Roman Durikovic (323-B, roman@u-aizu.ac.jp, http://www.u-aizu.ac.jp/∼roman)

Computational Topology of Polygonal Surfaces.

The angle defect of a polyhedron

A 3D solid which consists of a collection of polygons joined at their edges is called a **polyhedron**.

The **angle defect** at a vertex of a polyhedron is defined to be 2π minus the sum of the angles at the corners of the faces at that vertex. For example, at any vertex of a cube there are three angles of $\pi/2$, so the angle defect is $\pi/2$. One can visualize the angle defect by cutting along an edge at that vertex, and then flattening out a neighborhood of the vertex into the plane. A little gap will form where the slit is; the angle by which it opens up is the angle defect.

The **total angle defect** of the polyhedron is computed by adding up the angles defects at all the vertices of the polyhedron. For a cube, the total angle defect is $8 \times \pi/2 = 4\pi$.

Theorem 1 (Descartes's formula) *Denote the total angle defect of a polyhedron by T. Then*

$$T = 2\pi(V - E + F).$$

PROOF. We will try to cancel out the terms as much as possible, by grouping within polygons.

For each edge, there is -2π to allocate. An edge has a polygon on each side: put $-\pi$ on one side, and $-\pi$ on the other.

For each vertex, there is $+2\pi$ to allocate: we will do it according to the angles of the polygons at that vertex. If the angle of a polygon at the vertex is α, allocate α of the 2π to that polygon. This leaves something at the vertex: the angle defect.

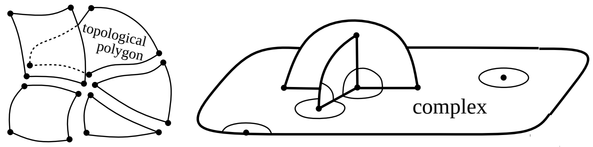
In each polygon, we now have a total of the sum of its angles minus $n\pi$ (where n is the number of sides) plus 2π (contribution of the faces). Since the sum of the angles of any polygon is $(n-2)\pi$, this is 0. □

WHAT IS TOPOLOGY ABOUT

Topology is a branch of geometry. Imagine a geometric figure, such as a circular disk, cut from a sheet of rubber and subjected to all sorts of twisting, pulling, and stretching. Any deformation of this sort is permitted, but tearing and gluing are forbidden. Mathematically these allowed distortions are called **continuous transformations**. **Topology** studies properties of figures that endure when the figures are subjected to continuous transformations. So topology has earned the nickname **rubber sheet geometry.**

Basic definitions

- Two shapes are called **topologically equivalent** if any one can be continuously transformed to any other.
- Let us call any figure topologically equivalent to a disk a **cell**.
- A cell is called a **topological polygon** when a finite number of points on the boundary of the cell are chosen as vertices.
- The sections of boundary in between the vertices are called **edges**.
- A **complex** is a geometric figure that can be constructed from polygons (cells) by gluing and pasting them together along their edges such that vertices are sewn to vertices and whole edges are sewn to whole edges.



Euler's formula

Theorem 2 (Euler's formula) *Given a complex topologically equivalent to a sphere, let F stand for the number of cells (called faces), E the number of edges, and V the number of vertices. Euler's formula states that*

$$V - E + F = 2.$$

PROOF. Let us remove one of the faces of the complex. The remainder is topologically equivalent to a cell and so can be flattened into a plane. The result is a cell in the plane divided into polygons. It remains to prove that for a complex equivalent to a cell we have

$$V - E + F = 1.$$

Let us triangulate the complex: divide each polygon into triangles by drawing diagonals. Each diagonal adds one edge and one face to the complex so the quantity $V-E+F$ is unchanged by this process. Finally let us remove triangles of the complex one by one starting with those on the boundary. There are two types of removal, depending on whether the triangle being removed has one or two edges on the boundary. Both the removals don't change the quantity $V - E + F$. Eventually it remains just one triangle for which $V - E + F = 1$. □

Regular polyhedra

A **regular polygon** is one with equal sides and equal angles: equilateral triangle, square, regular pentagon, regular hexagon, and so on. Clearly, there are infinitely many regular polygons, one for each n.

A **regular polyhedron** is one in which all faces have the same number of edges, and the same number of faces meet at each vertex. The regular polyhedra are called the **Platonic solids**.

Theorem 3 *There are only five regular polyhedra in 3D: tetrahedron, cube, octahedron, dodecahedron, icosahedron.*

PROOF. Since each edge belongs to two faces, $aF = 2E$. Similarly, since each edge has two vertices, $bV = 2E$. Euler's formula now gives

$$\frac{2E}{a} - E + \frac{2E}{b} = 2 \qquad \text{or} \qquad \frac{1}{a} + \frac{1}{b} - \frac{1}{2} = \frac{1}{E}, \tag{1}$$

where a, b, and E must be positive integers. Actually both a and b must be greater then two, they must also be less than six because the left-hand side of (1) is positive. Both a and b are positive integers and the solutions can be found by trial and error. The five Platonic solids are

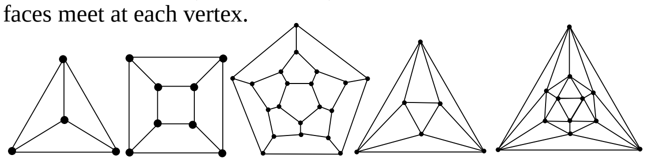
(1) the **tetrahedron** ($a = 3$, $b = 3$): each face has three sides, three faces meet at each vertex;

(2) the **cube** ($a = 4$, $b = 3$): each face has four sides, three faces meet at each vertex;

(3) the **octahedron** ($a = 3$, $b = 4$): each face has three sides, four faces meet at each vertex;

(4) the **icosahedron** ($a = 3$, $b = 5$): each face has three sides, five faces meet at each vertex;

(5) the **dodecahedron** ($a = 5$, $b = 3$): each face has five sides, three faces meet at each vertex.



The Euler characteristic

Let K be a complex with V vertices, E edges, and F faces. The Euler characteristic of K is

$$\chi(K) = V - E + F$$

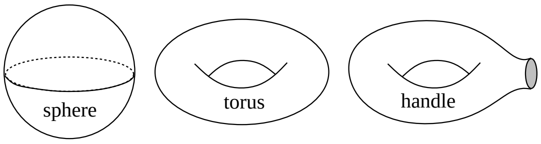
Surfaces

A **surface** is a complex in which every point has a neighborhood that is topologically equivalent to either an open disk or a half of an open disk. The points whose neighborhoods topologically equivalent to a half of an open disk form the surface **boundary**.

A sphere and a torus are the simplest examples of the surfaces without boundaries. A simple surface with boundary is the cylinder obtained from a rectangle by gluing together a pair of opposing edges.

Let a surface be represented by a complex K. The number $\chi(K)$ is called the Euler characteristic of the surface. Of course two different complexes may represent the same surface.

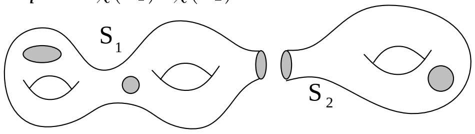
Theorem 4 *The Euler characteristic of a surface does not depend on representation of the surface as a complex.*



Theorem 5 *The Euler characteristics of a sphere, torus, and handle are given by*

$$\chi(\text{sphere}) = 2, \quad \chi(\text{torus}) = 0, \quad \chi(\text{handle}) = -1$$

Theorem 6 *Let S_1 and S_2 be two surfaces with boundaries and each surface has a boundary component equivalent to a circle. Gluing together S_1 and S_2 along that circles gives a surface whose Euler characteristic is equal to $\chi(S_1)+\chi(S_2)$.*



Theorem 7 *The Euler characteristic of a sphere with p handles and q holes is given by*

$$\chi = 2 - 2p - q$$

Triangulated surfaces

Let a triangulation of a surface consists of F triangles (faces), E edges, and V vertices. Suppose that the surface is topologically equivalent to a sphere with some number of handles and holes and has the Euler characteristic χ. Euler's formula gives

$$V - E + F = \chi$$

Let the surface have q holes with $B_1, B_2, \ldots, B_q$ vertices on the boundaries of the holes, $B = B_1 + \ldots + B_q$ be the total number of boundary vertices. The kth hole can be triangulated by adding $(B_k - 3)$ edges and $(B_k - 2)$ faces (triangles). Thus, if we add $\sum(B_k - 3) = B - 3q$ edges and $\sum(B_k - 2) = B - 2q$ triangles, we obtain a closed surface for which

$$3(F + B - 2q) = 2(E + B - 3q) \quad \text{or} \quad 2E = 3F + B$$

Combining it with Euler's formula we get

$$F = 2V - B - 2\chi$$

Usually for a triangulated model consisting of a large number of points, the number of vertices is much greater than the number of boundary vertices. Thus

$$F \approx 2V$$

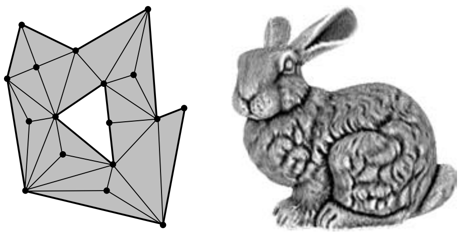
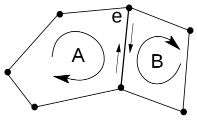


Figure 1: Left: a triangulated polygon with $V = 17$, $B = 12$, $E = 39$, $F = 22$, $\chi = 0$. Right: the Stanford bunny model consisting of 35947 vertices and 69451 triangles.

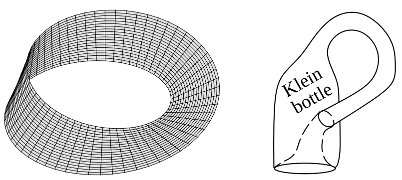
Orientation.

Let us consider a complex consisting of polygons. A polygon is said to be oriented if it is equipped with a circuit oriented either clockwise or counterclockwise. Two oriented adjacent polygons A and B are said to agree if the common edge e between the two polygons is oriented one way in the boundary of A and the other way in the boundary of B.



The complex $\mathcal{K}$ is **oriented** if $\mathcal{K}$ is is directed in such a way that directions of adjacent polygons always agree. A surface S is **orientable** if every complex equivalent to S is orientable.

The Möbius strip and Klein bottle are not orientable.



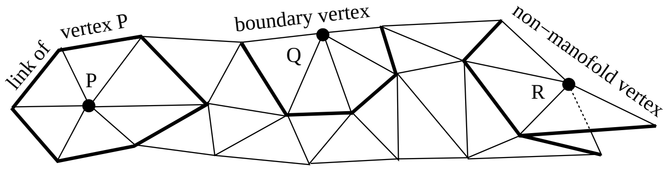
The Classification Theorem.

Theorem 8 *Every bounded, closed, connected surface with boundary is equivalent either to a sphere a sphere with h handles or a sphere with μ holes glued by Möbius strips, in any case with some number of disks removed.*

The number of handles h of an orientable surface is called the **genus** of the orientable surface. The number of Möbius strips μ of a nonorientable surface is called the **genus** of the nonorientable surface.

Surface Recognition Algorithms

An algorithm for recognition of a surface. Find the links of all vertices. If all of them turn out to be closed or unclosed polygonal lines, the complex is a triangulation of a surface.



Recognition of connectedness. Mark an arbitrary vertex. Then mark all the vertices joined by edges with the vertex. Continue the process until it ends. If all the vertices turn out to be marked, the complex is connected, otherwise it is not.

Recognition of orientability of a connected surface. Choose an arbitrary triangle of of the triangulation and orient it, i.e. indicate a circuit around its sides. Orient all the adjacent triangles with respect to the orientation, i.e. any two adjacent triangles induce opposite orientations on their common edge. Continue the process until either all triangles are oriented or a contradiction arises. If the process ends without contradiction, the surface is orientable.

Surface genus recognition. Calculate the Euler characteristic χ. Count the number of boundary components q. The number of handles, p, can be easily found from the formula $\chi = 2 - 2p - q$.

Problems

1. Consider a triangulation of a compact connected closed surface S. Show that
$$3F = 2E, \quad 2E \leq V(V-1), \quad V \geq \frac{7 + \sqrt{49 - 24\chi}}{2}.$$
If S is a sphere, then $V \geq 4$, $E \geq 6$, $F \geq 4$.

2. Consider a regular triangulation of a torus: the same number of triangles, say a, meet at each vertex. Determine a.

3. There are 20 points inside a square. They are connected by non-intersecting segments with each other and with the vertices of the square, in such a way that the square is dissected into triangles. How many triangles are there?

4. Consider a triangulation of a sphere such that exactly five edges meet at each vertex. Find the total number of faces (triangles).

5. Consider a triangulation of a sphere. Suppose the triangulation has only two types of vertices: where 5 edges meet and where 6 edges meet. Let V_5 be the number of vertices of the first type (at which exactly 5 edges meet) and V_6 be the number of vertices of the second type (at which exactly 6 edges meet). Determine V_5.

6. Consider a polyhedron topologically equivalent to a sphere. Suppose that the polyhedron has no triangles or 4-gones. Show that the polyhedron must have at least twelve 5-gones.

7. Show that for any triangulation of a torus
$V \geq 7$, $E \geq 21$, $F \geq 14$.