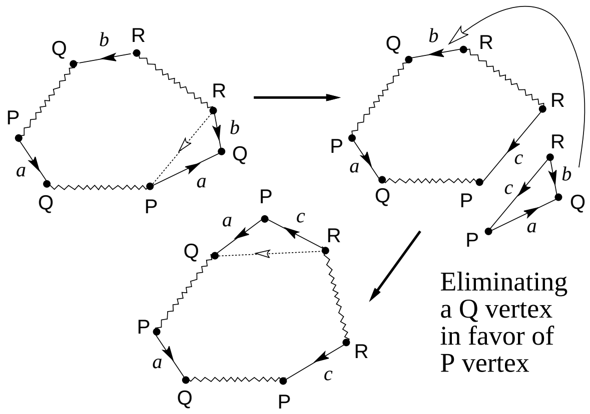
$$Xaa^{-1}Y \longrightarrow XY$$

3. Eliminating all but one vertex. Let us figure out how many different vertices there are in the planar diagram. The planar diagram in the left image of the figure below has three different vertices P, Q, R. Vertex P is the head of c and the tail of a. Vertex Q is the tail of both a and b. Vertex R is the head of both b and c.

The planar diagram of the right image of the figure below has only one vertex, since P is the head of a, which is the same as the tail of a, which is glued to the tail of c, which is glued to the head of b which is glued to the head of c, which is the tail of b.



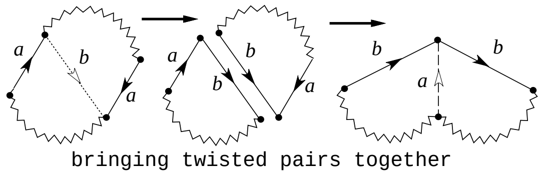
In the process depicted in the figure below, we want to change the Q vertex into a P vertex. Cut along the dotted line and reglue along b, the edge which contains the undesirable vertex Q, to obtain one more P vertex and one less Q vertex.



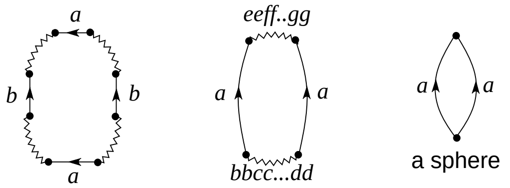
Repeat this step as necessary to eliminate all but one vertex.

4. Collecting twisted pairs. Algebraically this step can be written as

$$XaYaZ \longrightarrow XY^{-1}aaZ$$



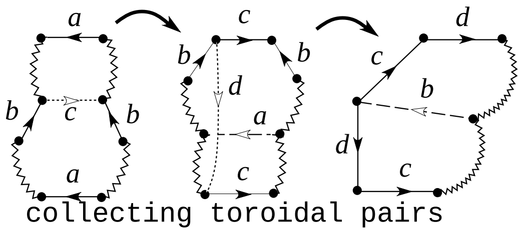
5. Collecting toroidal pairs. Suppose the planar model contains a toroidal pair. Then we claim that there is a second toroidal pair separating the edges of the first pair from each other, as in the left image of the figure below. To prove this, suppose there were no such separating pair. Then the remaining portions of the planar model consist of adjacent twisted pairs, as in the middle image of the figure below. Thus the model contains either two different vertices or contains only one pair of edges aa and represents a sphere (see the right image of the figure below).



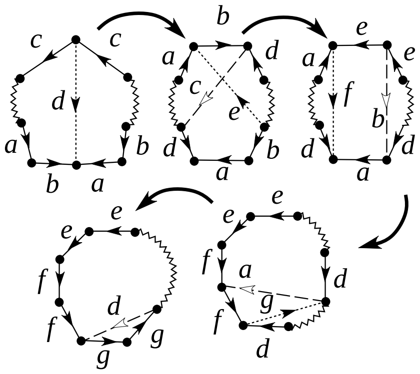
Algebraically collecting toroidal pairs can be written as

$$XbYaZb^{-1}Sa^{-1} \longrightarrow aba^{-1}b^{-1}YXSZ$$

The corresponding surgery is demonstrated below.



6. Toroidal and twisted pairs $\longrightarrow$ twisted pairs. The surgery transforming toroidal and twisted pairs in twisted pairs is shown below.



Classification.

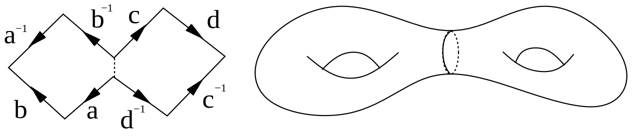
We have shown that any planar diagram of a closed, bounded, connected surface can be transformed to a diagram represented by a word of one of the two following forms:

$$W_1 = aa^{-1}$$
$$W_2 = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \dots a_k b_k a_k^{-1} b_k^{-1}$$
$$W_3 = c_1 c_1 c_2 c_2 \dots c_n c_n,$$

where k and n are positive integer numbers.

A planar diagram W representing a given surface S and having one of the above forms is called a *canonical fundamental diagram* of S.

We have shown that W_1 represents a sphere. It is easy to see that W_2 represents a sphere with k handles glued.



One can show that W_3 represents a sphere with n holes glued by Möbius strips.

Polygonal Surface Computer Representation

Coding positions of vertices and connectivity. A polygonal surface can be described by postions of its vertices and information how the vertices are connected with each other. For example, a unit cube consisting of 12 triangles can be described as follows

```
8             number of vertices
12            number of triangles
0 0 0         x0 y0 z0
1 0 0         x1 y1 z1
0 1 0         x2 y2 z2  coordinates
1 1 0         x3 y3 z3     of
0 0 1         x4 y4 z4   vertices
1 0 1         x5 y5 z5
0 1 1         x6 y6 z6
1 1 1         x7 y7 z7
3 0 1 3
3 0 2 3
3 0 4 6          connectivity information
3 0 2 6
3 0 1 5          3 0 1 5 means that vertices
3 0 4 5       (x0,y0,z0), (x1,y1,z1), (x5,y5,z5)
3 2 6 3             form a triangle
3 6 7 3
3 1 3 5
3 3 5 7
3 5 6 7
3 4 5 6
```

A simple representation of a triangulated surface. There are three primary data types: vertices, edges and faces. All the vertices are doubly linked into a list, as are all the edges, and all the faces. The ordering of the elements in the list has no significance. The vertex structure contains the coordinates of the vertex only. The edge structure contains pointers to the two vertices that are endpoints of the edge, and the pointers to the two adjacent faces. The face structure contains pointers to the three vertices forming the corners of the triangular face, as well as pointers to the three edges.

Winged-edge data structure (Baumgart, 1975). Each vertex points to an arbitrary one of its incident edges, and each face points to an arbitrary one of its bounding edges. The edge structure for e consists of eight pointers: to the two endpoints of e, v_0 and v_1; to the two faces adjacent to e, f_0 and f_1, left and right respectively of v_0v_1; and to four edges (the "wings" of e): e_0^- and e_0^+, edges incident to v_0 clockwise and counterclockwise of e respectively; and e_1^- and e_1^+, edges incident to v_1. See Fig. 1.

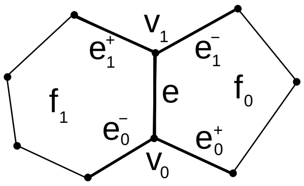


Figure 1: The winged-edge data structure.

For example, the edges bounding the a face f may be found by retrieving the sole edge e stored in the record of f, and then following the e_1^+ edges around f until e is again encountered.

Quad-edge data structure (Guibas & Stolfi, 1985). It is more complex representation but, in fact, it simplifies many operations and algorithms. It has the advantage of being very general, representing any subdivision, permitting distinctions between two sides of a surface, allowing the endpoints of an edge to be the same vertex, permitting dangling edges, etc.

Each edge record is part of four circular lists: for the two endpoints, and for the two adjacent faces. Thus it contains four pointers. See Fig. 2.

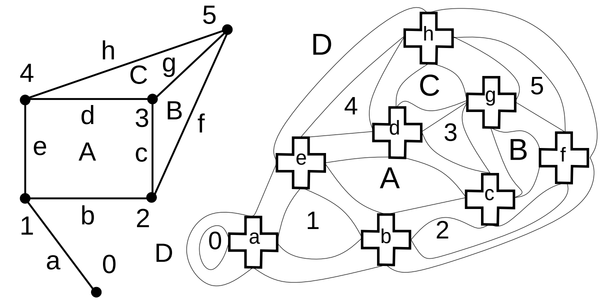


Figure 2: A graph and its associated quad-edge data structure.

Problems

1. Find canonical fundamental diagram of the surfaces represented by the diagrams
$$abcdec^{-1}da^{-1}b^{-1}e^{-1} \qquad ae^{-1}a^{-1}bdb^{-1}ced^{-1}c^{-1}.$$
2. Explain why
$$W_2 = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \dots a_k b_k a_k^{-1} b_k^{-1}$$
represents a sphere with k handles.
3. Find the Euler characteristic of the surface defined by the planar diagram $a\,b\,c\,a^{-1}b^{-1}c^{-1}$.
4. Find the Euler characteristic of the surface