Applied Computational Topology Lecture 2.
Surface Recognition: Algorithmic Problems

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Orientation.

Let us consider a complex consisting of polygons. A polygon is said to be oriented if it is equipped with a circuit oriented either clockwise or counterclockwise. Two oriented adjacent polygons A and B are said to agree if the common edge e between the two polygons is oriented one way in the boundary of A and the other way in the boundary of B.

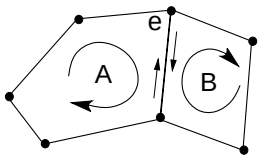

Figure 1:

The complex $\mathcal{K}$ is **oriented** if $\mathcal{K}$ is is directed in such a way that directions of adjacent polygons always agree. A surface S is **orientable** if every complex equivalent to S is orientable.

The Möbius strip and Klein bottle are not orientable.

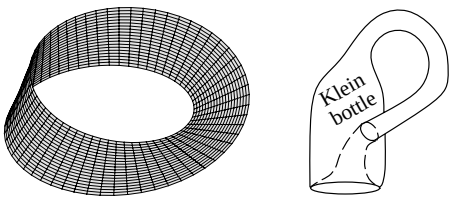

Figure 2: Möbius strip and Klein bottle

The Classification Theorem.

Theorem 1 *Two bounded, closed, connected surfaces are topologically equivalent if and only if they have the same Euler characteristic and orientability. An orientable, bounded, closed, connected surface is topologically equivalent to a sphere with h handles. A nonorientable, bounded, closed, connected surface is topologically equivalent to a sphere with with μ holes glued by Möbius strips.*

The number of handles h of an orientable surface is called the **genus** of the orientable surface. The number of Möbius strips μ of a nonorientable surface is called the **genus** of the nonorientable surface.

Theorem 2 *Every bounded, closed, connected surface with boundary is equivalent either to a sphere a sphere with h handles or a sphere with μ holes glued by Möbius strips, in any case with some number of disks removed.*

Simplicial complexes.

A zero-dimensional simplex is a vertex, an one-dimensional simplex is a segment, a two-dimensional simplex is a triangle, a 3-dimensional simplex is a tetrahedron, etc. A simplicial complex is a complex obtained by gluing simplexes together along their vertices, edges, triangles, etc.

A simplicial complex is given by the set of its vertices with specification of which vertices are connected by edges.

If P is a vertex of a simplicial complex $\mathcal{K}$, then the union of all simplexes of the complex $\mathcal{K}$, for which P is a vertex, is called the **star** of P and is denoted by $\mathrm{St}(P,\mathcal{K})$. The union of all those simplexes of the star $\mathrm{St}(P,\mathcal{K})$ for which P is not a vertex is called a **link** of P and is denoted by $\mathrm{link}(P,\mathcal{K})$

For a triangulation, the link of a vertex P is the collection of the edges opposite to P in all the triangles incident to P.

Conceptual algorithms.

1. An algorithm for recognition of a surface.

Q. How to understand whether a given two-dimensional simplicial complex is a triangulated surface or not?

A. Find the links of all vertices. If all of them turn out to be closed or unclosed polygonal lines, the complex is a triangulation of a surface.

2. Recognition of connectedness.

Q. How to understand whether a given two-dimensional simplicial complex is connected or not?

A. Mark an arbitrary vertex. Then mark all the vertices joined by edges with the vertex. Continue the process until it ends. If all the vertices turn out to be marked, the complex is connected, otherwise it is not.

3. Recognition of orientability of a connected surface.

Q. How to understand whether a given surface is orientable or not?

A. Choose an arbitrary triangle of of the triangulation and orient it, i.e. indicate a circuit around its sides. Orient all the adjacent triangles with respect to the orientation, i.e. any two adjacent triangles induce opposite orientations on their common edge. Continue the process until either all triangles are oriented or a contradiction arises. If the process ends without contradiction, the surface is orientable.

3. Surface genus recognition. The genus of a surface is the number of handles and holes in the orientable case and holes and twisted pairs (holes glued by the Möbius strips) in the non-orientable case.

Q. How to compute the number of handles and holes of a given connected oriented surface?

A. Calculate the Euler characteristic χ. Count the number of boundary components q. The number of handles, p, can be easily found from the formula $\chi = 2 - 2p - q$.

How to represent a polyhedral surface in a computer.

It is not obvious how best to represent a polyhedral surface in a computer and several sophisticated suggestions can be found the literature.

A simple representation of a triangulated surface. There are three primary data types: vertices, edges and faces. All the vertices are doubly linked into a list, as are all the edges, and all the faces. The ordering of the elements in the list has no significance. The vertex structure contains the coordinates of the vertex only. The edge structure contains pointers to the two vertices that are endpoints of the edge, and the pointers to the two adjacent faces. The face structure contains pointers to the three vertices forming the corners of the triangular face, as well as pointers to the three edges.

Winged-edge data structure (Baumgart, 1975). Each vertex points to an arbitrary one of its incident edges, and each face points to an arbitrary one of its bounding edges. The edge structure for e consists of eight pointers: to the two endpoints of e, v_0 and v_1; to the two faces adjacent to e, f_0 and f_1, left and right respectively of v_0v_1; and to four edges (the "wings" of e): e_0^- and e_0^+, edges incident to v_0 clockwise and counterclockwise of e respectively; and e_1^- and e_1^+, edges incident to v_1. See Fig. 3.

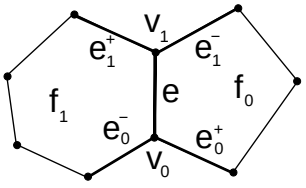

Figure 3: The winged-edge data structure.

For example, the edges bounding the a face f may be found by retrieving the sole edge e stored in the record of f, and then following the e_1^+ edges around f until e is again encountered.

Quad-edge data structure (Guibas & Stolfi, 1985). It is more complex representation but, in fact, it simplifies many operations and algorithms. It has the advantage of being very general, representing any subdivision, permitting distinctions between two sides of a surface, allowing the endpoints of an edge to be the same vertex, permitting dangling edges, etc.

Each edge record is part of four circular lists: for the two endpoints, and for the two adjacent faces. Thus it contains four pointers. See Fig. 4.

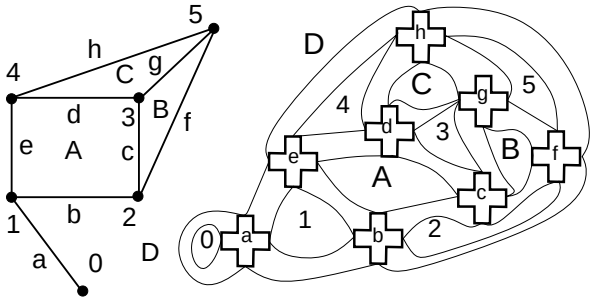

Figure 4: A graph and its associated quad-edge data structure.